The Role of the Euclidean *Reductio*

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What follows is a brief map of the role of *reductio ad absurdam* inferences in Euclidean practice. Restricting myself to the first book of the *Elements*, I shall argue that foundationalism in the Euclidean tradition provides a systematic, justified basis for *reductio* inference. In the first section of this paper, the content and diagrammatic attributions of the *reductio* arguments in Book I are catalogued and classified. The second section offers a way to understand all of these arguments as stable and justified by the tradition of Euclidean foundationalism. The final section attempts to address the objection that the role of the Euclidean *reductio* is an insignificant proper subset of modern practice.

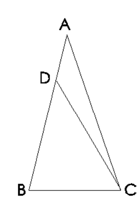
1. Euclidean Diagrams in *Reductio* Inferences

Recently, a role for diagrams in the practice of mathematical demonstration in ancient geometry has been suggested, mapped out, and fortified[1]. But if we are to understand the Euclidean diagram as a stable practice, we must understand its various roles as a justificatory tool. In this section, we examine the role of the diagram in conjunction with the inference rule of *reductio ad absurdam.* The structure of each *reductio* is briefly catalogued, and found to fall into a simple two-part classification. Again, for the purposes of this short paper, we shall restrict ourselves to Book I of Euclid's *Elements*.

1.1. Catalogue of Euclidean Reductios

In order to understand the structure of the *reductio* arguments of Book I, we must identify the *reductio* hypothesis, the contradiction that is produced, and the role of the diagram in this reasoning. In each of the following propositions, we thus identify: a) the *reductio hypothesis*, or the claim to be disproved; b) the *contradictory claim*, the first claim inferred that will later be contradicted, and c) the *contradicting claim*, the inference which is the opposite of the "contradictory claim", resulting in an impossibility. Additionally, it is noted whenever a **diagrammatic attribution** is employed in an inference; when no diagrammatic attributions are employed, the inference is "purely discursive." These structural comments may be readily verified in Euclid's *Elements*[2].

Proposition 6. In this demonstration, two angles are taken to be equal to each other ($\angle ABC$ and $\angle ACB$ in the diagram to the right).

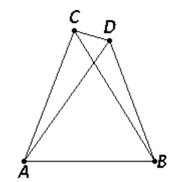


- The *reductio hypothesis* is that the sides opposite these angles are not equal. A particular point D is taken to lie properly between points A and B.
- The *contradictory claim* is that the triangle ABC is equal to the triangle DBC; it is arrived at through purely discursive inferences (no diagrammatic attribution).

[1]For example, see Manders 1995a, 1995b; Norman 2006.
[2]Each proposition discussed is taken from the standard English translation by Heath.

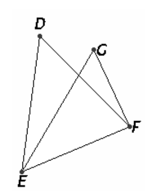
- The *contradicting claim* is that *ABC* is not equal to *DBC*. This is arrived at through **one diagrammatic attribution** that triangle *DBC* is a *part* of triangle *ABC*. The claim then follows from "parthood[3]" in the following way: by Common Notion 5 that the whole is greater than the part, and thus, Euclid concludes that *ABC* is not equal to *DBC*. This last inference is through an implicit understanding in the practice of equality relations, that "greater than" implies "not equal."

Proposition 7.



- The *reductio hypothesis* is that $AC = AD$, and that $BC = BD$ on the same side of a common base, as in the diagram to the right.
- The *contradictory claim* is that $\angle CDB$ is not equal to $\angle DCB$. This is arrived at through **two diagrammatic attributions:** first, that $\angle DCB$ is a part of $\angle ACD$, and then that $\angle ADC$ is a part of $\angle CDB$. It then follows from "parthood" (as in proposition 6) that $\angle DCB \neq \angle ACD$, and that $\angle ADC \neq \angle CDB$. But using Proposition 5 Euclid shows that $\angle ACD = \angle ADC$, and thus shows that $\angle DCB \neq \angle CDB$, which proves the contradictory claim.
- The *contradicting claim* is that $\angle CDB$ is equal to $\angle DCB$. It is arrived at discursively, through the use of proposition I.5.

