# How to time-reverse a quantum system

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**Abstract**

This paper examines why the the time reversal operator in quantum mechanics is antiunitary. In particular, I show how the antiunitarity of $T$ can be determined from symmetry considerations, which amount to the assumption that there is no preferred direction in spacetime. On this approach, it follows that ordinary quantum mechanics is indeed time reversal invariant, contrary to recent remarks by David Albert.

David Albert (2000) has recently brought the received view of time reversal into question. Although much has been said about his remarks on time reversal in electromagnetism, little has yet been said about his view of time reversal in quantum mechanics. This paper aims to supply evidence that, in spite of Albert's suggestions, ordinary quantum mechanics is indeed time reversal invariant.

In particular, I'll seek to establish *the antiunitarity of the time reversal operator*, a condition known to be sufficient for the time reversal invariance of ordinary quantum mechanics. In the

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following section, I'll review the meaning of time reversal and time reversal invariance. I'll then present three arguments for antiunitarity: one standard, one graphical, and (the most fundamental) one based on a symmetry principle. Finally, I'll provide evidence that this particular symmetry principle is very plausible, and particularly relevant for determining the symmetry operators in a physical theory.

# 1 Time Reversal in Quantum Mechanics

## 1.1 Time Reversal and Time Reversal Invariance

In spite of the suggestive name, 'time reversal' does not require much in the way of whacky metaphysics. In this paper, we'll identify time reversal with plain old boring 'motion reversal.'

Here's what's meant by that. Think of the set of models of a theory as defining a space of motions considered 'possible' according to that theory. The symmetries of the theory (in fancier terms, the theory's *covariance group*) provides an informative way to organize that space of motions. These symmetries allow us to relate possible trajectories – say, $\alpha(t)$ and $\beta(t)$ – to further possible trajectories – say, $\alpha'(t')$ and $\beta'(t')$ – by rotating, flipping, or sliding our space of motions about.

It's common practice to begin with a very general definition of time reversal along these lines, before further physical assumptions are used to characterize the concept more explicitly. For us, that general definition will be:

**Definition 1.** *Time reversal* is a mapping that sends each trajectory $\psi(t)$ through some state space to a trajectory $T\psi(-t)$, where $T$ is a symmetry operator on a theory's state space, commonly called the *time reversal operator*. The time reversal operator is what makes this definition so general. For example, it's normally missing in discussions of time reversal in classical mechanics, where it is chosen to be the identity. For now, we'll leave the explicit expression of this operator open. That leads us to:

**Definition 2.** A theory is *time reversal invariant* if, whenever $\psi(t)$ is a solution to the equations of motion (that is, whenever $\psi(t)$ is possible trajectory according to the theory), so is $T\psi(-t)$.

In the particular case of quantum mechanics, it's well known that the time-reversibility of the theory depends whether or not the time reversal operator is *unitary* or *antiunitary*. The goal of this paper is to draw attention to why it's got to be the latter. But first, let's take just a moment and review the meaning of these terms.

## 1.2 Unitarity and Antiunitarity

**Definition 3.** We say that a dense operator $U$ is *linear* if $U(a\psi + b\varphi) = aU\psi + bU\varphi$. We say that $U$ is *unitary* if $\langle U\psi, U\varphi\rangle = \langle\psi, \varphi\rangle$.

**Definition 4.** Let $K$ be the conjugation operator[1], which takes a vector $\psi$ to its conjugate $\psi^*$. We say that an operator $A$ is *antilinear* if $A = UK$ and $U$ is linear. We say that $A$ is *antiunitary* if $A = UK$ and $U$ is unitary[2].

Since $T$ is a symmetry operator – a bijection that preserves probabilities – it follows that linearity is equivalent to unitarity, and antilinearity is equivalent to antiunitarity. It's easy to show

[1]More generally, a conjugation operator $K$ is a Hilbert space operator such that $\langle K\psi, K\phi\rangle = \langle\psi, \phi\rangle^*$ and $K^2 = I$.

[2]Notably, the latter implies that $\langle A\psi, A\varphi\rangle = \langle\psi, \varphi\rangle^*$.

that whether or not one holds of any given operator does not depend on which representation is chosen[3]. We'll thus use these terms interchangeably.

Wigner's theorem guarantees that there are really only two candidate types of time-reversal operators in quantum mechanics.

