Physics Without Foundations?
A Remark on Euler's *Mechanica*

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"I am two with nature." –Woody Allen

1 Beyond the *Principia*

Leonhard Euler is generally credited with first laying the foundations for analytic mechanics. But in what way is Euler's *Mechanica* analytic? Certainly it includes little in the way of *argumentum ex phaenomenis*, to take Isaac Newton as a contrast case. For Newton, adequate warrant for the most important derivations issues forth from phenomenal experience[1]. Euler, on the other hand, reserves a significant role for rationalist reflection in his derivations. The aim of this paper is to lay some groundwork for understanding how deeply this contrast cuts.

A number of recent scholars have agreed that there are important distinctions between the mechanics of Newton and Euler, gesturing above all toward the high priority given to mathematical considerations in Euler's system. Jerome Ravetz isolated Euler's union of "operational" and "geometric" representations almost fifty years ago, in contrast with Newton's tendency to avoid particular representations "as much as possible" (Ravetz, 1961).

[1]For the purposes of this paper, it is sufficient to note that Newton's typical derivations were clearly guided by this principle; detailed examples are discussed in Stein (1990) and Harper (1990). However, it is important to bear in mind that not all of Newton's arguments proceeded in this way; for example, the existence of absolute and infinite space is warranted by metaphysical and theological considerations in Newton's system. McGuire (2007) has shown how this kind of claim may still be squared with Newton's more well-known principle of argument from the phenomena.

Home (1988) chronicles Euler's disagreement with and ultimate refutation of Newton's theory of optics. Gaukroger (1982), Harman (1988), and Hepburn (2006) have all challenged the characterization of Euler as a "Newtonian," with the latter two focusing on "the emergence of the concept of function of a variable" in Euler's work, as a new "primary target" that was absent in Newton[2].

What I hope to correct in this literature is the implication that Euler's *Mechanica* is founded purely on mathematical analysis, while at the same time filling in some much needed details about the foundation of Euler's system. While it is correct to revise the view that Euler's mechanics is merely a translation of Newton's *Principia*[3], or so I shall argue, one must not take this reaction to far. Indeed, such an implication should stir the intuitions of all but the most staunch rationalists; for how could we possibly gain a meaningful account of physical phenomena from a system of mechanics derived through introspective mathematical analysis alone? I claim that we need not. In the next two sections, I briefly lay out the foundational elements and derivation rules of the *Mechanica*, arguing that Euler's foundation for mechanics consists of three essential parts: a mathematically idealized language, an empirical language, and a rule for bridging between the two. This clarification of Euler's system offers a clear starting point for understanding its place in a larger context, and I thus conclude by pointing to one foundational level on which we must distinguish Euler's system from that of Newton.

2 Foundational Elements

We begin with a systematic exposition of Euler's fundamental linguistic elements, as they appear in the Chapter 1 of the *Mechanica*. But let us first observe one important point of interpretation, that Euler's presentation of the concepts need not occur in the text in strict order of logical priority. Indeed, given that mathematical derivation is the central methodological tool available to Euler, it is more natural to give priority to order of derivation rather than ordering in the text. Terms are often introduced, perhaps for purposes of pedagogical clarity, and then explicitely defined in subsequent paragraphs. An important example of this that will be discussed shortly is

[2]see Harman (1988, 76) and Hepburn (2006, 2).

[3]This apparently prevalent view is expressed by Clifford Truesdell (1960), and is later echoed by Fellmann (1988) and Home (1988).

