THREE MYTHS ABOUT TIME REVERSAL IN QUANTUM THEORY

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Abstract. We seek to dispell three myths about the mathematical underpinnings of the time reversal operator in quantum theory, both ordinary and relativistic, by showing how the time reversal operator can be built up from independently motivated assumptions, without appeal to classical mechanics.

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Introduction

Mackey [6, §2-2] characterized experiments as propositions with 'yes' or 'no' outcomes. This allows for a natural representation of the experiments of quantum theory, as forming a lattice of projection operators on a Hilbert space $\mathcal{H}$. A great deal of work in the mid-20th century has gone toward characterizing the mathematical underpinnings of quantum theory in this very general context, through results like the Spectral theorem, Gleason's theorem, Stone's theorem, as well as the characterization of specific quantum systems such as the canonical commutation relations.

Inspired by this approach, the present work seeks to articulate the mathematical underpinnings of the concept of 'time reversal.' In particular, we aim to preserve the clarity and generality established by Mackey's approach, while dispelling a number of popular myths about the foundations of time reversal. These myths include:

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Myth 1. *The preservation of transition probabilities $|\langle T\psi, T\phi\rangle| = |\langle\psi,\phi\rangle|$ is a definitional feature of time reversal, with no further physical or mathematical justification.* This myth is expressed in many presentations, in which a 'symmetry operator' is defined to be one that preserves transition probabilities, and time reversal is simply assumed to be an example of such an operator.

Myth 2. *The antiunitary character of time reversal can only be established in the context of a system with 'position' and 'momentum' degrees of freedom.* This myth arose from the (currently standard) primitive assumption that time reversal transforms $X \mapsto X$ and $P \mapsto -P$. A standard argument then shows that since X and P satisfy the canonical commutation relations, the time reversal operator cannot be unitary. Therefore, if $|\langle T\psi, T\phi\rangle| = |\langle\psi,\phi\rangle|$, then T is antiunitary by Wigner's theorem. (This argument is explicated in many standard texts; [8, §3.2].)

Myth 3. *The time reverse of the position and momentum degrees of freedom can only be justified by appeal to the transformation rules for their classical analogues.* When asked to justify the transformations $X \mapsto X$ and $P \mapsto -P$, authors often appeal to the myth that this is merely in order to comply with the transformation rules for the classical Hamiltonian analogues $\mathbf{x}$ and $\mathbf{p}$. This has led to some philosophers (notably Callender [3] and Albert [1]) to argue that the standard time reversal might not be justified at all, or at least stands on equal ground with other (possibly unitary) concepts of time reversal.

The present note aims to dispel these myths. We proceed in three stages.

In Stage 1, we recall a theorem of Uhlhorn, which shows how the preservation of transition probabilities $|\langle T\psi, T\phi\rangle| = |\langle\psi,\phi\rangle|$ can be justified by a very weak assumption about the nature of the time reverse of pure states; this implies as a consequence of Wigner's theorem that T is either unitary or antiunitary.

In Stage 2, we posit and argue for three further conditions on T, and prove that they are enough to establish that T must be antiunitary. No appeal to any particular representational features (such as the position and momentum commutation rules) are required for this result.

In Stage 3, we consider the specific context of representations involving position and momentum. We first prove to that a 'weak commitment' about T is enough to establish that there always exists some representation in which time reversal behaves in the standard way. We then show that two further 'strong commitments' about T are enough to uniquely establish the standard time reversal transformation for position and momentum.

What is lost in clinging to the myths? In a sense, what is lost is a deeper understanding of the nature of time according to quantum theory. Consider, by way of analogy, the canonical commutation relations $[Q, P] \subset i$. One might take this statement to be a primitive law of physics, and be done with it. Instead, Dirac suggested that its mathematical underpinnings lie in the structure of a classical (Poisson) algebra, together with the existence of a 'quantization' homomorphism onto an irreducible Hilbert space representation. Unfortunately, such a homomorphism does not exist for algebras of arbitrary classical observables (see [13, Ch.8-9], [9, §12.7]). But more importantly, there is a much more satisfying justification of the commutation relations available – more satisfying in that it does not necessarily require the assumption of classical laws in the formulation of quantum mechanics – which assumes little more than the homogeneity of physical space [5, §12.2]. For readers unfamiliar with this argument, here is a brief overview.