**Wigner's Theorem.** *If $T$ is Hilbert space bijection that preserves probabilities – that is, $|\langle T\psi, T\varphi\rangle| = |\langle \psi, \varphi\rangle|$ – then $T$ is either unitary or antiunitary.*

**Corrolary.** *The time reversal operator in quantum mechanics is either unitary or antiunitary.*

The antecedent of Wigner's theorem is guaranteed if $T$ is to represent time-reversal, since $T$ is a symmetry operator, and must preserve probabilities. So, we have at the outset a deep result showing that the time reversal operator must be one of two types. Which one is correct has recently become a matter of debate, which we will now discuss.

## 1.3 David Albert's Approach

After establishing the above theorem, Wigner (1931) took the additional (now standard) step of demanding that time-reversal be *anti*unitary. It is a well-known consequence that if $[T, H] = 0$, then the Schrödinger equation of ordinary quantum mechanics is time-reversal invariant. Moreover, the antecedent holds under nearly all physically reasonable circumstances. For example, in the position representation, when $T$ is taken to be complex conjugation, $[T, H] = 0$ is equivalent to the condition that $H$ include only real-valued potential fields[4].

[3]Technical caveat: this does assume that all representations are unitarily equivalent. However, in the context of ordinary quantum mechanics at issue in this paper, this equivalence is guaranteed by the Stone-von Neumann uniqueness theorem.

[4]See (Ballentine 1998, 381).

It thus came as a surprise when David Albert claimed that quantum mechanics (together with a host of other theories) “is *not* invariant under time reversal” (Albert 2000, pp. 14). Albert’s claim is not so outlandish. By Wigner’s theorem, it simply means that Albert’s time reversal operator must be *unitary*. And this makes sense, given that Albert’s general picture of time-reversal involves sending $t \mapsto -t$, and hence sending $\psi(t) \mapsto \psi(-t)$, with no conjugation involved.

Albert is right to point out that one’s choice of time-reversal operator requires justification. Most textbook accounts of time reversal, like Wigner’s account, typically just assume an antiunitary operator is the correct one. So, let us consider our choice, of whether $T$ is unitary or antiunitary, with a bit more care.

# 2 The antiunitarity of $T$

This section has three parts. We’ll begin with a well-known argument for antiunitarity, which I claim doesn’t quite work. We’ll then discuss a way to visualize antiunitarity, which seems a bit more plausible. Finally, I’ll give a symmetry argument, which I argue provides very good reason to think $T$ is antiunitary.

## 2.1 First indication: the classical limit

John Earman (2002) pointed out (in response to Albert) that if time reversal is going to reverse the position and momentum observables like their classical analogues, then $T$ *must* be antiunitary[5]. So, Albert’s unitary time reversal operator does not recover these classical transformation rules.

This is certainly a first indication that antiunitarity is desirable. However, this argument

[5] See (Ballentine 1998, 378) for a standard proof of this fact.

alone doesn't seem to provide a *fundamental* reason for $T$ to be antiunitary. After all, quantum mechanics is taken to be the more fundamental theory. Why should it matter what classical mechanics has to say?

One reason it matters is that, for the most part, classical descriptions must be recovered as limiting cases of quantum descriptions. In particular, we have Ehrenfest's theorem, which guarantees cases in which the operators $Q$ and $P$ display classical behavior. One might hope it follows that $Q$ and $P$ must therefore time-reverse classically, too[6].

This almost follows, but not quite. For example, consider a quantum system consisting of a single free particle. Ehrenfest's theorem says only that the expectation values $\langle Q\rangle$ and $\langle P\rangle$ form a solution to the classical Hamilton equations. Now, there is a formal property of these equations, which says that $\langle Q\rangle$ and $\langle -P\rangle$ must form a solution as well[7]. And indeed, this is normally called the 'time reversal' solution to Hamilton's equations. Nevertheless, the definitions alone provide no guarantee that time reversal means the same thing for the laws of *quantum mechanics* (and in particular for solutions to the Schrödinger equation).

Perhaps we could posit some kind of principle of consistency, which requires that $Q$ and $P$ to 'time reverse' consistently, in both Hamilton's equations and the Schrödinger equation. But instead of accepting such a principle a priori, let's try to develop better physical evidence that this is the case.

