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Physica D 198 (2004) 136–147
PHYSICA D
www.elsevier.com/locate/physd

The transition between quasi-static and fully dynamic for interfaces

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Received 6 April 2004; received in revised form 8 August 2004; accepted 20 August 2004

Communicated by Y. Nishiura

Abstract

Renormalization group and scaling theory have been used to determine the large time growth exponent for the characteristic length, $R(t)$, of an interface in the form $R(t) \sim t^{\beta}$. The exponent β is different in the two cases: quasi-static, in which the time derivative in the heat equation is suppressed, and the fully dynamic system. This paper examines the transition between the two regimes through an examination of the Green's function for elliptic equations as a limit of the fundamental solution for parabolic equations. The key interface equation can be written as a sum of two terms: the elliptic ($c = 0$) and parabolic. For $c = 0$, the exponent β can take on values in a continuous spectrum. As c takes on finite values, a unique exponent is selected from this spectrum.
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AMS subject classifications: 82C24; 82B24; 35K55

Keywords: Renormalization group; Interface dynamics; Quasi-static regime; Fully dynamic system; Green's function

1. Introduction

Renormalization and scaling techniques (RG) have been used successfully to determine the large time behavior of interfaces (see [6,1,2,8]). Throughout this paper, the basic interface equations we consider are given by

$$C_v T_t = K \Delta T \quad \text{in } \Omega \backslash \Gamma \tag{1.1}$$

$$l v_{\text{n}} = -K[\nabla T \cdot \hat{n}]_{-}^{+} \quad \text{on } \Gamma \tag{1.2}$$

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doi:10.1016/j.physd.2004.08.026

$$T = \frac{-\sigma_0}{[s]_{eq}}(\kappa + \alpha v_n) \quad \text{on } \Gamma \tag{1.3}$$

where C_v is the specific heat per unit volume, K is the thermal conductivity per unit volume, l is the latent heat per unit volume, σ_0 is the surface tension, $[s]_{eq}$ is the entropy difference per unit volume between phases, α is the dynamic undercooling and $[\cdots]_-^+$ is the difference in the limiting values between the two sides of the interface. The variables v_n and κ denote the (normal) velocity and the sum of the principle curvatures at a point on the interface, respectively. In addition, +/− and $\Gamma(t)$ denote the phase with higher/lower energy (which we call the liquid and solid) and the interface, respectively, and Ω is assumed to be $\Omega := \mathbb{R}^d$. A problem of both theoretical and practical importance involves the nature of the large time behavior of the interface. Jasnow and Vinals [5,6] used a quasi-static version of this model to study large time growth that is obtained from the difference between the interface position and a plane wave solution that is imposed through the boundary conditions. In particular, they used (1.1)–(1.3) with $C_v = \alpha = 0$, and found that the characteristic length, $R(t)$, of the self-similar system varies as t. Subsequently, Caginalp [1,2] used the full set of dynamic equations (1.1)–(1.3), with and without reference to a plane wave and found that the characteristic length varies as $R(t) \sim t^{1/2}$. An analysis by Merdan and Caginalp [8] for the quasi-static case considered the set of models that can be obtained from (1.1)–(1.3) and established the long term behavior for the characteristic length, finding a spectrum of characteristic length exponents from which a single one is selected (in the case $\alpha = 0$) with the imposition of boundary conditions that produce a plane wave (i.e., the Jasnow and Vinals result).

The characteristic length, $R(t)$, is the time-dependent length scale governing the morphology of late stage pattern growth. For example, it may be the radius of a circle which contains the pattern evolving self-similarly in time (see [7]). In many interesting physical situations, e.g. dentritic growth, the interface appears to have a stochastically self-similar behavior that is approached asymptotically for large time.

In other words, for some $\beta > 0$, one has $R(t) \sim t^\beta$ for large t. If we choose two large times $t_2 > t_1$, then magnifying the interface at time t_1 by factor $(t_2/t_1)^\beta$ will yield an interface that is stochastically equivalent to the actual interface at time t_2. Of course, since there is some randomness in the sidebranching arising from interface instability, the self-similarity will not be exact. In order to obtain the scaling relationships, we state the self-similarity in the exact form in (3.18) and (3.19).

An interesting feature of the quasi-static problem is the existence of a scaling regime in which surface tension is invariant. Since the fully parabolic problem always has zero surface tension as a fixed point, an examination of the transition between parabolic and elliptic is key to understand the crossover behavior of the characteristic length. For many materials (e.g. aluminium and other light metals), the heat diffusion is very rapid, and the quasi-static approximation is regarded as an accurate one. Consequently, it is of practical importance to understand this crossover behavior in $R(t)$ in order to gauge the significance of the surface tension for large time.

An important theoretical question remains, however, with this analysis, namely what is the nature of the transition between the different regimes, quasi-static and fully dynamic? What type of mathematical analysis underlies this transition, and can it facilitate other RG studies of dynamic situations? We address these questions (for $d > 2$) by (i) transforming the Eqs. (1.1)–(1.3) into a single equation for points on the interface using the fundamental solution for a parabolic equation, (ii) exploring the limit as the specific heat, C_v, approaches zero, i.e., the quasi-static limit, (iii) performing the RG procedure, and finally, (iv) understanding the transition between the scaling regimes.