Let Σ be a spacelike hypersurface containing a collection of Borel subsets Δ. Let $\{E_\Delta\}$ be the set of projection operators bearing the interpretation, 'an experimental event occurs in the open spatial region Δ.' To the extent that physical space is *homogeneous*, it makes no difference to the predictions of quantum theory if these regions are transformed by a continuous spatial translation $\Delta \mapsto \Delta + a$. One may thus posit the existence of a strongly continuous one-parameter group of unitary operators U_a (implementing 'spatial translations') such that that

$$U_a E_\Delta U_a^* = E_{\Delta+a},$$

where $\Delta + a = \{x : x - a \in \Delta\}$. By Stone's theorem, we may write $U_a = e^{iaP}$ for some self-adjoint operator P. The self-adjoint operator Q is defined in the usual way, $Q = \int_{-\infty}^{\infty} \lambda dE_\lambda$. It is then a simple matter to show that as a consequence,

$$e^{iaP} e^{ibQ} = e^{ia \cdot b} e^{ibQ} e^{iaP},$$

from which one can derive the relation $[Q, P]\psi = i\psi$ for all ψ in the common dense domain of Q and P (see [5, §12.2] for details). This justification of the commutation relations is not only more mathematically plausible than Dirac's quantization picture, but it provides a much deeper perspective on the commutation relations than one gets by simply positing them full stop, by illustrating the extent to which they are tied to a symmetry principle.

In what follows, I hope to provide a small but similar insight into the underpinnings of time reversal in quantum theory, some of which will make similar use of symmetry principles. To fix our notation, let me begin by stating a few elementary definitions.

Definition 1 (Ray Space)**.** Let $\mathcal{H}$ be a separable Hilbert space over the complex field $\mathbb{C}$. A *ray* of $\mathcal{H}$ is a set

$$\Psi = \{\phi : \phi = c\psi, |c| = 1\},$$

that is, the set of all complex unit multiples of some vector ψ. The set of rays of $\mathcal{H}$ forms a Hilbert space $\mathfrak{H}$, with its inner product inherited from $\mathcal{H}$,

$$\langle \Psi, \Phi \rangle := |\langle \psi, \phi \rangle|^2, \psi \in \Psi, \phi \in \Phi,$$

and which is called *ray space* of $\mathcal{H}$.

Definition 2 (Orthogonality $\perp$)**.** Two rays Ψ, Φ are *orthogonal* ($\Psi \perp \Phi$) if their inner product vanishes, $\langle \Psi, \Phi \rangle = 0$. A pair of projection operators P_1, P_2 are orthogonal if the rays in their respective subspaces $P_1\mathcal{H}, P_2\mathcal{H}$ are mutually orthogonal.

Definition 3 (Spectrum)**.** If A is a linear operator on $\mathcal{H}$, the *spectrum* of A is the set $\{\langle \psi, A\psi \rangle : \psi \in \mathcal{D}_A\}$, where $\mathcal{D}_A$ is the *domain of* A.

Definition 4 (Unitarity & Antiunitarity)**.** An operator $A : \mathcal{H} \to \mathcal{H}$ is *unitary* if it is linear and if $A^*A = AA^* = I$. It is *antiunitary* if it is antilinear and if $A^*A = AA^* = I$.

(Recall that A is *linear* if $A(a\psi + b\phi) = aA\psi + bA\phi$; it is *antilinear* if $A(a\psi + b\phi) = a^* A\psi + b^* A\phi$).

Definition 5 (Dynamical System)**.** A *dynamical system* is a pair $(\mathcal{H}, t \mapsto U_t)$, where $\mathcal{H}$ is a separable Hilbert space, and $t \mapsto U_t$ is a strongly continuous representation of $(\mathbb{R}, +)$ in terms of unitary operators on $\mathcal{H}$.

