Section Summary

- Permutations
- Combinations
- Combinatorial Proofs
Permutations

**Definition**: A *permutation* of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of *r* elements of a set is called an *r*-permutation.

**Example**: Let $S = \{1,2,3\}$.
- The ordered arrangement $3,1,2$ is a permutation of $S$.
- The ordered arrangement $3,2$ is a 2-permutation of $S$.
- The number of *r*-permutations of a set with *n* elements is denoted by $P(n,r)$.
  - The 2-permutations of $S = \{1,2,3\}$ are $1,2; 1,3; 2,1; 2,3; 3,1; and 3,2$. Hence, $P(3,2) = 6$. 
A Formula for the Number of Permutations

**Theorem 1:** If \( n \) is a positive integer and \( r \) is an integer with \( 1 \leq r \leq n \), then there are
\[
P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1)
\]
\( r \)-permutations of a set with \( n \) distinct elements.

**Proof:** Use the product rule. The first element can be chosen in \( n \) ways. The second in \( n - 1 \) ways, and so on until there are \( (n - (r - 1)) \) ways to choose the last element.

- Note that \( P(n,0) = 1 \), since there is only one way to order zero elements.

**Corollary 1:** If \( n \) and \( r \) are integers with \( 1 \leq r \leq n \), then
\[
P(n, r) = \frac{n!}{(n-r)!}
\]
Solving Counting Problems by Counting Permutations

Example: How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?
Solving Counting Problems by Counting Permutations

**Example:** How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

**Solution:**

\[ P(100,3) = 100 \cdot 99 \cdot 98 = 970,200 \]
Example: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?
Example: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Solution: The first city is chosen, and the rest are ordered arbitrarily. Hence the orders are:

\[ 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040 \]

If she wants to find the tour with the shortest path that visits all the cities, she must consider 5040 paths!
Example: How many permutations of the letters $ABCDEFGH$ contain the string $ABC$?
Example: How many permutations of the letters $ABCDEFGH$ contain the string $ABC$?

Solution: We solve this problem by counting the permutations of six objects, $ABC$, $D$, $E$, $F$, $G$, and $H$.

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$
Combinations

Definition: An \( r \)-combination of elements of a set is an unordered selection of \( r \) elements from the set. Thus, an \( r \)-combination is simply a subset of the set with \( r \) elements.

- The number of \( r \)-combinations of a set with \( n \) distinct elements is denoted by \( C(n, r) \). The notation \( \binom{n}{r} \) is also used and is called a binomial coefficient. (We will see the notation again in the binomial theorem in Section 6.4.)

Example: Let \( S \) be the set \{a, b, c, d\}. Then \{a, c, d\} is a 3-combination from \( S \). It is the same as \{d, c, a\} since the order listed does not matter.

- \( C(4,2) = 6 \) because the 2-combinations of \{a, b, c, d\} are the six subsets \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, and \{c, d\}. 
Combinations

**Theorem 2:** The number of \( r \)-combinations of a set with \( n \) elements, where \( n \geq r \geq 0 \), equals

\[
C(n, r) = \frac{n!}{(n-r)!r!}.
\]

**Proof:** By the product rule \( P(n, r) = C(n,r) \cdot P(r,r) \). Therefore,

\[
C(n, r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{(n-r)!r!}.
\]
Combinations

**Example**: How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a deck of 52 cards?

*This is a special case of a general result.*
Combinations

**Example:** How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a deck of 52 cards?

**Solution:** Since the order in which the cards are dealt does not matter, the number of five card hands is:

\[ C(52, 5) = \frac{52!}{5!47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960 \]

- The different ways to select 47 cards from 52 is

\[ C(52, 47) = \frac{52!}{47!5!} = C(52, 5) = 2,598,960. \]

*This is a special case of a general result.*
Combinations

**Corollary 2:** Let \( n \) and \( r \) be nonnegative integers with \( r \leq n \). Then \( C(n, r) = C(n, n - r) \).

**Proof:** From Theorem 2, it follows that

\[
C(n, r) = \frac{n!}{(n-r)!r!}
\]

and

\[
C(n, n - r) = \frac{n!}{(n-r)![(n-(n-r))!]} = \frac{n!}{(n-r)!r!}.
\]

Hence, \( C(n, r) = C(n, n - r) \).