2. The transition between the parabolic fundamental solution and the elliptic Green's function

We consider the heat equation with a source term, namely

$$u_t - \frac{1}{c}\Delta u = \frac{f}{c} =: g \tag{2.1}$$

on $\mathbb{R}^d$, where f is a smooth function of compact support and c is a constant. The fundamental solution can be written as

$$u(x,t) = \int_0^t \int_{\mathbb{R}^d} \left[\frac{4\pi}{c}(t-s)\right]^{-d/2} \exp\left(\frac{-c|x-y|^2}{4(t-s)}\right) \frac{f(y,s)}{c} \mathrm{d}^d y \, \mathrm{d}s = \int_0^t \int_{\mathbb{R}^d} G(x-y, t-s) \frac{f(y-s)}{c} \mathrm{d}^d y \, \mathrm{d}s \tag{2.2}$$

where the Green's function is given by

$$G(\xi,\tau) := \left(\frac{4\pi}{c}\tau\right)^{-d/2} \exp\left(\frac{-c|\xi|^2}{4\tau}\right). \tag{2.3}$$

For $f \in C^3$ ($\mathbb{R}^d$), we can write

$$f(y,s) = f(y,t) + (s-t)D_2 f(y,t) + \text{remainder term}, \tag{2.4}$$

where we use $D_2 f(y, t)$ to denote differentiation with respect to the second variable. Using this expression, we can write (2.2) as

$$u(x,t) = F_1(x,t) + F_2(x,t) \tag{2.5}$$

where

$$F_1(x,t) := \int_{\mathbb{R}^d} f(y,t) L_1(x-y,t) \mathrm{d}^d y, \tag{2.6}$$

$$L_1(x-y,t) := \int_0^t \left[\frac{4\pi}{c}(t-s)\right]^{-d/2} \exp\left(\frac{-c|x-y|^2}{4(t-s)}\right) \frac{\mathrm{d}s}{c} \tag{2.7}$$

and

$$F_2(x,t) := \int_{\mathbb{R}^d} D_2 f(y,t) L_2(x-y,t) \mathrm{d}^d y, \tag{2.8}$$

$$L_2(x-y,t) := \int_0^t (s-t) \left[\frac{4\pi}{c}(t-s)\right]^{-d/2} \exp\left(\frac{-c|x-y|^2}{4(t-s)}\right) \frac{\mathrm{d}s}{c}. \tag{2.9}$$

The integral L_1 will be split into two pieces, one of which will lead to the Green's function for the limiting elliptic equation. We first transform variables as

$$z := \frac{t-s}{c} \quad \text{and} \quad a := \frac{|x-y|^2}{4}. \tag{2.10}$$

Then we can write $L_1 := L_{1\infty} - L_{1R}$ with the two parts defined by

$$L_{1\infty} := \int_0^\infty (4\pi z)^{-d/2} \mathrm{e}^{-a/z} \, \mathrm{d}z \tag{2.11}$$

and

$$L_{1R} := \int_{t/c}^{\infty} (4\pi z)^{-d/2} e^{-a/z}\, dz. \tag{2.12}$$

Similarly for the L_2 integral, we write $L_2 := -L_{2\infty} + L_{2R}$ with the two parts defined by

$$L_{2\infty} := (4\pi)^{-d/2} c \int_{0}^{\infty} z^{-(d/2)+1} e^{-a/z}\, dz \tag{2.13}$$

and

$$L_{2R} := (4\pi)^{-d/2} c \int_{t/c}^{\infty} z^{-(d/2)+1} e^{-a/z}\, dz. \tag{2.14}$$

Recalling that the gamma function can be expressed as

$$\int_{0}^{\infty} x^n e^{-x}\, dx = \Gamma(n+1) \tag{2.15}$$

we write the main component of the L_i integrals as

$$I_{\infty}(n) := \int_{0}^{\infty} z^{-n} e^{-a/z}\, dz = a^{-n+1}\Gamma(n-1). \tag{2.16}$$

Similarly, the remainder part of the integral can be written, for large T, to leading order as

$$I_R(n) := \int_{T}^{\infty} z^{-n} e^{-a/z}\, dz = T^{-n+1}\left\{\frac{1}{n-1} - \frac{a}{nT} + \frac{a^2}{(n+1)T^2}\right\}. \tag{2.17}$$

Using these formulae to evaluate the L_i integrals, we obtain

$$L_{1\infty} = (4\pi)^{-d/2} a^{-(d/2)+1}\Gamma\left(\tfrac{1}{2}d - 1\right), \tag{2.18}$$

$$L_{2\infty} = (4\pi)^{-d/2} c a^{-(d/2)+2}\Gamma\left(\tfrac{1}{2}d - 2\right), \tag{2.19}$$

$$L_{1R} = (4\pi)^{-d/2}\left(\frac{t}{c}\right)^{-(d/2)+1}\left\{\frac{1}{(d/2)-1} - \frac{a}{dt/2c}\right\}, \tag{2.20}$$

$$L_{2R} = (4\pi)^{-d/2} c\left(\frac{t}{c}\right)^{-(d/2)+2}\left\{\frac{1}{(d/2)-2} - \frac{a}{((d/2)-1)t/c}\right\}. \tag{2.21}$$

Now using these integrals, we can rewrite $u(x, t)$ as a sum of four terms as follows:

