The Oscillator, Rewound

The Newtonian system I will describe involves no surface at all. We will obtain a mass on a spring (Figure 1) for which there are two possible evolutions: one in which the mass just sits there for all of time, and another in which it starts bouncing for no apparent reason.

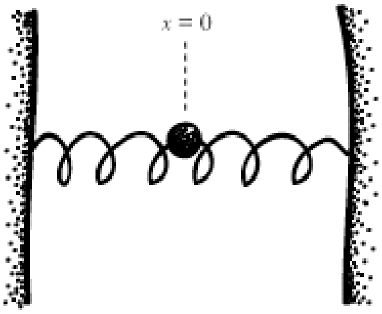

Figure 1: A mass and spring oscillator.

Take a spring with a unit mass in the middle that's sitting at rest. Set the origin of a coordinate system x at the center of the mass. Experience shows that the force on the mass (in a neighborhood of the origin) is that of the harmonic oscillator,

$$F = -kx,$$

where k is a positive constant describing the tension of the spring. Now, imagine that we wind the spring up, and then begin unwinding it again, at a rate determined by the equation,

$$k = 6 - 4t^2. \tag{1}$$

Imagine that this operation takes place over a fixed interval of time $t \in (-\sqrt{3/2}, \sqrt{3/2})$, in order to guarantee that k is always positive. Then the motion of the unit mass is described by Newton's equation $\ddot{x} = F$, which in our case comes out to be,

$$\ddot{x} = -(6 - 4t^2)x. \tag{2}$$

One obvious solution to this differential equation is $x(t) = 0$, for all time t. This represents an evolution in which the mass just sits there at rest. However, one can also show[1] that there is a second solution:

$$x(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ te^{-t^2} & \text{if } t \geq 0. \end{cases} \tag{3}$$

Sanity Check: A. The tension constant in equation (1) is positive and finite at all times during the winding process. **B.** The force field in our equation of motion (2) is well-defined and finite at all points (x, t) this system's phase space. **C.** The solution in equation (3) is C^∞. In fact, all solutions to this equation of motion are similarly smooth: if $x = A \neq 0$ at time $t = 0$, then $x(t) = -3At^2$ is the unique solution. **D.** The velocity in all solutions is finite, although it takes on arbitrarily large values in solution (3) after time $t = 0$.

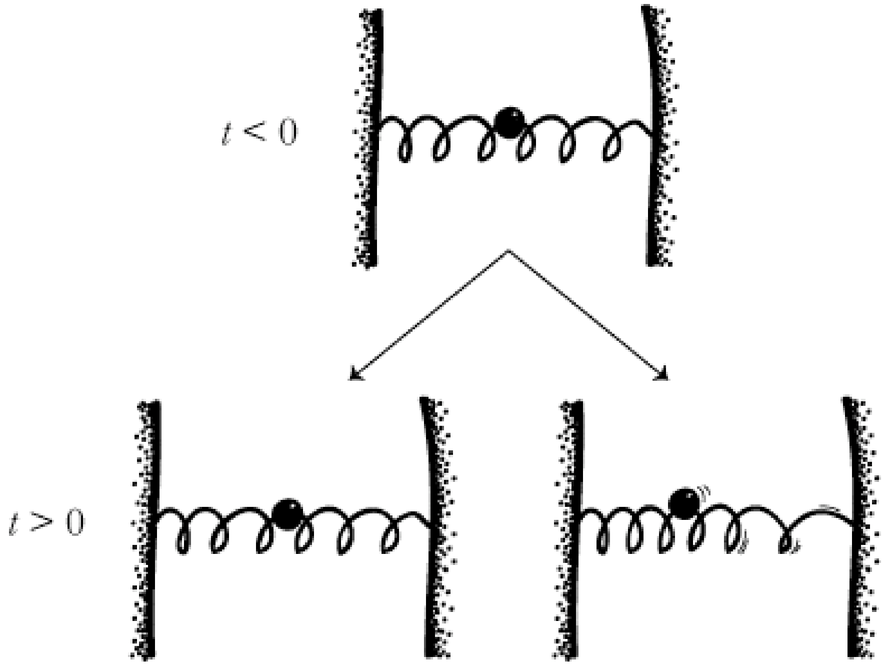

Figure 2: Two possible evolutions: one in which the mass remains at rest for all time, and the other in which it starts bouncing for no apparent reason.

So, determinism fails for this spring system: there is more than one Cauchy evolution for the initial conditions, $x = 0$, $t \leq 0$. In one case, the mass remains at rest, and in the other, the mass starts bouncing after time $t = 0$ (Figure 2).

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[1] If $x = te^{-t^2}$, then taking the derivative (and applying the chain rule) gives $\dot{x} = (\frac{1}{t} - 2t)te^{-t^2}$. By substitution, this implies that $\dot{x} = (\frac{1}{t} - 2t)x$. Taking a second derivative, we get then get that $\ddot{x} = -(6 - 4t^2)x$, as required.