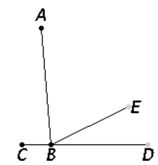
Proposition 8. Two triangles are taken to have each of their respective sides equal to one another (the "side-side-side" relationship).



- The *reductio hypothesis* is that these two triangles may be aligned on the same side of a common base, so that the other to sides do not coincide.
- The *contradictory claim* is that these triangles take the form of the reductio hypothesis in Proposition 7. This is arrived at purely discursively.
- The *contradicting claim* is simply that the reductio hypothesis of Proposition 7 is false, as was shown previously. Hence, there are **NO diagrammatic attributions** in this demonstration.

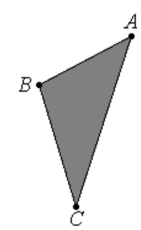
[3]This notion is discussed more thoroughly below, in the "Classification and Commentary".

Proposition 14. It is given that ∠*ABC* and ∠*ABD* sum to equal two right angles.



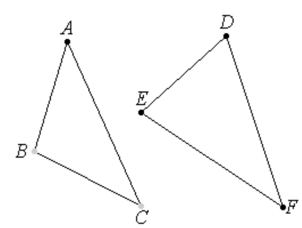
- The *reductio hypothesis* is that *CB* is "not straight" with *BD*.
- The *contradictory claim* is that ∠*ABE* equals ∠*ABD*, and is made through purely discursive argument.
- The *contradicting claim* is that ∠*ABE* is not equal to ∠*ABD*. This is concluded through **one diagrammatic attribution**, in which Euclid notes that ∠*ABE* is a *part* of ∠*ABD*. The claim then follows from "parthood," as in Proposition 6.

Proposition 19. One angle of a triangle, ∠*ABC*, is taken to be larger than another angle, ∠*BCA*.



- The *reductio hypothesis* is that line *AC* is not greater than *AB*.
- The *contradictory claim* consists of two additional facts, that *AC* is not less than *AB* and that *AC* ≠ *AB*. These conclusions are justified purely discursively.
- The *contradicting claim* is that given any two lines, one is either <, >, or = to the other. This is an implicit assumption about the meaning of equality and inequality relations between lines. Hence, there are **NO diagrammatic attributions** in this demonstration.[4]

Proposition 25. We are given two triangles, *ABC* and *DEF*, in which the *AB* = *DE*, *AC* = *EF*, and *BC* > *EF*.

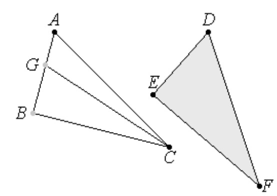


- The *reductio hypothesis*is that ∠*BAC* is not greater than ∠*DEF*.

[4]Neither this nor the following proposition make use of diagrammatic attributions. This is not surprising from a modern standpoint, for as Heath points out, each of these propositions is a "merely *logical* deduction" from previous ones (Heath 284 & 299). However, it is important to note that this relationship between diagram use and logical deduction may be misleading. For Heath also notes that proposition I.6 follows deductively from other propositions (Heath 256); yet as we have seen, Proposition I.6 nevertheless *does* make use of a diagrammatic attribution.

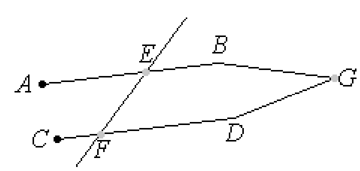
- The *contradictory claim*, similar to Proposition 19, is that that ∠*BAC* is not less than or equal to ∠*DEF* either. These conclusions are justified purely discursively.
- The *contradicting claim* is that given any two angles, one is either <, >, or = to the other. This is an implicit assumption about the meaning of equality and inequality relations between angles. Hence, there are **NO diagrammatic attributions** in this demonstration.

Proposition 26. There are two reductios given here, but the argument structure is symmetric, so only one is discussed in this commentary. We are given two triangles that have the "angle-side-angle" property. That is, ∠*ABC* equals ∠*DEF*, ∠*BCA* equals ∠*EFD*, and *BC* = *EF*.