Remark. Note that this conception of the dynamics is fully compatible with both ordinary quantum mechanics and relativistic quantum field theory; in the latter case, the group $t \mapsto U_t$ is defined relative to a given foliation of spacetime into spacelike hypersurfaces. Note also that Stone's theorem [2, Thm. 5.9.2] implies that if $(\mathcal{H}, t \mapsto U_t)$ is a dynamical system, then there exists a unique self-adjoint operator H that generates U_t. We will thus set $U_t = e^{-itH}$.

Definition 6 (T-reversal invariance)**.** Let $(\mathcal{H}, t \mapsto e^{-itH})$ be a dynamical system. Then for any bijection $T : \mathcal{H} \to \mathcal{H}$, we say that this dynamical system is *T-reversal invariant* just in case $Te^{-itH}T^{-1} = e^{itH}$, for all $t \in \mathbb{R}$.

Remark. This definition is sometimes motivated by the equivalent definition, that 'T-reversal' means if $\psi(t) = e^{-itH}\psi$ is a trajectory of our dynamical system, then so is $T\psi(-t) = e^{-itH}T\psi$.

Definition 7 (Weyl commutation relations)**.** Let $a \mapsto U_a$, $b \mapsto V_b$ be two strongly continuous unitary representations of $(\mathbb{R}^n, +)$. The triple $(\mathcal{H}, a \mapsto U_a, b \mapsto V_b)$ is called a *unitary representation of the Weyl commutation relations* if

$$U_a V_b = e^{a \cdot b} V_b U_a$$

for all $a, b \in \mathbb{R}^n$. The representation is *irreducible* if, whenever $\mathcal{H}' \subseteq \mathcal{H}$ is a subspace such that $(\mathcal{H}', a \mapsto U_a, b \mapsto V_b)$ is a unitary representation of the Weyl commutation relations, then $\mathcal{H}' = 0$ or $\mathcal{H}' = \mathcal{H}$.

Remark. Stone's theorem allows us to write $U_a = e^{iaP}$ and $V_b = e^{ibQ}$. For all ψ in the common dense domain of Q and P, it one can show that a continuous representation of the Weyl commutation relations is equivalent to a representation of the canonical commutation relations $[Q, P]\psi = i\psi$.

Stage 1: T is Unitary or Antiunitary

Wigner's Theorem establishes that, if the time reversal operator $\mathbf{T}$ on ray space preserves ray-inner products $\langle \mathbf{T}\Psi, \mathbf{T}\Phi \rangle = \langle \Psi, \Phi \rangle$, then there exists a Hilbert space operator T that implements $\mathbf{T}$ and preserves transition probabilities $|\langle T\psi, T\phi \rangle|^2 = |\langle \psi, \phi \rangle|^2$, and this T is either unitary or antiunitary. But why think that ray inner products are preserved? Is there any physical principle underlying this claim?

The question matters because, from a certain philosophical perspective, the *a priori* assumption that T preserves ray inner products is worrisome. In particular, it may appear on the face of it one is 'sneaking in' a certain kind of T-reversal invariance. To be sure, it is not T-reversal invariance according to the official definition (see Definition 6). But is it in some other sense? If so, then one would have every right to question the assumption of Wigner's theorem, given that the world seems to violate T-reversal invariance in other ways.

On the contrary, it turns out that Wigner's assumption can indeed be provided with a natural physical foundation. A 'lattice theoretic' approach to Wigner's theorem, stemming from the work of Uhlhorn [10], has shown that Wigner's assumption follows from a surprisingly weak claim about orthogonal rays[1]. The theorem can be stated as follows.