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^d} f(y,t) L_{1\infty}\, d^d y - \int_{\mathbb{R}^d} f(y,t) L_{1R}\, d^d y - \int_{\mathbb{R}^d} D_2 f(y,t) L_{2\infty}\, d^d y + \int_{\mathbb{R}^d} D_2 f(y,t) L_{2R}\, d^d y \\ &=: F_{1\infty}(x,t) - F_{1R}(x,t) - F_{2\infty}(x,t) + F_{2R}(x,t). \end{aligned} \tag{2.22}$$

Substitution then implies with $a_1 := (1/4)\pi^{-d/2}\Gamma((d/2)-1)$, $a_2 := (2/(d-2))(4\pi)^{-d/2}$ and $a_3 := (1/2d)(4\pi)^{-d/2}$

$$F_{1\infty}(x,t) = \int_{\mathbb{R}^d} f(y,t)(4\pi)^{-d/2}\frac{|x-y|^{-d+2}}{2^{-d+2}}\Gamma\left(\frac{d}{2}-1\right)\,\mathrm{d}^d y = a_1\int_{\mathbb{R}^d}|x-y|^{-d+2}f(y,t)\,\mathrm{d}^d y, \tag{2.23}$$

$$\begin{aligned} F_{1R}(x,t) &= \int_{\mathbb{R}^d} f(y,t)(4\pi)^{-d/2}\left(\frac{t}{c}\right)^{-(d/2)+1}\left\{\frac{1}{(d/2)-1}-\frac{|x-y|^2}{2dt/c}\right\}\mathrm{d}^d y \\ &= a_2\left(\frac{t}{c}\right)^{-(d/2)+1}\int_{\mathbb{R}^d} f(y,t)\mathrm{d}^d y - a_3\left(\frac{t}{c}\right)^{-d/2}\int_{\mathbb{R}^d}|x-y|^2 f(y,t)\,\mathrm{d}^d y, \end{aligned} \tag{2.24}$$

$$\begin{aligned} F_{2\infty}(x,t) &= \int_{\mathbb{R}^d} D_2 f(y,t)(4\pi)^{-d/2}c\left(\frac{|x-y|^2}{4}\right)^{-(d/2)+2}\Gamma\left(\frac{d}{2}-2\right)\,\mathrm{d}^d y \\ &= (4\pi)^{-d/2}\left(\frac{1}{4}\right)^{-(d/2)+2}\Gamma\left(\frac{d}{2}-2\right)c\int_{\mathbb{R}^d}|x-y|^{-d+4}D_2 f(y,t)\,\mathrm{d}^d y \\ &= \tfrac{1}{16}c\pi^{-d/2}\Gamma\left(\frac{d}{2}-2\right)\int_{\mathbb{R}^d}|x-y|^{-d+4}D_2 f(y,t)\,\mathrm{d}^d y, \end{aligned} \tag{2.25}$$

$$\begin{aligned} F_{2R}(x,t) &= \int_{\mathbb{R}^d} D_2 f(y,t)(4\pi)^{-d/2}c\left(\frac{t}{c}\right)^{-(d/2)+2}\left\{\frac{1}{(d/2)-2}-\frac{|x-y|^2/4}{(d/2-1)t/c}\right\}\mathrm{d}^d y \\ &= \frac{2(4\pi)^{-d/2}}{d-4}c\left(\frac{t}{c}\right)^{-(d/2)+2}\int_{\mathbb{R}^d} D_2 f(y,t)\,\mathrm{d}^d y - \frac{(4\pi)^{-d/2}}{2(d-2)}c\left(\frac{t}{c}\right)^{-(d/2)+1}\int_{\mathbb{R}^d}|x-y|^2 D_2 f(y,t)\,\mathrm{d}^d y. \end{aligned} \tag{2.26}$$

We have thereby verified the following:

Proposition 2.1. *Solutions to the heat equation with source term, f/c* (*i.e.*, (2.1)), *can be written to leading order in large time as the sum*

$$u = F_{1\infty} - F_{1R} - F_{2\infty} + F_{2R} \tag{2.27}$$

where F_{iR} *and* F_∞, $i = 1, 2$, *are defined by* (2.23)–(2.26).

In the next section, we will apply Proposition 2.1 to describe the transition between two regimes: quasi-static and fully dynamic system.

3. Application to interface equations

We write Eqs. (1.1) and (1.3) as a single equation [see [9] and [2]]

$$T_t - \frac{1}{c}\Delta T = -\frac{\tilde{l}}{2c}\varphi_t \tag{3.1}$$

where $c := C_v/K$ which is equivalent to $1/D$ in [2] and $\tilde{l} := l/K$ where l is the latent heat per unit volume. In the equation above, $\varphi(x, t)$ is a "*phase*" variable that has the value +1 in the liquid phase and −1 in the solid phase. The derivative of φ is then interpreted in the weak or generalized sense, and can be approximated by smooth functions as described below. By using the formulation (3.1), we can treat the phase change as a source term with support along the interface, $\Gamma(t)$. Following [2], we can rewrite (3.1) by using the Green's function representation [see (2.2)] with $f(y, s) := (-1/2)\tilde{l}\varphi_s(y, s)$ as follows:

$$T(x,t) = \int_0^t \int_{\mathbb{R}^d} \left[\frac{4\pi}{c}(t-s)\right]^{-d/2} \exp\left(\frac{-c|x-y|^2}{4(t-s)}\right)\left(\frac{-\tilde{l}}{2c}\right)\varphi_s(y,s)\mathrm{d}^d y\,\mathrm{d}s$$
$$= \int_0^t \int_{\mathbb{R}^d} G(x-y, t-s)\left(\frac{-\tilde{l}}{2c}\right)\varphi_s(y,s)\mathrm{d}^d y\,\mathrm{d}s, \tag{3.2}$$