[6]Thanks to NAME REMOVED FOR BLIND REVIEW for pointing this out.

[7]This is a consequence of the fact that if the pair $(q(t), p(t))$ is a solution to Hamilton's equations, then so is the pair $(q(-t), -p(-t))$.

## 2.2 Second indication: visualizing antiunitarity

Visualizing a simple example may help illustrate why time reversal seems to involve conjugation, and hence an antiunitary operator.

Consider a plane wave with unit angular frequency, given by the wave function:

$$\psi_\varphi(\mathbf{x}, t) = e^{i\mathbf{p}\mathbf{x} - it}.$$

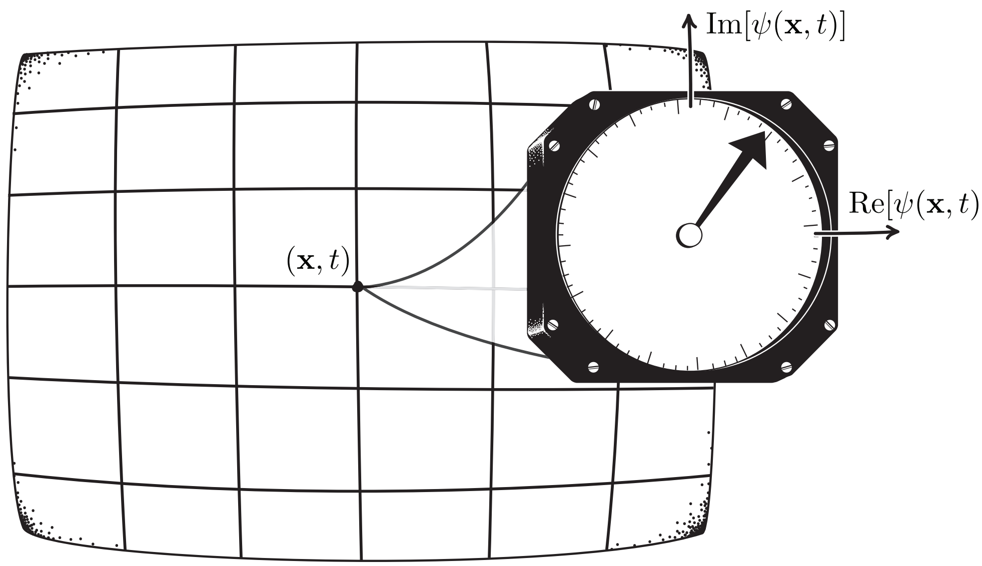


*Figure 1: A wave function can be visualized as the assignment of a 'dial' value (an amplitude) to each point in spacetime.*

Physically, we might take $\psi$ to describe a bead on a string with some given initial momentum. But let's focus for now on one fact about $\psi$: part of its job is to assign a complex coefficient $e^{it}$ to each little region of spacetime. We normally interpret this coefficient as the *phase* of the plane wave. The phase indicates how much of a cycle the wave has completed at that point, with respect to an initial time $t = 0$ in a coordinate system. But for now, just think of the value of $e^{it}$ as a point on the complex unit circle.

Such a value can be represented by an arrow pointing in a particular direction on a dial, as in Figure 1. In fact, we can trace through our wave function's assignment of dials to a point in space, which changes over time. This will correspond to the arrow moving around smoothly on the face of the dial – the changing phase of the plane wave. Now, suppose $\psi$ assigns dials in such a way that, as we pass from time $t_1$ to $t_2$ at some fixed region in space, the arrow moves clockwise from a vertical-up position to a vertical-down position. How do our two candidate time-reversal operators transform this situation?