Theorem (Uhlhorn)**.** *Let* $\mathbf{T}$ *be any bijection on the ray space* $\mathfrak{H}$ *of a separable Hilbert space* $\mathcal{H}$ *with dimension greater than 2, such that* $\Psi\perp\Phi$ *if and only if* $\mathbf{T}\Psi\perp\mathbf{T}\Phi$*. Then*

$$\langle\mathbf{T}\Psi,\mathbf{T}\Phi\rangle = \langle\Psi,\Phi\rangle\,.$$

Moreover, there exists a unique (up to a constant) $T:\mathcal{H}\to\mathcal{H}$ *that implements* $\mathbf{T}$ *on* $\mathcal{H}$*, in the sense that* $\psi\in\Psi$ *iff* $T\psi\in\mathbf{T}\Psi$*, and which satisfies* $|\langle T\psi,T\phi\rangle| = |\langle\psi,\phi\rangle|$ *for all* $\psi,\phi\in\mathcal{H}$*.*

Discussion. Uhlhorn's premise ($\Psi\perp\Phi$ iff $T\Psi\perp T\Phi$) encodes the assumption that 'mutual orthogonality' is a property that depends only on the nature of states. In particular, it does not depend on dynamical structures such as temporal evolution or the direction of time, which might plausibly be changed under the operation T. The intuition here is that, if two states of affairs like "the electron was found in orbital 1" and "the electron was found in orbital 2" are mutually exclusive, then this fact is independent of anything to do with the dynamical evolution of the electron, and thus certainly independent of whether time is running forward or in reverse. This provides a clearer account of why the preservation of transition probabilities does not in fact 'sneak in' any interesting kind of T-reversal invariance. Rather, it follows from the very plausible assumption that the orthogonality of states is unaffected by T.

It is worth highlighting that the requirement of dimensionality greater than 2 is a necessary condition for Uhlhorn's theorem[2]. This may preclude the application of such a result to certain idealized Hilbert spaces of dimension 2. However, since most realistic quantum systems are thought to have a momentum (or energy-momentum) degree of freedom with a continuous spectrum, the applicability of the result remains extremely general.

In the remainder of this paper, we will adopt Uhlhorn's very weak perspective on time reversal, and thus presume that the implementation T of the time reversal operator is either unitary or antiunitary.

Stage 2: T is Antiunitary

The time reversal operator is standardly taken to be antiunitary, and not unitary. In the following proposition, we motivate this conclusion by assuming that a certain (very bare) kind of quantum system might 'in principle' exist. The proof follows a strategy suggested by Wigner [12, §20]. It may be viewed as justification for view that time reversal is antiunitary; or, more weakly, it may be viewed as an illustration

[1]See Varadarajan [11], especially Theorem 4.29, for an extended treatment in this approach. I thank David Malament for drawing this to my attention, and Keith Hannabuss directing me to the Uhlhorn reference.

[2]A simple counterexample if $\mathcal{H}$ has dimension 2 is the following. Let Ψ, Φ be orthogonal rays, and let T be the mapping that exchanges Ψ and Φ, but is the identity on all other rays. Then T is orthogonality-preserving, but not angle-preserving. I thank John Norton for this counterexample.

of skeptics of the standard view (cf. Albert [1] and Callender [3]) are committed to giving up.

Proposition 1. *Let T be a unitary or antiunitary bijection on $\mathcal{H}$. Suppose there exists at least one densely-defined self-adjoint operator H on $\mathcal{H}$ such that, if $(\mathcal{H}, t \mapsto e^{-itH})$ is a dynamical system, then the following conditions hold.*

(i) *(positive spectrum) $0 \leq \langle \psi, H\psi \rangle$ for all $\psi \in \mathcal{D}_H$.*
(ii) *(non-triviality) H is not the zero operator.*
(iii) *(invariance) The dynamical group $t \mapsto e^{-itH}$ is T-reversal invariant.*

Then T is antiunitary.