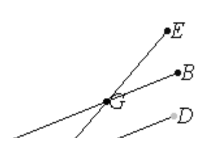
- The *reductio hypothesis* is that *AB* ≠ *DE*.
- The *contradictory claim* is that ∠*BCG* is equal to ∠*BCA*. This inference is made purely discursively.
- The *contradicting claim* is that ∠*BCG* is not equal to ∠*BCA*. This is justified by **one diagrammatic attribution**, that ∠*BCG* is a *part* of ∠*BCA*. The claim then follows by "parthood", as in the comment on Postulate 6.

Proposition 27. We are given a line *EF* intersecting two other lines *AB* and *CD* as in the diagram shown here, so that the alternate angles are equal.



- The *reductio hypothesis* is that *AB* and *CD* are not parallel.
- The *contradictory claim* is that the exterior angle ∠*AEF* is equal to the interior angle *EFG*. This argument is made purely discursively.
- The *contradicting claim* is that an exterior angle is always *greater than* an the opposite interior angles, which is justified by postulate I.16. Thus, there are **NO diagrammatic attributions** made in this argument.

Proposition 29. Two lines *AB* and *CD* are taken to be parallel, and a third line *EF*

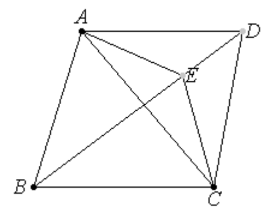


intersects them at points *G* and *H*, respectively.

- The *reductio hypothesis* is that $\angle AGH \neq \angle GHD$ (these are the alternate angles).
- The *contradictory claim* is that *AB* and *CD* must meet at a point. This is argued purely discursively.
- The *contradicting claim* is that *AB* and *CD* cannot meet at a point because they are parallel, by Definition 23 of parallel lines. Thus, there are **NO diagrammatic attributions** made in this argument.

Proposition 39. Two equal triangles are taken to be on the same side of a common base.

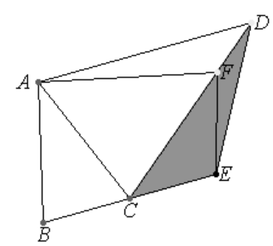
- The *reductio hypothesis* is that the line *AD* connecting the peaks of the two triangles is not parallel to the base *BC*.
- The *contradictory claim* is triangle *DBC* is equal to *EBC*. This is shown entirely discursively.



- The *contradicting claim* is that triangle *EBC* is not equal to *DBC*. This is made through the **diagrammatic attribution** that *EBC* is a *part* of triangle *DBC*. The claim then follows by "parthood", as in the comment on Postulate 6.

Proposition 40.[5] The bases of two equal-area triangles are taken to form a straight line.

- The *reductio hypothesis* is that the line *AD* connecting the peaks of these triangles is not parallel to *BE*.
- The *contradictory claim* is that the triangle *DCE* is equal to the triangle *FCE*. This is thrown through entirely discursive argument.



- The *contradicting claim* is that triangle *FCE* is not equal to *DCE*. This is made through the **diagrammatic attribution** that *FCE* is a *part* of triangle *DCE*. The claim then follows by "parthood", as in the comment on Postulate 6.

1.2. Classification and Commentary

[5]Heath notes that "Heiberg has proved" Proposition 40 was probably not in the original Euclidean text, but was rather added by a later writer (Heath 338). However, this proposition will nevertheless be counted in our analysis, as the methodology employed seems to have proceeded in the Euclidean tradition.

Each of the eleven *reductio* arguments of Book I fall into one of two broad categories. The first is composed of those demonstrations that do not make use of diagrammatic attributions (Propositions 8, 19, 25, 27, and 29); the second is composed of those demonstrations that do (Propositions 6, 7, 14, 26, 39, and 40). Examples of these two categories of *reductio* are more or less evenly distributed throughout the book, suggesting no particular preference for one over the other on the part of Euclid.