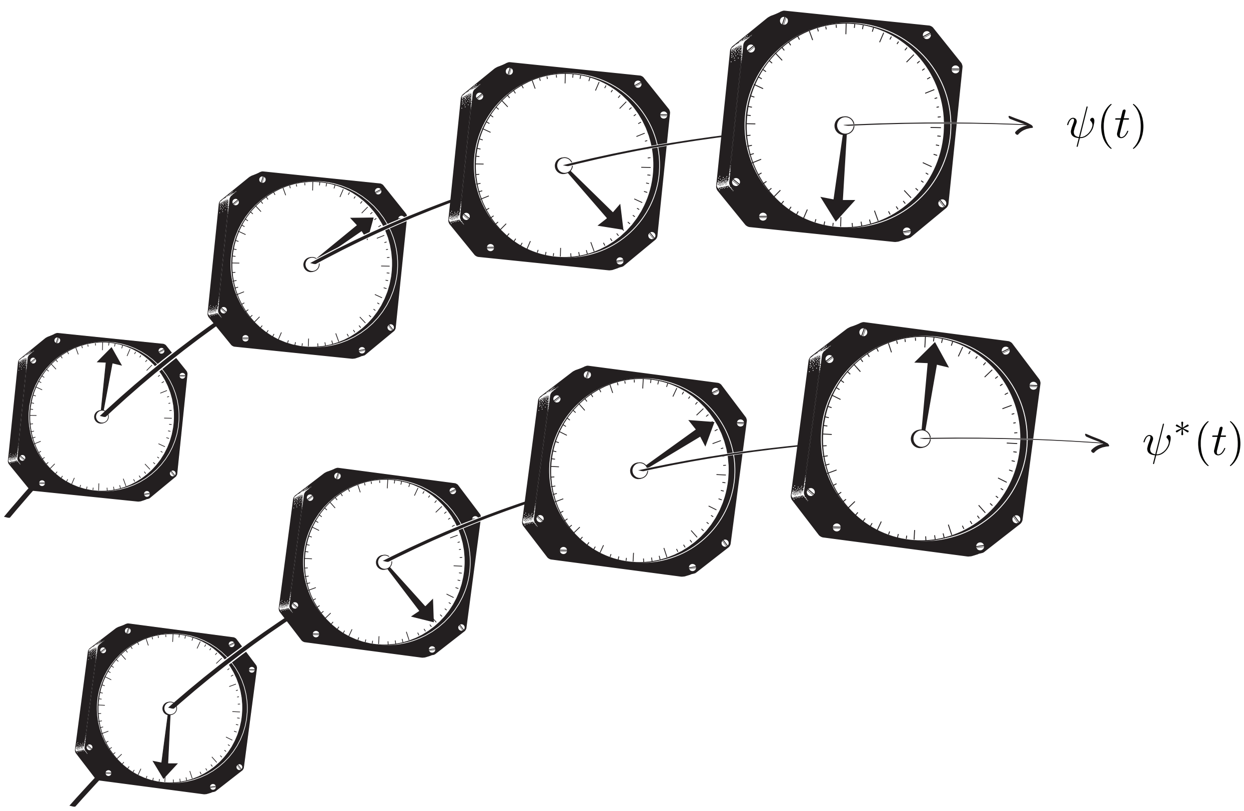


*Figure 2: Conjugating the wave function has the effect of 'flipping' the arrow about the real axis of the dial. The resulting conjugate wave function corresponds to a dial that spins in the reverse direction.*

Albert's unitary time-reversal operator does not change the picture at all. That's because the amplitude $\psi_\phi(\mathbf{x}, t)$ is given by an inner-product, say $\langle\varphi', \varphi\rangle$. But if time-reversal is unitary, then by definition $T$ preserves inner products: $\langle T\varphi', T\varphi\rangle = \langle\varphi', \varphi\rangle$.

On the other hand, Wigner's antiunitary operator has the effect of *conjugating* the amplitude at each point, since antiunitarity implies $\langle T\varphi', T\varphi\rangle = \langle\varphi', \varphi\rangle^*$. In terms of our picture, this $T$ has the effect of 'flipping' the arrow on each dial face about the real axis, as shown in Figure 2. As a consequence, our description changes to one in which the arrow moves *counter-clockwise*, from a vertical-down position to a vertical-up position. The resulting transformed dial is just a reversed description of how the phase changes over time. This illustrates one intuition behind Wigner's picture of time-reversal in quantum mechanics: antiunitary operators are precisely what's needed to reverse a changes in phase.

Now, there is an obvious objection to all this is, in the well-known chorus: 'Phase isn't physical!' And it's true that the phase factor for our plane wave is an artifact – it depends only on our choice of coordinate system. However, that doesn't mean phase is unphysical: a *difference* in phase, such as that arising from a pair of plane waves, is absolutely measurable. In such cases, this picture of time reversal – the antiunitary picture – is apparently the most natural way to visualize it.