Proof. By the invariance condition (iii) that $Te^{-itH}T^{-1} = e^{itH}$ and the definition of the exponential, one has that

$$Te^{-itH}T^{-1} = e^{T(-itH)T^{-1}} = e^{itH}.$$

Since e^{itH} is unitary, Stone's theorem implies its self-adjoint generator is unique. Hence, the self-adjoint operators H and $T(\mp H)T^{-1}$ are in fact the same generator (with the $-$ or $+$ depending on whether T is unitary or antiunitary, respectively). This can be expressed as

$$T(-itH)T^{-1}\psi = itH\psi \tag{1}$$

for all $\psi \in \mathcal{D}_H$ (note that T is a bijection, and therefore $\mathcal{D}_H \subseteq \mathcal{D}_T = \mathcal{H}$). Now, suppose for reductio that T is unitary, and hence linear. Then (1) implies that $-itTHT^{-1} = itH$, and hence that $THT^{-1} = -H$. Moreover, if T is unitary, then we may write

$$\langle \psi, H\psi \rangle = \langle T\psi, TH\psi \rangle = -\langle T\psi, HT\psi \rangle$$

for all $\psi \in \mathcal{D}_H$. But since H has a non-negative spectrum by (i), both $\langle \psi, H\psi \rangle$ and $\langle T\psi, HT\psi \rangle$ are non-negative, and so

$$0 \leq \langle \psi, H\psi \rangle = -\langle T\psi, HT\psi \rangle \leq 0.$$

Thus, $\langle \psi, H\psi \rangle = 0$ for all $\psi \in \mathcal{D}_H$. But since H is densely defined, this is only possible if H is the zero operator, contradicting our assumption (ii). Therefore, T cannot be unitary, and must instead be antiunitary. □

Discussion. The expression of this proposition applies equally to ordinary quantum mechanics and to relativistic quantum field theory. The first premises (i)-(ii), about the existence of a non-zero positive operator, are trivial. Less trivial is the claim that there exists such an operator that is T-reversal invariant (iii). However, there may be good reason to think that the condition holds for free particle systems; that is, there might in principle exist a system described by representations of the Weyl commutation relations, with a dynamics generated $H = (1/2\mu)P^2$, such that $T(itH)T^{-1} = -itH$. In this case it is certainly difficult to imagine, in the absence of any interactions whatsoever, how T-reversibility could be violated.

There may also be good philosophical reason to simply *posit* that such a system is in principle possible. Namely, one would like the ability to to say what's so strange about interactions normally considered T-violating (such as neutral kaon decay; see [8, §9] for an overview). If our concern philosophical concern is whether or not 'nature cares about the direction of time' in some *new and interesting sense* when such interactions are present, then the possibility of some T-reversal invariant system seems to be required, in order to have a basis for comparison to these new and interesting cases.

Finally, note that there is an alternative formulation of the proposition, in which the positive spectrum requirement (i) is replaced with this alternative:

(i*) *the spectrum of H has an arbitrary lower bound but no upper bound.*

An obvious analogue to the argument above may then be carried out to show that T is antiunitary[3]. This alternative may be more plausible than requiring the lower bound to be set at 0 as in (i), since a positive H generates the same dynamics as $H - rI$ for any $r \in \mathbb{R}$, even though the latter is bounded from below by $-r$. Moreover, we can generally expect H to be unbounded from above, in any realistic system admitting creation and annihilation operators.

Stage 3: Position and Momentum

The above discussion provides a very general perspective on the nature of time reversal, in which the lattice of propositions (and the algebra of observables it generates) is not constrained in any way. In this context, I have shown that there is still good reason to take T to be antiunitary. However, little more can be said about T beyond the claim.

On the other hand, suppose one wishes to characterize a particular physical system, such as a Galilean 'particle,' or a free relativistic Bose field. Then one may wish to ask how time reversal transforms the position and momentum degrees of freedom in that system, characterized by unitary groups U_a and V_b satisfying the canonical commutation relations in Weyl form. The answer to this question depends on the level of commitment that one wishes to bring to the nature of time reversal. Here, we illustrate two such levels of commitment.

The first option is conservative. Suppose that one is willing to commit to no more than the assumption that the ray mapping $\mathbf{T}$ is an *involution*; that is, that $\mathbf{T}$ is the kind of operation that if applied twice, acts identically on states: $\mathbf{T^2} = \mathbf{I}$. This is the natural mathematical characterization of what it means to be a *reversal* of a quantum state. This is not enough[4] to uniquely fix the particular form of T. However, it is enough to ensure that there always *exists* a representation in which T behaves in the standard way, as shown by the following.