where the Green's function G is given by (2.3). We now use the results of the previous section on (3.2) prior to integrating across the interface. With f defined as the source term generated by the latent heat, we apply Proposition 2.1 to (3.2) so that

$$\text{right-hand side of (3.2)} = F_{1\infty} - F_{1R} - F_{2\infty} + F_{2R} \tag{3.3}$$

with the functions $F_{1\infty}$, F_{1R}, $F_{2\infty}$ and F_{2R} defined by (2.23)–(2.26). As in [2], we want to integrate across the interface. The F_{1R}, $F_{1\infty}$ terms involve φ_s while the others involve φ_{ss}.

The function $\varphi_t(x, t)$ will vanish outside of the interfacial region. Across the interface, it will behave like a *delta function*. In order to exploit these features and to integrate across the interface, we define a local coordinate system $(r, \vec{\sigma})$ in a narrow region along the interface. Here, r is a signed normal to the interface (positive toward the liquid phase) while $\vec{\sigma}$ is the tangential vector. Note that for a sufficiently thin region (of width δ) containing the interface, the local coordinate system $(r, \vec{\sigma})$ can be defined unambiguously. With this notation, we can express the normal velocity, v_n, at each point on the interface as

$$v_\mathrm{n} = -r_t(x,t) \tag{3.4}$$

We approximate the order parameter $\varphi(x, t)$ by a function $\Phi(x, t)$ that varies only in the normal direction (i.e., r) so that

$$\varphi(x,t) \cong \Phi(r(x,t)) \tag{3.5}$$

The time derivative is then given by

$$\varphi_t(x,t) \cong \Phi_t(r(x,t) = r_t\Phi_r(r(x,t)) = -v_n\Phi_r(r(x,t)). \tag{3.6}$$

We now integrate across a sufficiently thin crosssection of the interface where Φ makes its transition from −1 to +1. For sufficiently small δ, one obtains

$$\int_{-\delta}^{\delta} \Phi_r(r(x,t)) = 2 \tag{3.7}$$

while integrating the time derivative across the interface results in

$$\int_{-\delta}^{\delta} \varphi_t(y,t)\,\mathrm{d}r \cong \int_{-\delta}^{\delta} -v_n\Phi_r(r(y,t))\,\mathrm{d}r \cong -2v_\mathrm{n}. \tag{3.8}$$

Note that the derivatives of φ vanish just outside of the interfacial region so that we can perform the integral in the normal direction thereby reducing the integral over $\mathbb{R}^d$ to one over $\Gamma(t)$. Using the identities above, we can evaluate each part of the right-hand side of (3.2). The first of these is evaluated as:

$$\begin{aligned} F_{1\infty}(x,t) &= a_1 \int_{\mathbb{R}^d} |x-y|^{-d+2} \left(\frac{-\tilde{l}}{2} \varphi_t(y,t) \right) \mathrm{d}^d y = \frac{-\tilde{l}}{2} a_1 \int_{\mathbb{R}^d} |x-y|^{-d+2} \{ -v_\mathrm{n}(y,t) \Phi_r(r(y,t)) \} \, \mathrm{d}^d y \\ &= a_1 \tilde{l} \int_{\Gamma(t)} |x-y|^{-d+2} v_\mathrm{n}(y,t) \, \mathrm{d}^{d-1}\sigma_y. \end{aligned} \tag{3.9}$$

This term is identical to the term obtained from the elliptic case. Next, we use the same idea on the F_{1R} term

$$F_{1R}(x,t) = a_2 \tilde{l} \left(\frac{t}{c} \right)^{-(d/2)+1} \int_{\Gamma(t)} v_\mathrm{n}(y,t) \, \mathrm{d}^{d-1}\sigma_y - a_3 \tilde{l} \left(\frac{t}{c} \right)^{-d/2} \int_{\Gamma(t)} |x-y|^2 v_\mathrm{n}(y,t) \, \mathrm{d}^{d-1}\sigma_y. \tag{3.10}$$

The terms $F_{2\infty}$ and F_{2R} have $D_2 f$ terms that lead to φ_{tt} terms. If the temporal change in the velocity is small, one can write, recalling that Φ is a function of $r(x, t)$,

$$\varphi_{tt}(x,t) \cong \Phi_{tt}(r(x,t)) = -\left(\frac{\partial v_\mathrm{n}}{\partial t} \right) \Phi_r + v_\mathrm{n}^2 \Phi_{rr} \cong v_\mathrm{n}^2 \Phi_{rr}(r(x,t)) \tag{3.11}$$

so that integration across the interface yields

$$\int_{-\delta}^{\delta} \varphi_{tt}(y,t) \, \mathrm{d}r \cong -v_\mathrm{n}^2 \int_{-\delta}^{\delta} \Phi_{rr}(r(y,t)) \, \mathrm{d}r = 0. \tag{3.12}$$