## 2.3 A Symmetry Argument

Let's finally turn to a more fundamental argument for the antiunitarity of $T$, which relies on a symmetry principle. For better or for worse, our approach will proceed exactly backwards from the way that time-reversal is usually presented.

On the usual textbook approach, one assumes that the time-reversal operator $T$ is Wigner's (antilinear) operator. One then dutifully proves that the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi = H\psi$$

is time-reversal invariant under ordinary conditions (such as when $T$ commutes with $H$). In contrast, in this paper, we make the observation that under those same ordinary conditions, the implication goes the other way as well. In particular, given a system with no forces or interactions, the assumption that the Schrödinger equation is time-reversal invariant implies that $T$ is antilinear. This is a consequence of the following fact (proved in Section 5).

**Proposition 1.** *Suppose $T$ is an operator that commutes with the free-particle Hamiltonian. If the Schrödinger Equation with this Hamiltonian is time reversal invariant, then $T$ is antiunitary.*

In other words: if there are independently-motivated reasons to believe the premises of Proposition 1, then it will follow that $T$ is antiunitary. And then, Wigner will be windicated.

Moreover, the demand that $[T, H] = 0$ nothing special: it was also required in the usual textbook derivations of the converse proposition discussed above. And certainly, $[T, H] = 0$ is trivially true when $T$ is the identity, as it is in classical mechanics. But the basic idea holds more generally: the free-particle Hamiltonian has only a kinetic energy term, and a symmetry operator can't have an effect on that.

The other premise of this Proposition 1 is more interesting. It's an example of a general principle, which is perhaps interesting enough to merit naming.

**Principle.** *Free Motion Symmetry (FMS):* In the absence of forces and interactions, the group of spacetime symmetries transforms the laws of physics covariantly.

The basic idea is simple (although we will discuss it more below): in the absence of any forces or interactions, there should be no preferred direction in space and time. If that's the case, then a symmetry $S$ of spacetime must leave the laws of physics invariant – otherwise, the laws of physics would pick out a preferred direction in space and time. This leads immediately to the following:

**Corrolary.** *For the free-particle Hamiltonian, the ordinary Schrödinger equation is time reversal invariant.*

Notably, FMS does indeed restrict the debate about time reversal from the outset. It says that while we may question whether or not our theories are time reversible when interactions are turned on, there is no question when interactions are turned off. Theories without interactions are time reversible.

However, the upshot is that if we can convince ourselves of FMS, then Proposition 1 guarantees that time reversal is antiunitary – and hence that ordinary quantum mechanics is time reversal invariant. I believe there are several good reasons to buy into this principle, and discuss them in the next section.

# 3 Why Believe Free Motion Symmetry?

## 3.1 No preferred direction

In the absence of any forces, potentials or interactions, the only structure available to pick out a preferred direction (in time or space) is empty spacetime itself. Thus, if the solutions to our equations of motion favor one direction over another, then that direction must be distinguished by empty spacetime. But, according to a very plausible physical principle, empty spacetime *has no* *preferred direction.* So our solutions can't either – that's the idea behind FMS[8].

The same story can be put another way: imagine a theory describing no potentials, no interactions, just a single particle evolving through spacetime. There is a frame of reference in which that particle is described at rest. And, if we rotate or flip the background spacetime in an appropriate way, then we'll get back exactly the same description again of that rest particle. The spacetime is empty of any distinguishing forces or interactions according to this theory, so there is nothing to distinguish the flipped situation from the original. Therefore, if the original particle's motion is a solution for this theory, then so is the motion described after the time-reversal.

Still, FMS is a physical principle: it lives (and dies) by the sword of experience. However, there are at least two further reasons to think this physical principle is actually correct.

## 3.2 Wigner's program

FMS has been used by physicists with much success, and seemingly no failures. Most recently, FMS has been advocated by Robert Sachs, who calls it a condition of 'kinematic admissibility,' and uses it as the foundation for his considerations of time reversal:

> [We] must avoid using properties of the forces or interactions that determine the dynamics, because it is the transformation properties of the dynamic equations that we seek to determine. Since the kinematics are those properties of the motion that are independent of the dynamics, we may accomplish this objective by requiring that the concept of an admissible transformation be formulated in kinematic terms. In order

[8]Of course, some theories (such as the theory of weak interactions) do allow for interactions in which various spacetime symmetries are violated. This is perfectly compatible with FMS, which only posits the lack of a preferred direction *when interactions are turned off.*

> to express explicitly the independence between the kinematics and the nature of the forces, we require that the transformations leave the equations of motion invariant *when all forces or interactions vanish.* (Sachs 1987, 7.)