Proposition 2. *Let $\mathcal{H}$ be a separable Hilbert space of infinite dimension. Suppose there exists a bijection $T : \mathcal{H} \to \mathcal{H}$ satisfing the following two conditions.*

(i) *(antiunitarity)* $\langle T\psi, T\phi\rangle = \langle \psi, \phi\rangle^*$ *for all* $\psi, \phi \in \mathcal{H}$.
(ii) *(involution)* $T^2 = c$, *where* $c \in \mathbb{C}_{unit}$.

Then there exists a unitary representation $(\mathcal{H}, a \mapsto U'_a, b \mapsto V'_b)$ of the Weyl commutation relations on $\mathcal{H}$ such that $TU'_aT^{-1} = U'_a$ and $TV'_bT^{-1} = V'_{-b}$.

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[3]I thank David Wallace for pointing this out to me.

[4]For every orthonormal basis $\varphi_1, \varphi_2, \varphi_3, \ldots$ of $\mathcal{H}$, there exists a distinct operator T that is an antiunitary involution. Namely, let $T = K$ be the *conjugation operator* with respect to that basis, defined by the property $K(a_1\varphi_1 + a_2\varphi_2 + \ldots) = a_1^*\varphi_1 + a_2^*\varphi_2 + \ldots$.

Proof. We use the Schrödinger representation $(L^2(\mathbb{R}), a \mapsto U_a, b \mapsto V_b)$ to construct the desired representation. Let K be the conjugation operator defined by $K\psi(x) = \psi^*(x)$. Recall that K is an antiunitary operator such that $K^2 = I$, $KU_aK = U_a$, and $KV_bK = V_{-b}$. Moreover, since T is antiunitary, it follows that the product TK is unitary. So, we can always set $T = AK$ for some unitary A, given that then $AK = (TK)K = T$. Note moreover that from $cI = T^2 = AKAK$, it follows that $KAK = cA^*$.

Since $KAK = cA^*$, he have also that $cKAK = A^*$. Plugging the latter into the former gives us $KAK = c(cKAK) = c^2KAK$, from which we conclude that $c = \pm 1$. As a matter of convenience, we will now treat the case of $c = 1$; the case of $c = -1$ follows in exactly the same way.

It follows from our assumptions that exists a well-defined (though not unique) 'square root of A' operator $\tilde{A}$, which has the properties that $\tilde{A}^2 = A$ and $K\tilde{A}K = \tilde{A}^*$. To check this, let us write A in its spectral resolution,

$$A = \int_0^{2\pi} e^{i\lambda} dE_\lambda$$

where $\Delta \mapsto E_\Delta$ is a projection-valued measure on Borel sets of the interval $(0, 2\pi)$, and E_λ denotes the projection corresponding to the interval $(0, \lambda)$. Since A is regular and the square-root function $f(e^{i\lambda}) = e^{i\lambda/2}$ is holomorphic in a neighborhood of the complex unit circle, we may apply the functional calculus, and define the operator

$$\tilde{A} := f(A) = \int_0^{2\pi} e^{i\lambda/2} dE_\lambda.$$

Applying the definition of f, we see that $f(e^{i\lambda})^2 = (e^{i\lambda/2})^2 = e^{i\lambda}$, and it follows that $\tilde{A}^2 = f(A)^2 = A$. Moreover, since $KAK = cA^*$, we see immediately from the definition of $\tilde{A}$ that $K\tilde{A}K = c\tilde{A}^*$.

Now let $(L^2(\mathbb{R}), a \mapsto U'_a, b \mapsto V'_b)$ be a new representation, in which

$$U'_a = \tilde{A}U_a\tilde{A}^*, \quad V'_b = \tilde{A}V_b\tilde{A}^*.$$

Since $\tilde{A}$ is unitary, this is a unitary representation of the Weyl commutation relations. But $\tilde{A}^2 = A$, so $A\tilde{A}^* = \tilde{A}$. Therefore, applying $T = AK$ to both sides of U'_a and V'_b, we have that