Thus, the terms $F_{2\infty}$ and F_{2R} can be neglected, since the integral across the interface vanishes. Replacing the temperature, $T(x, t)$, on the left-hand side of the Eq. (3.2) by (1.3) (for the points (x, t) on the interface), we rewrite (3.2) as

$$\begin{aligned} -\frac{\sigma_0}{[s]_\mathrm{eq}} [\kappa(x,t) + \alpha v_\mathrm{n}(x,t)] &= a_1 \tilde{l} \int_{\Gamma(t)} |x-y|^{-d+2} v_\mathrm{n}(y,t) \, \mathrm{d}^{d-1}\sigma_y - a_2 \tilde{l} \left(\frac{t}{c} \right)^{-(d/2)+1} \int_{\Gamma(t)} v_\mathrm{n}(y,t) \, \mathrm{d}^{d-1}\sigma_y \\ &\quad + a_3 \tilde{l} \left(\frac{t}{c} \right)^{-d/2} \int_{\Gamma(t)} |x-y|^2 v_\mathrm{n}(y,t) \, \mathrm{d}^{d-1}\sigma_y. \end{aligned} \tag{3.13}$$

Dividing the variables in the equation above by appropriate reference length, L_0, and time, T_0, scales, etc., we convert all constants and variables in (3.13) to their dimensionless counterparts, and write the equation entirely in dimensionless variables. Recalling that $\tilde{l} := l/K$ where l is the actual latent heat (per unit volume) and following [8], we define

$$d_0 := \frac{\sigma_0/[s]_\mathrm{eq}}{l/C_v} \quad \text{and} \quad d_0' := \frac{\sigma_0/[s]_\mathrm{eq}}{l/K} \tag{3.14}$$

where d_0 is the true capillarity length and both incorporate the surface tension. Using now (3.14) together with the dimensionless units, we see that (3.13) has the form

$$\begin{aligned} -d_0'[\kappa(x,t) + \alpha v_\mathrm{n}(x,t)] &= a_1 \int_{\Gamma(t)} |x-y|^{-d+2} v_\mathrm{n}(y,t) \, \mathrm{d}^{d-1}\sigma_y - a_2 \left(\frac{t}{c} \right)^{-(d/2)+1} \int_{\Gamma(t)} v_\mathrm{n}(y,t) \, \mathrm{d}^{d-1}\sigma_y \\ &\quad + a_3 \left(\frac{t}{c} \right)^{-d/2} \int_{\Gamma(t)} |x-y|^2 v_\mathrm{n}(y,t) \, \mathrm{d}^{d-1}\sigma_y. \end{aligned} \tag{3.15}$$

Next, we implement a renormalization procedure to (3.15).

Stage 1: For any $b > 0$ and $\lambda \in \mathbb{R}$, the algebraic substitution bx for x and $b^{-\lambda}t$ for t in (3.15) leads to

$$\begin{aligned} &-d_0'[\kappa(bx, b^{-\lambda}t) + \alpha v_n(bx, b^{-\lambda}t)] \\ &\quad = a_1 \int_{\Gamma(b^{-\lambda}t)} |bx-y|^{-d+2} v_n(y, b^{-\lambda}t)\, \mathrm{d}^{d-1}\sigma_y - a_2 \left(\frac{b^{-\lambda}t}{c}\right)^{-(d/2)+1} \int_{\Gamma(b^{-\lambda}t)} v_n(y, b^{-\lambda}t)\, \mathrm{d}^{d-1}\sigma_y \\ &\qquad + a_3 \left(\frac{b^{-\lambda}t}{c}\right)^{-d/2} \int_{\Gamma(b^{-\lambda}t)} |bx-y|^2 v_n(y, b^{-\lambda}t)\, \mathrm{d}^{d-1}\sigma_y. \end{aligned} \tag{3.16}$$

Next, define the new variables $y' = y/b$ and $\sigma_{y'} = \sigma_y/b$ (so that $d^{d-1}\sigma_y = b^{d-1}d^{d-1}\sigma_{y'}$) and (3.16) has the form

$$\begin{aligned} &-d_0'[\kappa(bx, b^{-\lambda}t) + \alpha v_n(bx, b^{-\lambda}t)] \\ &\quad = a_1 \int_{by' \in \Gamma(b^{-\lambda}t)} |bx-by'|^{-d+2} v_n(by', b^{-\lambda}t) b^{d-1}\, \mathrm{d}^{d-1}\sigma_{y'} - a_2 \left(\frac{b^{-\lambda}t}{c}\right)^{-(d/2)+1} \\ &\qquad \times \int_{by' \in \Gamma(b^{-\lambda}t)} v_n(by', b^{-\lambda}t) b^{d-1}\, \mathrm{d}^{d-1}\sigma_{y'} + a_3 \left(\frac{b^{-\lambda}t}{c}\right)^{-d/2} \\ &\qquad \times \int_{by' \in \Gamma(b^{-\lambda}t)} |bx-by'|^2 v_n(by', b^{-\lambda}t) b^{d-1}\, \mathrm{d}^{d-1}\sigma_{y'}. \end{aligned} \tag{3.17}$$

Note that the surface integral in (3.17) is over those points for which $y \in \Gamma(b^{-\lambda}t)$, which is identical (algebraically) to $by' \in \Gamma(b^{-\lambda}t)$. The latter will be equivalent to $y' \in \Gamma(t)$ upon assuming single scale self-similarity in (3.18) below.