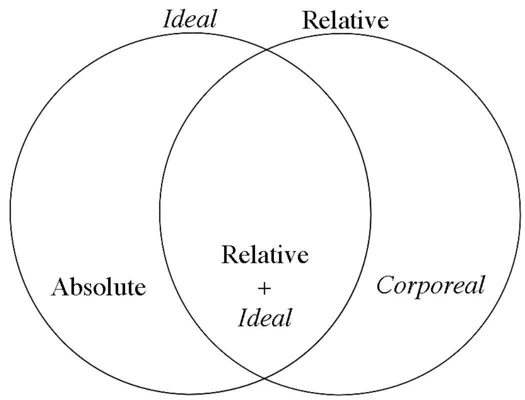

Figure 1: Absolute and relative spaces in relation to Euler's two *ontological categories.*

Euler's concept of motion, which despite being the very first word in the very first chapter, is quickly seen to be derivative of other concepts, and thus not a primitive term for Euler. To avoid possible confusion on such matters, we shall begin with the underived, "primitive" features of the system and work up.

Euler's approach begins with two broad ontological categories, which I call the *ideal* and the *corporeal*, shown above in figure 1. The former is the category human idealization, consisting entirely of imagined mathematical entities and relationships, which are obtained by "abstracting mentally." Euler, like Descartes, holds that we cannot form a definite idea of the true, absolute (infinite) space or its boundaries. Yet he goes well beyond Descartes in permitting himself to engage nevertheless in quantitative considerations of absolute space, construed as a mathematical idealization. In other words, "we imagine an infinite space" in order to do mechanics, independently of any facts about the "true" absolute space (Euler, [1736] 1912, §8). Accordingly, a body imagined in this space is said to have absolute position. On the other hand, Euler's considerations of empirical phenomena are always carried out in a separate, "corporeal" category, in which bodies are understood in "a finite space and corporeal limits," of which we can have phenomenal experiences (*ibid*, §7).

Within this framework, Euler builds his system of mechanics using exactly four primitive notions: space, body, position, and time. Each of these notions is assumed to have its own inherent properties. The notion of a *space* (ideal or corporeal) plays the role of a reference frame in the system, which Euler assures by requiring all spaces to have fixed boundaries. Since there is only one absolute space, any other space or object that serves as such a reference frame is said to be "relative" (Euler, [1736] 1912, §4). A *body* is then able to stand in a particular relation to the boundaries of a space, called "occupying a position" in that space (*ibid*, §2). *Position* may be understood here as a particular metric relation, which assigns a unique value to each body in a space, analagous to the position that a point occupies in geometric space. Finally, *time* is introduced as a continuous parameter on a body, used to construct a notion of uniform motion[4].

Every other concept that Euler deals with in this first chapter is built up from these four primitive concepts. For example, equipped with a notion of position that is well-defined on bodies, Euler is able to quantitatively define motion to be a change in position (Euler, [1736] 1912, §1), and later define uniform motion to be equal motion over equal intervals of time (*ibid*, §17). Like the primitive terms, each of these defined concepts may be considered in both ideal and corporeal considerations. However, a concept considered in a corporeal space (or any other relative space) does not automatically have the same properties as a concept considered in absolute space, such as absolute position, absolute motion, or absolute velocity. I shall now argue that this separation is not only crucial, but the very crux around which Euler's method of derivation is built.

3 Derivation Rules

Euler's derivations must certainly have been crafted with the greatest of care, as he indicates finding this aspect of Newton's *Principia* to be among its greatest shortcomings: "although I seemed to myself to have understood the solutions to many problems, still I could not solve other problems that differed even a little (Euler, [1736] 1912, 13). Euler's alternative methodology presented in Chapter 1 allows one to make inferences not only about the ideal

[4]The fact that time is a primitive notion in Euler's system may be obscured by his derivation of time equations such as $t = (1/250) \int dx/\sqrt{v}$ in Chapter II§This equation does indeed provide a tool for the measurement of time, but it is not a derivation of the notion itself. This is evidenced by Euler's earlier use of time in defining velocity (see Chapter I Definition 5).

and the corporeal, but also about the relationship between the two.