Sachs is drawing on the powerful way that FMS was employed by Eugene Wigner, to whom Sachs's book is dedicated. Here's a brief sketch of how Wigner used this idea.

Picking up on the spherical symmetry of the hydrogen atom's potential well, Wigner (1931) used the rotational symmetry group of a sphere to determine the angular momentum properties of this system[9]. To make this strategy work, Wigner had to assume that all of the symmetries of the potential well *really are relevant* in determining the angular momentum properties of a system. In particular, he assumed that no preferred orientation is imposed by the background spacetime – or anything other than the potential field itself – that might impose an asymmetry in the system.

Wigner's assumption here is very similar to that of Free Motion Symmetry. Indeed, when one turns to situations with no potential wells at all, Wigner's assumption is *precisely* FMS: a vanishing potential well has all the symmetries of the background spacetime; thus, all the symmetries of the spacetime are relevant in determining which quantum states are admissible. This was famously the central idea behind Wigner's famous (Wigner 1939) analysis of quantum particles. In that paper, it will be recalled, Wigner indexed quantum particles according to the parameters picked out by the irreducible representations of the Poincaré group[10].

[9]In particular, Wigner observed that the the generators of a Hilbert space representation of $SO(3)$ satisfy the angular momentum commutation relations, and can in fact be identified as angular momentum observables for a discrete energy level.

[10]In fact, FMS is part of a *family* of symmetry principles powerfully employed by Wigner. (Wigner 1964, 302) states that there are three classes of so-called 'geometric' symmetry principles: the equivalence of all directions in spacetime; the symmetries of the laws of nature; and the independence of the laws of nature from any uniform state of

Wigner's program is one area in which Free Motion Symmetry was used with great success. If this is an indication that FMS is correct, then as a consequence of our discussion in the previous section, it indicates that $T$ is antinuitary, too – and hence that ordinary quantum mechanics is time reversal invariant.

### 3.3 Natural generalization to other theories

Free motion symmetry also turns out to be very generally applicable. It not only works in various other theories of physics – it seems to recover the usual notion of time reversal in those theories. This suggests that FMS is not just a trick that happened to work for quantum mechanics, but a general, more fundamental principle.

Take a simple example: consider a single classical particle obeying Hamilton's equations of motion:

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}$$
$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}.$$

Then we have the following fact (the proof of which is in Section 5):

**Proposition 2.** *Suppose Hamilton's equations are time reversal invariant for the free particle Hamiltonian. Then* $Tq = a_1 q + a_2 t + a_3$ *and* $Tp = b_1 p + b_2$.

This proposition might seem surprising: FMS allows time reversal to have a much more 'general form' in classical mechanics than the familiar $q \mapsto q$ and $p \mapsto -p$. The reason is, we have not yet posed any formal constraints on the way $T$ should behave, such as the one that Wigner's

motion.

theorem provides. This is not particularly worrisome, but rather teaches us something interesting about time reversal. In particular, we have the following simple corrolary.

**Corrolary.** *Suppose in addition that $T$ is an involution ($T^2 = 1$). Then $Tq = q$ and $Tp = p$ – i.e., $T$ is the identity.*

In other words: posing a very plausible constraint, that applying $T$ twice takes us 'back to where we started,' is enough to recover the standard time reversal operator. Moreover, this simple one-particle case easily generalizes to multiple particles. So as it turns out, there is nothing special about the application of FMS in quantum mehanics: it applies to all theories for which a Hamiltonian formulation can be given.

Of course, the limitations of this approach should also be emphasized: FMS appears capable only of characterizing the way that position and momentum behave under symmetry transformations. It does not appear capable of determining how more exotic observables behave – observables like spin, isospin, or flavor. In fact, it seems inevitable that additional physical assumptions must be introduced to understand time reversal in these contexts. For example, the spin observable couples to electromagnetic fields. One might thus draw on some independently characterization of how these fields behave under time reversal – such as that given by Malament (2004) – to determine how the spin observable reverses.