Stage 2: We assume the single scale self-similarity (see [2]), i.e., all physical lengths, ξ, and all physical time measurements, Ξ, in the problem scale as

$$\xi(bx, b^{-\lambda}t) = b\xi(x,t) \quad \text{and} \quad \Xi(bx, b^{-\lambda}t) = b^{-\lambda}\Xi(x,t) \tag{3.18}$$

respectively, so that

$$b\kappa(bx, b^{-\lambda}t) = \kappa(x,t) \quad \text{and} \quad v_n(bx, b^{-\lambda}t) = b^{1+\lambda} v_n(x,t). \tag{3.19}$$

Note that (3.18) implies $by' \in \Gamma(b^{-\lambda}t)$ is equivalent to $y' \in \Gamma(t)$.

Stage 3: Using the self-similarity (3.19), we rewrite (3.17) as

$$\begin{aligned} &-d_0'[b^{-1}\kappa(x,t) + \alpha b^{1+\lambda} v_n(x,t)] \\ &\quad = a_1 \int_{y' \in \Gamma(t)} b^{-d+2}|x-y'|^{-d+2} b^{1+\lambda} v_n(y',t) b^{d-1}\, \mathrm{d}^{d-1}\sigma_{y'} - a_2 b^{-\lambda(-(d/2)+1)} \left(\frac{t}{c}\right)^{-(d/2)+1} \\ &\quad \times \int_{y' \in \Gamma(t)} b^{1+\lambda} v_n(y',t) b^{d-1}\, \mathrm{d}^{d-1}\sigma_{y'} + a_3 b^{d\lambda/2} \left(\frac{t}{c}\right)^{-d/2} \int_{y' \in \Gamma(t)} b^2|x-y'|^2 b^{1+\lambda} v_n(y',t) b^{d-1}\, \mathrm{d}^{d-1}\sigma_{y'}. \end{aligned} \tag{3.20}$$

Collecting the factors of b above, and rewriting, we have

$$-\frac{d_0'}{b^{3+\lambda}}[\kappa(x,t)+\alpha b^{2+\lambda}v_{\rm n}(x,t)] = a_1\int_{y'\in\Gamma(t)}|x-y'|^{-d+2}v_{\rm n}(y',t)\,{\rm d}^{d-1}\sigma_{y'} - b^{(\lambda+2)((d/2)-1)}a_2\left(\frac{t}{c}\right)^{-(d/2)+1} \times\int_{y'\in\Gamma(t)}v_{\rm n}(y',t)\,{\rm d}^{d-1}\sigma_{y'} + b^{(d/2)(\lambda+2)}a_3\left(\frac{t}{c}\right)^{-d/2}\int_{y'\in\Gamma(t)}|x-y'|^2 v_{\rm n}(y',t)\,{\rm d}^{d-1}\sigma_{y'}. \tag{3.21}$$

Next, we rescale the physical parameters in order to make the new equation above similar to the original Eq. (3.15). Recalling that

$$a_1=\frac{1}{4}\pi^{-d/2}\Gamma\left(\frac{d}{2}-1\right),\qquad a_2=\frac{2}{d-2}(4\pi)^{-d/2}\quad\text{and}\quad a_3=\frac{1}{2d}(4\pi)^{-d/2} \tag{3.22}$$

and rewriting (3.21) in terms of

$$\begin{aligned} I_1(x,t) &:= a_1\int_{\Gamma(t)}|x-y'|^{-d+2}v_{\rm n}(y',t)\,{\rm d}^{d-1}\sigma_{y'},\\ I_2(x,t) &:= -a_2\int_{\Gamma(t)}v_{\rm n}(y',t)\,{\rm d}^{d-1}\sigma_{y'},\\ I_3(x,t) &:= a_3\int_{\Gamma(t)}|x-y'|^2 v_{\rm n}(y',t)\,{\rm d}^{d-1}\sigma_{y'} \end{aligned} \tag{3.23}$$

we have the interface equation

$$-\frac{d_0'}{b^{3+\lambda}}[\kappa(x,t)+\alpha b^{2+\lambda}v_{\rm n}(x,t)] = I_1(x,t)+b^{(\lambda+2)((d/2)-1)}\left(\frac{t}{c}\right)^{-(d/2)+1}I_2(x,t)+b^{(d/2)(\lambda+2)}\left(\frac{t}{c}\right)^{-d/2}I_3(x,t). \tag{3.24}$$

Note that $d>2$, so that exponents of (t/c) are negative in the last two terms, i.e., involving I_2 and I_3. Hence, in the limit as c approaches zero, these last two terms vanish, leaving only the term involving I_1, so that one retrieves the equation for the elliptic problem [8].

Our aim is to rescale the parameters d_0', α and c so that Eq. (3.24) is identical to the original Eq. (3.15). This is accomplished by rescaling d_0' and α as:

$$d_0'\to\frac{d_0'}{b^{3+\lambda}}\quad\text{and}\quad\alpha\to\frac{\alpha}{b^{-2-\lambda}}. \tag{3.25}$$

In order to scale c appropriately, we need to satisfy the pair of identities:

$$b^{(\lambda+2)((d/2)-1)}\left(\frac{b^q}{c}\right)^{-(d/2)+1}=\left(\frac{1}{c}\right)^{-(d/2)+1}, \tag{3.26}$$

$$b^{(d/2)(\lambda+2)}\left(\frac{b^q}{c}\right)^{-d/2}=\left(\frac{1}{c}\right)^{-d/2}, \tag{3.27}$$

for some $q \in \mathbb{R}$. The value $q = -2 - \lambda$ satisfies both of these equations so that the scaling

$$c \to \frac{c}{b^{-2-\lambda}} \tag{3.28}$$

together with the scaling above (3.25) for d_0' and α renders Eq. (3.24) into the original (3.15).

