On the Automorphism Towers of Some Lie Groups

Bryan W. Roberts

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Abstract

We prove that $SU(n)$ and $O(n,m)$ have non-trivial automorphism towers whenever (n,m) are non-trivial.

1 Main Results

Our results rest on a simple definition and a helpful lemma.

Definition. A group G has a *non-trivial automorphism tower* iff $G \not\simeq \operatorname{Aut} G$.

Lemma [Helpful]. Let ζG be the center of a group G, and let $\operatorname{Out} G$ be its outer automorphism group. If $|\zeta G| \neq |\operatorname{Out} G|$, then $G \not\simeq \operatorname{Aut} G$.

The helpful lemma has a simple proof for finite groups using Lagrange's Theorem. The general (possibly infinite case) is proven in the next section. This lemma allows us to give short proofs of two theorems.

Theorem 1. *Every non-trivial Special Unitary Group $SU(n)$ $(n > 1)$ has a non-trivial automorphism tower.*

Proof. Let $n > 1$. It can be shown that $\zeta SU(n)$ is a cyclic group generated by γI, where γ is the nth root of unity [1, 124]. Thus, $|\zeta SU(n)| = n$.

Now turn to $\operatorname{Out} SU(n)$. There are two cases to consider. First, let $n = 2$. Then every automorphism of $\operatorname{Out} SU(n)$ is trivial, so $|\operatorname{Out} SU(n)| = 1$. Second, let $n > 2$. Then one can show that $|\operatorname{Out} SU(n)| = 2$ [2, 154]. In both cases, $|\zeta SU(n)| \neq |\operatorname{Out} SU(n)|$. Therefore, by the Helpful Lemma, $SU(n) \not\simeq \operatorname{Aut} SU(n)$. So $SU(n)$ has a non-trivial automorphism tower. $\square$

Theorem 2. *Every Orthogonal group $O(m,n)$ that is non-trivial* $(m \geq 1, n \geq 0)$ *has a non-trivial automorphism tower.*

Proof. Let's refer to an arbitrary orthogonal group $O(m,n)$ as simply O for short. We first consider the outer automorphisms of O. It can be shown that every automorphism of an orthogonal group is an inner automorphism [3, 161]. So $\operatorname{Aut} O \simeq \operatorname{Inn} O$. Therefore, $\operatorname{Out} O := \operatorname{Aut} O/\operatorname{Inn} O$ is the trivial group, and $|\operatorname{Out} O| = 1$.

Now consider the center of O. There are two cases, both of which yield a non-trivial center. If $n = 0$, then $|\zeta O| = 2$ (the group $O(3)$ is an example of this). In a vector space representation, the two elements in the center are the identity transformation and the 'flip' transformation $\mathbf{x} \mapsto -\mathbf{x}$. On the other hand, if $n \neq 0$, then $|\zeta O| = 4$ (the Lorentz group $O(3, 1)$ is an example of this). In this case, the center consists of the identity transformation, plus three flips: $(\mathbf{x}, \mathbf{t}) \mapsto (-\mathbf{x}, -\mathbf{t})$, $(\mathbf{x}, \mathbf{t}) \mapsto (-\mathbf{x}, \mathbf{t})$, and $(\mathbf{x}, \mathbf{t}) \mapsto (\mathbf{x}, -\mathbf{t})$.

Therefore $|\operatorname{Out} O| \neq |\zeta O|$, and so by the Helpful Lemma, $O \not\simeq \operatorname{Aut} O$. Thus, O has a non-trivial automorphism tower. □

Remark. There appears to be a generalization of this result that applies to many Lie groups at once, instead of treating each one individually as in the above two theorems. For example, one can also exhibit non-trivial automorphism towers for the Poincaré and Galilei groups, and for the rotation groups $SO(n)$.

2 Proof of the Helpful Lemma

Lemma [Helpful]. Let ζG be the center of a group G, and let $\operatorname{Out} G$ be its outer automorphism group. If $|\zeta G| \neq |\operatorname{Out} G|$, then $G \not\simeq \operatorname{Aut} G$.

We prove the contrapositive of the lemma; namely, if $G \simeq \operatorname{Aut} G$, then $|\zeta G| = |\operatorname{Out} G|$. The proof is given by way of three propositions. The first proposition shows that if G/H is a quotient group, then there is a bijection from $H \times G/H$ to G. The second proposition assumes that $G \simeq \operatorname{Aut} G$, and exhibits a bijection from $\zeta G \times \operatorname{Inn} G$ to $\operatorname{Out} G \times \operatorname{Inn} G$. The final proposition then shows how a bijection is induced from ζG to $\operatorname{Out} G$.

Proposition 1. *Let G/H be a quotient group. There exists a bijection $\varphi : H \times G/H \to G$.*

Proof. We begin by observing that for any $g \in G$, $h \in H$, the mapping $\beta_g(h) : H \to gH$ defined by

$$\beta_g(h) = gh \tag{1}$$

is a bijection. This expresses an elementary fact about cosets over a subgroup H: they all have cardinality $|H|$. To give a quick proof of the claim : since $gh = gh'$ implies that $h = h'$, $\beta_g(h)$ is injective. And since every $gh \in gH$ has some $h \in H$ as its pre-image, $\beta_g(h)$ is also surjective, and we have a bijection.