Perhaps more interestingly, all of the diagrammatic attributions of this second category are of the same kind: the inference is that a line, angle, or figure is a *part of* another such diagrammatic element. (A triangle is part of another triangle, etc.) Furthermore, each of these inferences is accompanied by the following two assumptions, in order to complete the *reductio*. The first is that "the whole is greater than the part" (Common Notion 5), and the second is that "greater than or less than implies not equal" (an implicit assumption about equality). For example, in Proposition 6 Euclid makes the attribution that triangle *DBC* is a part of triangle *ABC*. From Common Notion 5, it may then be inferred that triangle *ABC* is greater than triangle *DBC*. Finally, Euclid infers, this implies *ABC* is not equal to *DBC*, and is able to produce the desired contradiction. The implicit inferences are fundamental (indeed, "foundational") to the Euclidean tradition, and although Euclid does not state them outright, their existence is far from speculative. We shall see why in the next section.

Thus, the diagrams of the Euclidean *reductio* appear to have a surprisingly consistent character, in Book I of the *Elements*. If the explicit and implicit assumptions deployed in these arguments turn out to be well justified by the practice, the method of the Euclidean diagram in *reductio* arguments may turn out to be a very reliable tool indeed. So let us now turn to the means through which these arguments are justified.

2. *Reductios* and the Euclidean Foundation

2.1. Is Euclidean Geometry Well Founded?

Euclidean Geometry has sustained a good deal of criticism over the centuries, to say the least. Discontent over the parallel line axiom dates back to Posidonius, and in recent centuries, logicians leveled serious complaints about the reliability of the practice. This section shall try to respond to the particular criticism that Euclidean demonstrations are often not sufficiently justified. After a brief discussion of foundationalism in the Euclidean tradition, it will be argued that the Euclidean *reductios* are much more robustly justified than these criticisms suggest, once they are understood to have both an "explicit" and an "implicit" foundation. An application of this understanding will then be given for the *reductio* arguments discussed in the previous section.

2.2. What Is The Euclidean Foundation?

Euclidean Geometry is sometimes described, in the spirit of Hilbert Program, as a "foundational" approach. This description is reasonable, insofar as it captures the fact that Euclid prefaces each book propositions with a body of definitions, postulates, and common notions, which he takes to be assertable without justification. A further useful aspect of the description is that this body plays a central role in justifying the inferences made in a Euclidean demonstration. However, it would be extremely presumptuous to assume that this body of claims played a role for ancient that was identical to the role that formal systems played for 19th and 20th century mathematicians. The fact that these practices are separated by over 2000 years should be a first clue that this is the case; the "missing postulates" parts of Euclidean arguments, as pointed out by Heath in the early 20th century, should be a second.

The Euclidean practice may be rescued from much of this confusion once we realize that the explicit bodies of definitions, postulates, and common notions do not exhaust the Euclidean foundation. There is another aspect to the Euclidean foundation, which is *not* explicitely stated. Every tradition takes for granted its share of primitive notions, those notions that "go without saying" in the practice, that are so obvious as to require neither justification nor even explication. Such notions are perhaps even the "most foundational" notions of the practice, insofar as they operate tacitly behind all others. Thus, when judging a system of justification within a practice, we clearly must be as sensitive as possible to the notions that are implicitly understood by members of that practice. The practice of Euclidean geometry should be no exception. Once certain notions are taken on board as tools implicitly available in a Euclidean demonstration, the scope of what proofs are "justified" is significantly broadened.

This is not to suggest that we speculate vaguely about the enigmatic assumptions of a long-lost tradition. Such an approach admits the danger of misrepresenting the tradition. Rather, the contention here is that many parts of the implicit Euclidean foundation may be precisely determined from Euclid's *Elements*. There may be many ways of discerning these assumptions; however, in what follows, we shall focus on the ones that may be read directly off of the definitions, postulates, and common notions. These will provide us with a significant, though necessarily incomplete, understanding of the implicit part of the Euclidean foundation.