In summary, to obtain the new interface Eq. (3.24) above from the original Eq. (3.15), we have followed the following two steps. We first used a set of algebraic substitutions (i.e., $bx \to x$ and $b^{-\lambda}t \to t$, etc.). Second, we rescaled all physical parameters in accordance with (3.18). In order to render the Eqs. (3.24) and (3.15) identical, we also rescale the parameters d_0', α and c in accordance with (3.25) and (3.28).

Since length scales in accordance with (3.18), the characteristic length scale in the problem, R, satisfies the transformation identity

$$R(t; d_0', \alpha, c) = bR\left(b^{\lambda}t; \frac{d_0'}{b^{3+\lambda}}, \frac{\alpha}{b^{-2-\lambda}}, \frac{c}{b^{-2-\lambda}}\right). \tag{3.29}$$

Step 4: Recall that the calculations are valid for any $b > 0$ and any real valued parameter λ. We can eliminate t in the first variable of R by selecting $b = t^{-1/\lambda}$ (with λ still arbitrary) so that the characteristic length now satisfies

$$R(t; d_0', \alpha, c) = t^{-1/\lambda}R\left(1; \frac{d_0'}{t^{-(3+\lambda)/\lambda}}, \frac{\alpha}{t^{(2+\lambda)/\lambda}}, \frac{c}{t^{(2+\lambda)/\lambda}}\right). \tag{3.30}$$

For any value of λ, relation (3.30) depends on t through the $t^{-1/\lambda}$ factor as well as at least one of the last three variables of $R(1; ., ., .)$ above. We assume nonsingular behavior of R as any of these approaches zero. This is often a reasonable assumption due to the existence of appropriate special solutions in the limiting cases.

Noting that $d_0' = d_0/c$, where d_0 is the true capillarity length, we observe that $d_0' \to 0$ implies $d_0 \to 0$ for both of the cases we consider (c finite as well as $c \to 0$). We examine the possible values of λ in terms of the scaling of d_0', α and c, prior to considering the scaling in terms of the actual capillarity length, d_0.

For $\lambda = -2$, both α and c are invariant, and one recovers from (3.30) the scaling of the full parabolic regime [2], namely $R(t) \sim t^{1/2}$. For $0 > \lambda > -2$, one has $\alpha/t^{(2+\lambda)/\lambda} \to \infty$ and $c/t^{(2+\lambda)/\lambda} \to \infty$ which is physically unrealistic. In other words, the fixed points of the RG transformation would correspond to infinite values of α and c. Similarly, for $\lambda > 0$ or $\lambda < -3$, one obtains a similar unphysical limit, since one has $d_0'/t^{-(3+\lambda)/\lambda} \to \infty$. For $-3 < \lambda < -2$, one has $d_0'/t^{-(3+\lambda)/\lambda} \to 0$, $\alpha/t^{(2+\lambda)/\lambda} \to 0$ and $c/t^{(2+\lambda)/\lambda} \to 0$, i.e., this λ represents a limit in which both c and α are represented by their fixed point of zero, and the capillarity length approaches zero faster than c (since $d_0 := cd_0'$). Note that for all values of λ above, the parameter d_0' approaches either zero or infinity. The only exception is the value $\lambda = -3$ for which d_0' is invariant, $\alpha/t^{(2+\lambda)/\lambda} \to 0$ and $c/t^{(2+\lambda)/\lambda} \to 0$. In particular, $\lambda = -3$ corresponds to the limit in which c has a fixed point at zero and d_0' is invariant (but d_0 still goes to zero, since $d_0 := cd_0'$). The characteristic growth is $R(t) \sim t^{1/3}$ in this case. Note that in the quasi-static case, the only value for which one has the true surface tension, σ_0, as an invariant corresponds to $R(t) \sim t^{1/3}$.

Next, we consider the values of λ in terms of the capillarity length, d_0, itself. Eq. (3.24) is identical to the original unscaled equation (3.15), using $d_0' = d_0/c$, upon rescaling the quantities:

$$d_0 \to \frac{d_0}{b}, \qquad \alpha \to \frac{\alpha}{b^{-2-\lambda}}, \qquad c \to \frac{c}{b^{-2-\lambda}}. \tag{3.31}$$

Using again the fact that the characteristic length scales in accordance with the self-similarity (3.18), one sees that (3.29) is replaced by

$$R(t; d_0, \alpha, c) = bR\left(b^{\lambda}t; \frac{d_0}{b}, \frac{\alpha}{b^{-2-\lambda}}, \frac{c}{b^{-2-\lambda}}\right). \tag{3.32}$$

Choosing once again the value $b = t^{-1/\lambda}$, one obtains a relation analogous to (3.30). We summarize the results as follows.