Now, for every $g \in G$, there exists one bijection β_g. So we can say that each element h in a given coset gH corresponds to one bijection β_h. Let us pick out just one such bijection for each coset gH, and call it $\beta_{\bar{g}}$. (In the case that $|G/H|$ is infinite, this will generally require the axiom of choice.) This associates a unique mapping $\beta_{\bar{g}}$ with each coset gH. The desired bijection $\varphi : H \times G/H \to G$ is then the following:

$$\varphi(h, gH) = \beta_{\bar{g}}(h).$$

To see that φ is injective, suppose that $\beta_{\bar{g}}(h) = \beta_{\bar{g}'}(h') \in G$. The cosets of H partition G. So $\beta_{\bar{g}}(h) = \beta_{\bar{g}'}(h')$ lies in a unique such coset, call it gH. By definition, there is just one bijection $\beta_{\bar{g}}$ on gH. So $g = g'$, and we have that $\beta_{\bar{g}}(h) = \beta_{\bar{g}}(h')$. But $\beta_{\bar{g}}$ was constructed so as to be injective, and so it follows that $h = h'$. Therefore $(h, gH) = (h', g'H)$, and we have an injection.

To see that φ is surjective, let $g \in G$. We know that $\beta_{\bar{g}}$ is surjective, so there must exist some $h \in H$ such that $\beta_{\bar{g}}^{-1}(g) = h$. Thus,

$$\varphi(h, gH) = \beta_{\bar{g}}(h) = \beta_{\bar{g}} \circ \beta_{\bar{g}}^{-1}(g) = g.$$

So φ is also surjective, and we have a bijection as claimed. □

Proposition 2. *Let G be any group. If $G \simeq \operatorname{Aut} G$, then there is a bijection $\Phi : \zeta G \times \operatorname{Inn} G \to \operatorname{Out} G \times \operatorname{Inn} G$.*

Proof. It follows from Noether's first isomorphism theorem that $\operatorname{Inn} G \simeq G/\zeta G$ [4, Proposition 1.5.3]. So by Proposition 1, there exists a bijection $\varphi_1 : \zeta G \times \operatorname{Inn} G \to G$. Similarly, we have by definition that $\operatorname{Out} G := \operatorname{Aut} G/\operatorname{Inn} G$. So by Proposition 1, there is also a bijection $\varphi_2 : \operatorname{Out} G \times \operatorname{Inn} G \to \operatorname{Aut} G$. Finally, by assuming that $G \simeq \operatorname{Aut} G$, we get a bijection $\varphi^* : G \to \operatorname{Aut} G$.

We now make the definition:

$$\Phi := \varphi_2^{-1} \circ \varphi^* \circ \varphi_1.$$

The mapping $\Phi : \zeta G \times \operatorname{Inn} G \to G \to \operatorname{Aut} G \to \operatorname{Out} G \times \operatorname{Inn} G$ is the composition of three bijections. Therefore, Φ itself is a bijection, thus proving the proposition. □

Proposition 3. *If $\varphi : A \times C \to B \times C$ is a bijection, then there exists a bijection $\bar{\varphi} : A \to B$.*

Proof. Let $\bar{\varphi}$ be defined implicitly: $(\bar{\varphi}(a), c) = \varphi(a, c)$. To see that $\bar{\varphi}$ is injective, let $\bar{\varphi}(a) = \bar{\varphi}(b)$. Then $\varphi(a, c) = \varphi(b, c)$, so since φ is injective, $a = b$. To see that $\bar{\varphi}$ is surjective, let $b \in B$. Then since φ is a surjection, there is some $a \in A$ such that $\varphi(a, c) = (b, c) = (\bar{\varphi}(a), c)$. Therefore, $\bar{\varphi}$ is a bijection. □

With these three propositions in place, we may now give a simple proof of the Helpful Lemma. Suppose that $G \simeq \operatorname{Aut} G$. Then by Proposition (2), there is a bijection $\Phi : \zeta G \times \operatorname{Inn} G \to \operatorname{Out} G \times \operatorname{Inn} G$. By Proposition (3), this induces a bijection $\bar{\Phi} : \zeta G \to \operatorname{Out} G$. Therefore, $|\zeta G| = |\operatorname{Out} G|$, and we have proved the lemma.

References

[1] P. Fendley. Integrable Sigma Models. In C. Changrim Ahn, editor, *Integrable Quantum Field Theories and Their Application, Proceedings of the APCTP Winter School*, pages 108–178. Singapore: World Scientific Publishing Co. Pte. Ltd., 2001.

[2] C. D. Froggatt and H. B. Nielsen. *Origin of Symmetries.* Singapore: World Scientific Publishing Co. Pte. Ltd., 1991.

[3] I. M. Gel'fand, R. A. Minlos, and Z. Ya. Shapiro. *Representations of the rotation and Lorentz groups and their applications.* New York: The Macmillan Company, 1963.

[4] Derek J. S. Robinson. *A Course in the Theory of Groups.* New York: Springer-Verlag New York, Inc., 2nd ed edition, 1996.