2.3. Elements of an Implicit Foundation

Euclid clearly expected his readers to come into the Elements with a certain amount of prerequisite knowledge. For example, if Euclid was to communicate to his audience his desired meaning for a definition, he would have had to use words that they already understood. Thus, in the *definiens* of each definition, we may safely assume that Euclid used only notions that he could take to be "common knowledge" among his readers. However, while the *definiendum* has long been understood to be an essential part of the Euclidean foundation, the notions contained in the *definiens* have been largely overlooked. These and other implicit notions make up a significant part of the foundation of Euclidean geometry. Let us explore some elements of an implicit foundation. In the definitions, postulates, and common notions of Book I, these implicit notions may be divided into three groups: objects, attributes, and practices. Euclid took each of them as given, tacitly expecting that readers in his geometric tradition would too, as they worked through the Elements.

First, a set of practices was taken as prerequisite. In other words, Euclid's readers were expected to be able to faithfully execute a certain number of activities. These included the ability to *produce* (Def 23) a line, and produce it *continuously* (Def 23) or *indefinitely* (Post 2); to *add* (C.N. 2) or *subtract* (C.N. 3) one figure from another; and to *cut off* (Def 18) a portion of a figure.

Second, a group of geometric objects were taken as given. Of the most basic, we are to understand a *length* (Def 2), perhaps as one might understand a length of rope, and that such a length has two *extremities* (Def 3). The related notions of a *distance* (Post 3) and of a *side* of a plane figure (Def 21) are also implicitly assumed, as is the notion of an *angle* (Def 21) and of the *remainder* (C.N. 3) of a geometric subtraction.

Finally, certain attributes were assumed to be applicable to these geometric objects and their derivatives. Some of these attributes apply to a particular object, such as having *breadth* (Def 2) or being *finite* (Post 2). Other attributes express a relationship between two or more objects. For example, we are to understand how certain figures could have the attribute of *being set upon* (Def 10), *falling upon* (Def 15), *lying evenly with* (Def 7), *coinciding with* (C.N. 4) or *meeting* (Post 5) another figure. Additionally, it is given that certain equivalence relations can be expressed between figures: an angle can be *less than* (Def 12), *greater than* (Def 11), or *equal to* (Def 10) another angle; two lines can be *equal* or *unequal* (Def 20). And crucially, we are expected to understand when two figures stand in a particular "parthood" relation, in which one figure is the *part* (Def 1) and the other is the *whole* (C.N. 2). This attribution is of particular interest to us with respect with the *reductios* of the previous section; let us now see how an understanding of "parthood" as an implicit element of the Euclidean Foundation is essential for a complete understanding of the justification of these arguments.

2.4. Application to the Euclidean *Reductio*

Heath has suggested in several places that "Euclid has no right" to make certain assumptions apparently made in the Elements (for example, Heath 242). But in geometry as in law, a "right" is relative to a tradition. As we have just seen, the Euclidean foundation contains a rich implicit part that Heath may not have recognized from his standpoint in the early 20th century. And as we shall now show, Heath's claim does not apply in the case of the *reductio* arguments categorized in the previous section.

In particular, let us examine Heath's claim that "[s]ome postulate is necessary to justify [the] tacit assumption" in Proposition 6 (Heath 256), and Proposition 7 (Heath 260). Heath is concerned about the situations in which Euclid infers that one element in a diagram is “lesser than” another element, with no apparent justification. He assumes that Euclid takes this to be implied by a fact about "betweenness", perhaps influenced by Hilbert's axiomatization of Euclidean geometry. However, no such postulate is lacking if we accept the above presentation. For as we have just seen, "parthood" is itself an implicit element of the foundation provided in the Euclidean foundation. We would not be faithfully representing Euclidean practice if we ignored one of Euclid's foundational assumptions, that his readers can infer when one element in a diagram is a part of another element. Indeed, this is the only kind of diagrammatic attribution made in the *reductio* arguments of Book I; all other inferences are purely discursive, and follow from explicit elements of the foundation[6]. Thus, once the notion of "parthood" is properly accepted as an element of the Euclidean foundation, Heath's worry dissolves. Propositions 6 and 7, like the other *reductios* of Book I, enjoy a robust level of justification within the tradition of Euclidean geometry.