$$\begin{aligned} TU'_aT^{-1} &= (AK)\tilde{A}U_a\tilde{A}^*(KA^*) \\ &= A\tilde{A}^*(KU_aK)\tilde{A}A^* \\ &= \tilde{A}U_a\tilde{A}^* = U'_a \end{aligned}$$

and

$$\begin{aligned} TV'_bT^{-1} &= (AK)\tilde{A}V_b\tilde{A}^*(KA^*) \\ &= A\tilde{A}^*(KV_bK)\tilde{A}A^* \\ &= \tilde{A}V_{-b}\tilde{A}^* = U'_{-b}, \end{aligned}$$

which is the desired transformation. □

Discussion. Write $U_a = e^{iaP}$ and $V_b = e^{iaQ}$, where P and Q are self-adjoint operators. Proposition 2 can then be seen to show the existence of a representation in which time reversal has the familiar effect, $Q \mapsto Q$ and $P \mapsto -P$.

Here is the lesson of this result. As with any operator, the particular form of an antiunitary involution T is different in different representations. However, Proposition 2 shows that we can always find some representation in which it behaves just like time reversal, in that it sends $Q \mapsto Q$ and $P \mapsto -P$. Since all such representations are unitarily equivalent (and hence empirically equivalent) in ordinary quantum mechanics by the Stone-von Neumann theorem [5, §12-3], this suggests that any antiunitary involution is 'just as good' as the standard time reversal operator.

Nevertheless, one may still wish to ask a more informative question. Namely, suppose that we *fix* a representation of the canonical commutation relations. Is there then some way to uniquely determine the action of the time reversal operator on Q and P? Such a uniqueness result is available, if one adopts two further assumptions about the nature of T.

Proposition 3. *Let $(\mathcal{H}, a \mapsto U_a, b \mapsto V_b)$ be a strongly continuous irreducible unitary representation of the Weyl commutation relations. Suppose there exists a bijection $T : \mathcal{H} \to \mathcal{H}$ satisfying the following conditions.*

(i) *(antinitarity) $\langle T\psi, T\phi\rangle = \langle \psi, \phi\rangle^*$ for all $\psi, \phi \in \mathcal{H}$*
(ii) *(involution) $T^2 = c$ where $c \in \mathbb{C}$ and $|c| = 1$*
(iii) *(homogeneity) $TU_aT^{-1} = U_a$*
(iv) *(spatial slice) $TE_\Delta T^{-1} = E_{f(\Delta)}$*

where $\Delta \mapsto E_\Delta$ is the spectral measure of Q and f is a bounded bijection on Borel sets of $\mathbb{R}$. Then $TV_bT^{-1} = V_{-b}$.

Proof. Writing Q in its spectral decomposition, we can see from the spatial slice condition (iv) that

$$\begin{aligned} TQT^{-1} &= \int_{-\infty}^{\infty} \lambda dTE_\lambda T^{-1} = \int_{-\infty}^{\infty} \lambda dE_{f(\lambda)} = \int_{-\infty}^{\infty} f^{-1}(\lambda) dE_\lambda \\ &= f^{-1}(Q). \end{aligned}$$

Since Q has a simple spectrum, $f^{-1}(Q)$ commutes with Q [2, Thm. 5.8.6]. So, the quantity $(Q - f^{-1}(Q))$ commutes with Q as well. As it turns out, this same quantity also commutes with P. To see this, we first note that from antiunitarity (i) and homogeneity (iii), it follows that $TPT^{-1} = -P$. So, by applying T to the commutation relations, we have that

$$\begin{aligned} & T[Q,P]T^{-1}\psi = TiT^{-1}\psi \\ \Rightarrow\ & [TQT^{-1}, TPT^{-1}]\psi = -i\psi \\ \Rightarrow\ & -[f^{-1}(Q), P]\psi = -i\psi = -[Q,P]\psi \end{aligned}$$

for all ψ in the common dense domain of Q and P. It follows that $(Q - f^{-1}(Q))P = P(Q - f^{-1}(Q))$ on this domain. But since the representation is irreducible, Schur's lemma implies that the only operators commuting with both Q and P are multiples of the identity. So, we may write

$$Q - f^{-1}(Q) = kI$$

on the the domain of Q. We now apply the involution condition (ii) to show that k must vanish identically. Namely, we write

$$\begin{aligned} Q &= T^2 Q T^{-2} = T f^{-1}(Q) T^{-1} = T(Q - kI)T^{-1} \\ &= TQT^{-1} - k^* I \\ &= Q - (k + k^*)I. \end{aligned}$$