For Euler, a result in mechanics is genuine only when achieved "using an analytic method" (Euler, [1736] 1912, 15). This certainly includes inferences within absolute space, Euler's special mathematical idealization. Of course, simple logical inference in building up concepts like motion and velocity is permissible in any context, be it ideal or corporeal, absolute or relative; Euler gives no second thought to this kind of construction from previously established concepts. However, I claim that this is the *only* kind of reasoning allowed in considerations of purely relative motion, including all problems that take place in the physical world. Euler's insistence that we rely only on "clear and distinct knowledge" prohibits any further inferences outside of an idealized context. On the other hand, idealized reasoning about absolute space does include an additional kind of inference that might be called "rational reflection," which typically involves a claim about what is or is not possible in Euler's idealization.

To take an example, let us observe how this rational reflection is deployed in the first theorem of Chapter 1, which says that a body cannot experience an instantaneous change in position, but must rather travel through a series of intervening points. Euler begins with a proof in the case that the body lies in absolute space, and proceeds by asserting the self-evident truth of several claims, arrived at through some rational reflection:

> *In the case of absolute motion:* If the body reached the last location all at once from the first, it would be necessary for it to be destroyed in the first and immediately produced *de novo* in the last, and this cannot happen by natural laws unless a miracle should occur. It thus proceeds from the first into some next and from this into a following, until finally it arrives at the end. (Euler, [1736] 1912, §13)

In particular, Euler views the application of natural laws to ultimately receive its warrant in the context of idealized, absolute space; for Euler and Descartes alike, a natural laws issue forth from rational reasoning. However, here Euler has refused to commit to the existence of absolute space; this would seem to elicit some difficulty in extending these laws to the real (corporeal) world. As Euler points out in numerous places, "it is a fallacious" to conflate results about a situation in absolute space with results about a similar situation in relative space[5]. In such a state of affairs, a natural law would be of little use in mechanics if it were never extended to relative spaces in the corporeal world. Thus, in order to bridge this gap, Euler must tell us something about when a result derived in idealization of absolute space agrees with the physical world; he must give a bridge principle between absolute space and all other relative spaces, including the corporeal ones. This is exactly what Euler provides in the first corollary to Definition 3, in the form of a criterion for inferring that a state of motion is reference-frame independent:

> Relative motion and rest agree with absolute motion and rest when the space or body with respect to which motion and rest are judged are *in fact at rest* with respect to boundless and infinite space (Euler, 1736, §10, emphasis added).

It seems necessary to interpret Euler's use of "boundless and infinite space" here to mean "idealized, absolute space"; this agrees with his claim in §8 that "we do not claim to be given an infinite space of this sort," and is consistent with his later use of the definition[6]. This definition then has a straightforward interpretation, as our reasoning takes place entirely within a mathematical idealization. Note that it is possible to create an arbitrary reference frame in Euler's system by taking any body or group of bodies in (idealized) absolute space to be our relative space, and judging the position of bodies with respect to that. Definition §10 may then be understood to assert that a body in the relative space is at rest in the absolute space if and only if the relative space is itself at rest in the absolute space; this is an obvious corollary to the definition of relative motion, a point which is discussed in (Hepburn, 2006, 28). In this case, it makes sense to simply equate the relation of being "in fact at rest" with being "at rest," since both relata are part of the same mathematical idealization[7].

However, the interpretation is not so simple if the relative space being considered is a corporeal one, as in Euler's example from this same definition

[5]For example, see Euler ([1736] 1912, sections §58 and §69).

[6]Euler's use of the Latin *immensum* is typically translated as "boundless," but should be here be recognized as synonymous with "infinite" (see, for example, Euler's use of the word in §4). On the other hand, Euler uses *termini* to refer to the fixed boundaries of a space. Unlike their English translations, these two Latin terms are not mutually exclusive, and are often used in conjunction in Euler's description of absolute space.