2.5. Conclusion

It has been suggested that "Euclid's enunciations not infrequently leave something to be desired in point of clearness and precision" (Heath 248). However, it is in no way clear that these points of desirability are equally applicable from the standpoint of the ancient practitioner of Euclidean geometry. Many precise foundational assumptions, though implicit, may be discerned and admitted as parts of the Euclidean method. Further analysis of the kind given above may indeed show that many other demonstrations of the

[6] See section 1.2 above.

Euclidean practice are, like the Euclidean *reductio*, justified by a sound and stable tradition.

3. A Critique of "Content Imperialism"

Having now seen the classification of Euclidean *reductios* and how they may be justified, let us now discuss the more general question of the philosophical import of these arguments as an independent practice. Each of the theorems of Euclidean geometry is accepted within modern mathematics, but they are not justified in the same way. What does this say about the significance of justifying a *reductio* in the Euclidean tradition?

3.1. The "Content Imperialist" Objection

Perhaps the most difficult critique of the Euclidean justification of *reductio* arguments is that it is irrelevant in light of modern practice. This view stems from the realization that the informational content of Euclidean geometry has been captured by modern mathematical logic. Mathematicians now work in abstract spaces in which the space of Euclidean geometry is merely a special case, $\mathbf{R}^2$. Logicians have similarly taken Euclidean geometry under their wing, interpreting the theorems of Euclid as consequences of an extension of formal logic[7]. This kind of view may be justly titled "Content Imperialism," and attributed the slogan: "The content of Euclidean practice is *merely* a part of logic." The *merely* indicates the general attitude that more abstract structures are in some sense "better" or "more advanced" than their specific representations.

The consequence of this view of concern to us here is that if Euclidean geometry is taken to be *merely* a part of logic, then the story we have told in previous sections is irrelevant; the Euclidean *reductio* need only be justified by modern practice. This purpose of this section is to present two theses suggesting that the Content Imperialism view is wrong. The first is a negative thesis, which says that Content Imperialism is incoherent,

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[7]These interpretations famously came in the form of a second and first order axiomatization, due to Hilbert and Tarski, respectively.

because its preference for "more abstract" structure is underdetermined. The second is a positive thesis, which says that despite being *content* equivalent to parts of modern theory, the justification provided by Euclidean foundationalism nevertheless offers a significant contribution to mathematics.

3.2. Can One Commit to the Highest Abstraction?

Setting aside ancient geometry for the moment, let us discuss the slogan and the claim of Content Imperialism in one paradigm example; namely, the relationship between group theory and its representations[8]. In this context, the Content Imperialist is inclined to dismiss familiar mathematical objects such as the real numbers or the invertible matrices, in favor of the group structures that these objects instantiate, respectively. Since the informational content of these objects is captured by the more abstract structure, the Content Imperialist suggests we follow a rule of "commitment to the greatest abstraction," and relegate the mere representations to a role of insignificance.

My claim is that it is far from clear that such a rule may be coherently followed. One difficulty that has been suggested is that in some cases, less abstract structures can lead to the proof of a theorem about a more abstract structure[9]. Such a results suggest that commitment to the highest abstraction may not be desirable for pragmatic reasons, in that it seems to neglect a fruitful path to mathematical understanding. However, there is a deeper worry about this rule. Because for a given mathematical object, it is not always clear which abstraction we should commit to.

Consider the example of the integers 0 through 6 under addition. Presumably, the Content Empiricist wants to dismiss these notions as *mere* representations, and instead commit to the more abstract group theoretic structure that they instantiate, Z_7. But if a rule of "commitment to the highest abstraction" is to be faithfully followed, we surely cannot stop there! The automorphism group of Z_7, which is Z_6, is an abstraction of the original group. It also seems to be more general, in that Z_6 is also the automorphism

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[8]Weyl is a classic contributor to what I am calling Content Imperialism in this domain (see Weyl xix-xxii).

[9]Thanks so much for a really interesting and eye-opening class, Ken.

group of at least three other groups, Z_9, Z_{14}, and Z_{18}[10]. But we may continue in this way: by the same rule, one should apparently go even more abstract, and commit to the automorphism group of Z_6, which is Z_2. It now begins to seem like the level of abstraction that we are being asked to commit to is not as clear as was supposed.