Proposition 3.1. *Under the assumption of single scale self-similarity, the characteristic length satisfies*

$$R(t; d_0, \alpha, c) = t^{-1/\lambda} R\left(1; \frac{d_o}{t^{-1/\lambda}}, \frac{\alpha}{t^{(2+\lambda)/\lambda}}, \frac{c}{t^{(2+\lambda)/\lambda}}\right). \tag{3.33}$$

In terms of d_0' (rather than d_0), this relation is expressed as (3.30). The finite valued fixed points are given by

$\lambda = -2$	α and c are invariant, $d_0' \to 0$	$R(t) \sim t^{1/2}$
$-3 < \lambda < -2$	$\alpha,\ c,\ d_0' \to 0$	$R(t) \sim t^{-1/\lambda}$
$\lambda = -3$	d_0' is invariant, α and $c \to 0$	$R(t) \sim t^{1/3}$
$\lambda < -2$	$d_0,\ \alpha,\ c \to 0$	$R(t) \sim t^{-1/\lambda}$

Note that the relations $R \sim t^{\beta}$ are under conditions of nonsingular R.

Remark. An examination of the values of λ (in (3.30)) also leads to the conclusion that $\lambda > 0$ is unphysical, since it corresponds to the fixed point $d_0 \to \infty$. Also, $\lambda = -2$ is the parabolic limit in which α and c are invariant while d_0 approaches the fixed point at zero. For $-2 < \lambda < 0$, both $\alpha/t^{(2+\lambda)/\lambda}$ and $c/t^{(2+\lambda)/\lambda}$ approach infinity which is physically unrealistic.

The situation may be summarized as follows. For $c := 0$, one has the quasi-static problem in which the characteristic length, $R(t)$, can have a range of large time behavior given by $R(t) \sim t^{-1/\lambda}$, where the values of λ are selected from $[-3, -2)$ (assuming nonsingular R). As we increase c from values that are negligible in comparison with $t^{-1/\lambda}$ for $\lambda \in [-3, -2)$ to values that are $O(1)$, the quasi-static regime is replaced by the fully dynamic (i.e., parabolic) regime in which a single value of λ, namely -2, is selected. The characteristic length has large time behavior that is uniquely specified by $R(t) \sim t^{1/2}$.

4. Conclusions

Renormalization group methods (RG) have been successful in determining key exponents in physics [3,4]. The application of RG to dynamic problems in applied mathematics, such as interface problems, poses an important challenge, and offers the potential to address the nature of large time behavior. An important link between the dynamic and static regimes is manifested in the transition between solutions to Eqs. (1.1)–(1.3) to this set of equations with (1.1) replaced by $\Delta T = 0$. We have considered this problem in the general case with $d > 2$. The methodology involves an understanding of the transition between the fundamental solution to the parabolic equation and the Green's function for the elliptic equation. By writing the fundamental solution as a sum of two parts, the first of which is the Green's function, and then approximating for large t/c, we can write an equation for points on the interface with similar properties.

This interface equation can then be analyzed using the RG methods, and considered in the limit as the specific heat, C_v (and consequently $c = C_v/K$ in Eq. (3.24)), approaches zero. Eq. (3.24) displays the interface equation in terms of the three parts involving I_1, I_2 and I_3. The part involving I_1 does not involve c, and is identical to the term one would obtain from the elliptic equation (i.e., $\Delta T = 0$). The other two terms involve terms c/t to a positive power. In order to obtain a scaling relation for the characteristic lengths, we need to find a rescaling of the parameters d_0', α and c so that the interface Eq. (3.24) will be identical to the original, i.e., (3.15) (without any rescaling). The terms involving I_2 and I_3 have factors of b that can both be eliminated by transforming c to $c/b^{-2-\lambda}$, while d_0' and α are transformed by (3.25). Since b is arbitrary, we can substitute $b = t^{-1/\lambda}$ where λ is to be determined. Confirming earlier results, we find that for any finite value of c, one has $\lambda = -2$, leading to the large time behavior of $R(t) \sim t^{1/2}$ for the characteristic length. For $c := 0$, one has the quasi-static (elliptic) problem that has been shown to have a spectrum of large time behavior $R(t) \sim t^{-1/\lambda}$ for $\lambda \in [-3, -2)$. This arises directly from the I_1 term in Eq. (3.24). The transition from $c := 0$ to finite c provides a selection of the specific large time behavior characterized by $t^{1/2}$.

Specifically, this is manifested in the balancing of the coefficients (involving c) of I_2 and I_3 to render the equation in the same form as the original (3.15).

Many problems in materials science have been simplified through the use of quasi-static formalisms such as replacing the heat equation, $u_t = \Delta u$, by Laplace's equation, $\Delta u = 0$. In many cases, this appears to be justified due to the very rapid heat conduction (particularly in metals such as aluminium) that leads to a very small u_t term shortly after the introduction of a constant heat source. However, our RG analysis indicates that the large time behavior of the quasi-static solution may differ significantly from the fully dynamic system. The methodology used in this paper illuminates the transition between the quasi-static approximation and the fully parabolic problem, suggesting that there may be a significant difference between $c = 0$ and small c in the large time behavior. These ideas have potential application to other materials science and applied mathematical problems in which the quasi-static approximation is of practical and/or theoretical importance.

Historically, RG methods were developed and understood in the context of equilibrium problems such as the divergence of exponents of physical measurables in statistical mechanics. The generalization of this methodology to dynamic problems would be of significance in a broad spectrum of applied mathematical problems.

This generalization, however, is a difficult issue that requires the integration of other applied mathematical methods with RG. In this paper, we have presented a perspective in understanding this transition.

As methodology is developed for these two regimes, the most challenging problem may be the understanding of the transition and crossover behavior between the short term asymptotics that were treated by linear stability theory and the long term asymptotics that were studied through a RG approach.

Acknowledgement

The authors are grateful to an anonymous referee for helpful comments and careful review.

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