[7]A clear example of this is found in Chapter 1, Proposition 11.

of "taking the earth as this space." At first blush, this would seem to challenge Euler's separation of two distinct ontological categories a few sections earlier. For how could a corporeal space be at rest in any sense with respect to a mathematical idealization? Although one might try to pin Euler as an idealist here, this is an unlikely solution. Alternatively, one might charge him with dishonesty in his characterizing absolute space as an idealization that need not exist, looking forward to the argument that Euler ([1765] 1975) makes thirty years later. However, it seems unlikely that Euler intended this to be a part of the *Mechanica*, as even his later arguments reveal some reservation.

Furthermore, I claim that there is a better way to solve this puzzle. My contention is that the predicate *in fact at rest* should be stripped of any prior metaphysical assumptions, just like all the other terms of Euler's system, and recognized instead as no more than an empty parameter. In other words, why not just take Euler's definition here at face value, as a condition that denotes the agreement between relative and absolute motion, just in case the claim that "a relative space is in fact at rest" is true? Despite this weakened reading, Euler's definition would still maintain its power by retaining the implicit assumption that a motion in any relative space (including the corporeal ones) can either agree or disagree with some arrangement in Euler's idealized absolute space. If a given relative motion agrees with an arrangement of absolute motion, then any property about the absolute motion can be imported directly into the relative context. If that relative motion disagrees with absolute motion, then it can be made to agree with some suitably constructed idealization of a moving relative space, returning us to the "straightforward interpretation" given above. Either way, the result is the existence of properties derived in a purely idealized context that hold independent of reference frames, ideal and corporeal alike.

This, I believe, is the most faithful representation of Euler's use of the definition; it is a kind of bridge principle, which allows one to bridge the gap between Euler's mathematical idealizations and the physical world. It is certainly fitting that the work Euler ([1736] 1912, 13) called his great "treatise on motion" should project a series of analytic results onto the world through none other than the concept of motion. Indeed, a discussion of such bridge principles in the general context of Euler's physics would provide an interesting topic for further research, although it is sufficient for the purposes of this paper to note that in Euler's foundation, given in the Chapter 1 of the *Mechanica*, the two theorems applicable to the physical world are both ultimately derived from this principle, as shown in figure 2 below.

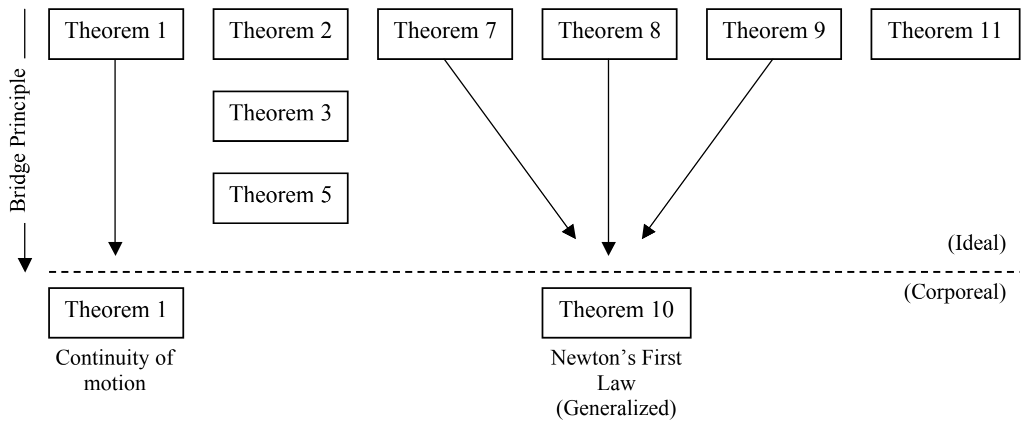

Figure 2: Each of Euler's theorems applicable to the corporeal world is derived from one or more theorems about mathematical idealizations, by way of the bridge principle given in §10.