Nevertheless, the great generality of FMS suggests it deserves fair consideration as a foundation for time reversal.

# 4 Conclusion

Through a non-standard definition of what counts as a state, David Albert concluded that "the dynamical laws that govern the evolutions of quantum states in time cannot possibly be invariant under *time-reversal*" (Albert 2000, 132). Albert is right to question the standard textbook presentation of time-reversal in quantum mechanics. As we have seen above, justifying this presentation is not a trivial matter. However, I hope to have shown that some well-motivated justification can be given, on the basis of quantum (not classical) mechanics. If this is right, then the dynamical laws of ordinary quantum mechanics may turn out to be time-reversal invariant, after all.

# 5 Appendix: Proofs of Propositions

**Proposition 1.** *Suppose $T$ is an operator that commutes with the free-particle Hamiltonian. If the Schrödinger Equation with this Hamiltonian is time reversal invariant, then $T$ is antiunitary.*

*Proof.* Let $\psi(t)$ be any solution to the Schrödinger equation: $i\hbar\frac{\partial}{\partial t}\psi = H\psi$, where $H$ is the free-particle Hamiltonian. We assume the Schrödinger equation is time reversal invariant, so $T\psi(-t)$ is also a solution. Thus, since $T$ commutes with $H$,

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}T\psi(\mathbf{x},-t) &= HT\psi(\mathbf{x},-t) \\ &= TH\psi(\mathbf{x},-t) \end{aligned} \tag{1}$$

Moreover, substituting $t \mapsto -t$ into the original Schrödinger equation implies:

$$\begin{aligned} H\psi(-t) &= i\hbar\frac{\partial}{\partial(-t)}\psi(-t) \\ &= -i\hbar\frac{\partial}{\partial t}\psi(-t). \end{aligned} \tag{2}$$

Plugging (2) into the RHS of (1), we thus have that

$$i\hbar\frac{\partial}{\partial t}T\psi(-t) = -Ti\hbar\frac{\partial}{\partial t}\psi(-t) \tag{3}$$

But by Wigner's theorem, $T = UK$, where $K$ is the identity (if $T$ is unitary) or $K$ is conjugation (if $T$ is antiunitary). Since $\frac{\partial}{\partial t}$ is not an operator, this means that Equation (3) is only satisfied if $T$ conjugates the complex scalar $i$. Therefore, $T$ is antilinear. □

**Proposition 2.** *Suppose Hamilton's equations are time reversal invariant for the free particle Hamiltonian. Then* $Tq = a_1q + a_2t + a_3$ *and* $Tp = b_1p + b_2$.

*Proof.* For a free-particle Hamiltonian, our equations of motion are now:

$$\frac{dq}{dt} = \frac{p}{m}$$
$$\frac{dp}{dt} = 0.$$

Substituting $t \mapsto -t$ into the original Hamilton equations implies that:

$$\begin{aligned} -\frac{d}{dt}q(-t) &= \frac{1}{m}p(-t) \\ \frac{d}{dt}p(-t) &= 0. \end{aligned} \tag{4}$$

But we assume that since $q(t)$ and $p(t)$ are a solution, so are $Tq(-t)$ and $Tp(-t)$. Therefore,

$$\frac{d}{dt}Tq(-t) = \frac{1}{m}Tp(-t) \tag{5}$$

$$\frac{d}{dt}Tp(-t) = 0 = -\frac{d}{dt}p(-t) \tag{6}$$

The equations in (6) are satisfied whenever $Tp = b_1p+b_2$, proving the latter part of the proposition. To prove the former part, we take the derivative of (5) to get:

$$\frac{d^2}{dt^2}Tq(-t) = \frac{1}{m}\frac{d}{dt}Tp(-t).$$

But we know from (6) that the RHS vanishes. So by (4), $\frac{d^2}{dt^2}q(-t)$ also vanishes. Integrating, we find that these two second order equations are satisfied whenever $Tq = a_1q + a_2t + a_3$, so we are done. □

**Corrolary.** *Suppose in addition that $T$ is an involution ($T^2 = 1$). Then $Tq = q$ and $Tp = p$ – i.e., $T$ is the identity.*

*Proof.* The proof of the corollary is a simple consequence of the constraint that $T^2 = 1$. □

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