Thus, $k = -k^*$, and so k is either pure imaginary or zero. But $TQT^{-1} = f^{-1}(Q)$ is self-adjoint, so k cannot be pure imaginary. Therefore, $k = 0$, and $TQT^{-1} = f^{-1}(Q) = Q$. Since $V_b = e^{ibQ}$, we thus have that

$$TV_bT^{-1} = e^{-ibTQT^{-1}} = e^{-ibQ} = V_{-b}.$$

□

Discussion. Antiunitarity (i) follows from our previous arguments. The involution condition (ii) characterizes what is meant by a 'reversal.' More interesting are the conditions of homogeneity (iii) and spatial slice (iv). The label 'homogeneity' stems from the interpretation of $U_a = e^{iaQ}$ as the group of spatial translations on a given hypersurface, discussed in the introduction to this paper. This interpretation is underwritten by the fact that, if $\Delta \mapsto E_\Delta$ is the (unique) spectral measure of Q, then

$$U_a E_\Delta U_a^* = E_{\Delta - a}.$$

In other words, U_a transforms the projection corresponding to the spatial region Δ to that corresponding to the 'translated' region $\Delta - a$. The homogeneity condition $[T, U_a] = 0$ then amounts to the claim that spatial translation followed by time reversal is equivalent to time reversal followed by spatial translation. In effect, it assumes the operation of time reversal does not to pick out any preferred region of a spatial hypersurface; T is independent of any particular spatial region in which it might be applied. This would seem to be desirable of any characterization of the reversal of time.

The spatial slice condition (iv) is motivated similarly. One does not want to allow that time reversal involve *evolution* in time; rather, it should take place on a single spatial slice. (In fact, it can be checked that the operator $e^{-itH}K$, where K is the 'standard' time reversal operator, satisfies conditions (i)-(iii), but does not in general satisfy (iv).) Since E_Δ can be interpreted as the proposition that a certain experimental event occurs in the spatial region Δ, the spatial slice condition says that, while time reversal might transform the region Δ in which the event occurs, it does not evolve it forward or backward in time.

Note finally that, in the context of relativistic quantum field theory, there is some subtlety in the interpretation of the projections E_Δ as capturing propositions about 'space.' For example, Malament [7] and Halvorson and Clifton [4] have shown that under very weak conditions, such projections will fail to satisfy a natural condition of localizability, namely that if Δ_1 and Δ_2 are disjoint open regions in the same spacelike hypersurface, then

$$E_{\Delta_1} E_{\Delta_2} = E_{\Delta_2} E_{\Delta_1} = \mathbf{0}.$$

However, these challenges to localizability do not themselves challenge the postulate of a more general spatial measure $\Delta \mapsto E_\Delta$ appearing in Proposition 3.

Conclusion

Apart from revealing some common mythology, the results above suggest that the meaning of time reversal depends on the level of commitment we wish to bring to its mathematical underpinnings. These commitments can be characterized in three stages. The first stage of commitment is that the direction of time should not determine whether or not two states are mutually exclusive; we capture this by demanding T preserve of orthogonality, which was shown to underly the properties of unitarity and antiunitarity. The second stage is the possibility of at least one (non-trivial) dynamical system that is time reversal invariant; this was found to underlie antiunitarity in particular. The third stage is that time reversal is an involution, and neither picks out a preferred region in space nor involves time translation; this determines the way time reversal transforms position and momentum. Some may still wish to get off the train at various points along the way. However, the suggestion that time reversal might *not* be antiunitary [3], or that it might *not* transform Q and P in the standard way [1], is to give up some very plausible assumptions about the concept, which may be more of a burden than has yet been recognized.

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