Of course, the Content Imperialist may come back by claiming that the rule of commitment to the highest abstraction does indeed halt at the level of Z_2. There are no non-trivial automorphism groups of Z_2, and so we cannot abstract any further using automorphism groups. So this must be the structure that we should commit to.

There are two problems with this result. The first is that we have lost most of the original understanding we gained about the integers 0 through 6 through abstracting to the level of groups. It is much more difficult to see how Z_2 contributes meaningfully to our understanding of this representation, and so it seems silly that we have been led to commit only to this simple structure. The second problem is that there is no guarantee that the process of taking successive automorphism groups will necessarily terminate, as Z_7 terminated with Z_2. As far as I know, it is an open question whether or not the process of taking successive non-trivial automorphism groups in this way could proceed indefinitely, either by forever increasing in order or else by cycling. Either case would render the rule of commitment to the highest abstraction impossible. And although an example is not readily forthcoming to the present author, the open possibility is enough to cast even more doubt on the Content Imperialist's commitment.

Although none of these worries make Content Imperialism unsalvageable, they are enough to show one show one conspicuous weakness of the view. Content Imperialism seems to have neglected the role of *understanding* as a central *desideratum* in mathematical practice. In the objections just raised, we saw that the view has the potential to neglect, diminish, or render indeterminate our understanding of simple mathematical notions. As we shall now see, the view also misses a significant contribution that was uniquely provided by the Euclidean practice.

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[10]The order of these automorphism groups may be calculated by the fact that the order of Aut(Z_n) is equal to $\varphi(n)$, the Euler function of n. Furthermore, each automorphism group discussed here is cyclic, because Aut(Z_n) is cyclic whenever $n = 2, 4, p^k$, or $2p^k$. Thus, all of the automorphism groups discussed here may be readily verified.

3.3. A Distinctive Contribution of the Euclidean Practice

It has recently been insightfully suggested that the Euclidean tradition, despite being "informationaly equivalent" to a subset of formal logic, nevertheless has the capacity to make distinctive contributions to mathematical practice[11]. And although a complete analysis of this capacity is outside the scope of this paper, one apparent contribution must be mentioned in the spirit of the previous two sections. My suggestion is that the very character of the Euclidean tradition itself licensed the long and fruitful practice of demonstration through *reductio ad absurdam*. What follows is a brief sketch of why I take this to be the case.

A distinct though implicit requirement that we may clearly infer from the Euclidean tradition is the insistence on jettisoning unstable foundational claims. The ancient practice of Euclidean geometry, characterized by particular sets of definitions, postulates, common notions, and derivative theorems, forms a startlingly stable theory. This fact is verified in the modern interpretation's proof of its consistency, and further witnessed by two millennia of internal consistency. So it would be unreasonable to suppose that such a successful body could have been spontaneously composed. On the contrary, the Euclidean foundation is the result of a long and careful practice of assuming claims followed by the removal offending elements that led to impossibility. Obviously, to make this claim historically sound requires much more research in the development of pre-Euclidean geometry; however, I nevertheless take it to be a reasonable, and indeed likely, possibility.

Now, the practice of hypothesis and rejection is closely wedded to the inference of *reductio ad absurdam*. The former is a broad value that may has spanned many epochs; the latter is a precise practice that hardened into a rule through regular and stable repetition. Yet they are both essentially the same principle to the mathematician who deploys the implicit values and preconceptions of his tradition in practice. This principle is something that I take to be a very distinctive contribution of the Euclidean practice, independent of its “mere” content, as belittled by the Content Imperialist.

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[11] Ken Manders, class discussion.

All of this has certainly not been to degrade the brilliant achievements in modern mathematics in capturing and uniting Euclidean geometry with the theories that followed it. Rather, it is intended as a small contribution to the thesis promoted recently by Manders that the Euclidean tradition rests on a much firmer foundation that has previously been admitted. The Euclidean *reductio*, like many of the modes of inference licensed by Euclidean geometry, has been licensed by a deep and far-reaching practice of justification.

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