Returning to the example of Euler's first theorem, we may then see how this principle plays out: two cases are considered, first the case that the relative space is "in fact at rest," followed by the case that it is not. In both cases, Euler is able to construe the situation so as to make use of the result previously derived in the case of absolute motion:

> *In the case of relative motion:* If a body, which lies in a place in infinite space, is in fact at rest, the preceding reasoning applies (§10). But if it should be in motion, it must itself also move through particular intervening places and therefore relative motion will also be successive and take place through particular intervening places. (Euler, [1736] 1912, §13)

Euler's foundation for mechanics may thus be summarized as follows: we a rich environment for deriving theorems and solving on mathematical idealizations, a language for use in considerations of the corporeal world, and a bridge principle that projects the former onto the latter. The result is a system of derivation in which the ideal and the corporeal are ontologically distinct and yet intimately connected. Although it might still be tempting to resist the last connection, and take Euler's definition in §10 to be a subset of his mathematical system, it would be a mistake to characterize §10 as an analytic claim. What Euler has provided us in this definition is a relation between the mathematically ideal and the corporeal, through a relation between the absolute and the relative, which licenses us to apply conclusions about one to the other. This principle is an aspect of Eulers system that is more reminiscent of what we would today call a "bridge principle" between theory and world. Although such an appellation cannot be used without some anachronism, it is to some extent an indication that Euler's foundation for mechanics was well ahead of its time. And, I believe, it certainly does more justice than an eighteenth century umbrella-term like "analytic."

4 Conclusion

It is easy to underestimate the significance of principle §10, given Euler's casual introduction of it as a corrolary to a definition. Indeed, it is possible that Euler himself preferred to downplay its significance; he evidentaly fancied himself to have simply laid out the "general properties of motion" and then proceeded to "derive the universal laws of nature" through an analytic process (Euler, [1736] 1912, 14). But the role this principle plays in the foundation for mechanics that Euler actually provides tells a different story.

Does this help us understand how Euler's theory of mechanics differs from that of Newton? As many of the authors previously mentioned have pointed out, there are many respects in which these two systems differ, especially with respect to their starting assumptions and the ontological notions that they take to be primitive. However, to making the additional claim that they are *different theories* is a notoriously tricky business that must be approached with care.

The first difficulty lies in the fact that two theories with different axiom sets can in some cases yield identical theorems. In such a case, the two theories would appear to be effectively identical. Whether or not the mechanics of Euler and Newton are an example of this is certainly not clear, but it cannot be taken for granted either.

A second difficulty is that in some cases, two theories generated from ontologically distinct starting points can turn out to be identical in all important respects. Henry Callendar first exhibited an example of this, by showing that an eighteenth century conception of caloric generates a theory that is practically identical to the subsequent, dynamical theory of heat (Callendar, 1910)[8]. Again, although the analogy here to Euler and Newton

[8]Callendar later argued *en force* that it is actually preferable to adopt the material,

need not hold in all respects, it is reason enough to resist taking ontological primitives to be the primary locution for distinguishing their theories.

The alternative perspective that I have suggested we adopt in studying Euler's *Mechanica* is one that takes it to be, as Euler did, a system of derivation. After all, Euler's hope in writing this work was to present a system that permits the reader to independently solve new problems, even if they "differ a little" from the examples provided in the text. The comparison of Euler's system to that of Newton is made significantly easier from this perspective. While Newton's derivations are ultimately licensed from the phenomenal world, Euler's derivations draw on a more subtle schema for their justification, in combining both idealized mathematical reasoning and bridge principles. In other words, insofar as these theories may be understood to be systems of derivation, Euler's theory of mechanics is clearly distinct from that of Newton, because the derivation rules are different.

Even though Euler's derivations arise in large part from analytic considerations of mathematical idealizations, it should come as some relief that these are not the only considerations on which the *Mechanica* is founded. Euler's analytic reasoning is indeed wedded to the physical world in a principled way, and thank goodness. One could scarcely make sense of a foundation without physics, especially one that achieved such long-lasting and resounding success.

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