This result can be proved without using algebraic manipulation.
Combinatorial Proofs

**Definition 1:** A *combinatorial proof* of an identity is a proof that uses one of the following methods.

- A *double counting proof* uses counting arguments to prove that both sides of an identity count the same objects, but in different ways.
- A *bijective proof* shows that there is a bijection between the sets of objects counted by the two sides of the identity.
Combinatorial Proofs

Here are two combinatorial proofs that
\[ C(n, r) = C(n, n - r) \]
when \( r \) and \( n \) are nonnegative integers with \( r < n \):

- **Bijective Proof**: Suppose that \( S \) is a set with \( n \) elements. The function that maps a subset \( A \) of \( S \) to \( \overline{A} \) is a bijection between the subsets of \( S \) with \( r \) elements and the subsets with \( n - r \) elements. Since there is a bijection between the two sets, they must have the same number of elements.

- **Double Counting Proof**: By definition the number of subsets of \( S \) with \( r \) elements is \( C(n, r) \). Each subset \( A \) of \( S \) can also be described by specifying which elements are not in \( A \), i.e., those which are in \( \overline{A} \). Since the complement of a subset of \( S \) with \( r \) elements has \( n - r \) elements, there are also \( C(n, n - r) \) subsets of \( S \) with \( r \) elements.
Combinations

**Example:** How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.

**Solution:** By Theorem 2, the number of combinations is

\[ C(10, 5) = \frac{10!}{5!5!} = 252. \]

**Example:** A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission?

**Solution:** By Theorem 2, the number of possible crews is

\[ C(30, 6) = \frac{30!}{6!24!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593,775. \]
Binomial Coefficients and Identities

Section 6.4
Section Summary

- The Binomial Theorem
- Pascal’s Identity and Triangle
- Other Identities Involving Binomial Coefficients (not currently included in overheads)
Powers of Binomial Expressions

**Definition:** A *binomial* expression is the sum of two terms, such as \( x + y \). (More generally, these terms can be products of constants and variables.)

- We can use counting principles to find the coefficients in the expansion of \((x + y)^n\) where \(n\) is a positive integer.
- To illustrate this idea, we first look at the process of expanding \((x + y)^3\).
- \((x + y) (x + y) (x + y)\) expands into a sum of terms that are the product of a term from each of the three sums.
- Terms of the form \(x^3, x^2y, xy^2, y^3\) arise. The question is what are the coefficients?
  - To obtain \(x^3\), an \(x\) must be chosen from each of the sums. There is only one way to do this. So, the coefficient of \(x^3\) is 1.
  - To obtain \(x^2y\), an \(x\) must be chosen from two of the sums and a \(y\) from the other. There are \(\binom{3}{2}\) ways to do this and so the coefficient of \(x^2y\) is 3.
  - To obtain \(xy^2\), an \(x\) must be chosen from one of the sums and a \(y\) from the other two. There are \(\binom{3}{1}\) ways to do this and so the coefficient of \(xy^2\) is 3.
  - To obtain \(y^3\), a \(y\) must be chosen from each of the sums. There is only one way to do this. So, the coefficient of \(y^3\) is 1.
- We have used a counting argument to show that \((x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3\).
- Next we present the binomial theorem gives the coefficients of the terms in the expansion of \((x + y)^n\).
Binomial Theorem

**Binomial Theorem**: Let $x$ and $y$ be variables, and $n$ a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$ 

**Proof**: We use combinatorial reasoning. The terms in the expansion of $(x+y)^n$ are of the form $x^{n-j} y^j$ for $j = 0, 1, 2, \ldots, n$. To form the term $x^{n-j} y^j$, it is necessary to choose $n-j$ $x$s from the $n$ sums. Therefore, the coefficient of $x^{n-j} y^j$ is $\binom{n}{n-j}$ which equals $\binom{n}{j}$. ▶
Using the Binomial Theorem

Example: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution: We view the expression as $(2x +(-3y))^{25}$. By the binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^j.$$  

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j=13$.

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!12!} 2^{12}3^{13}.$$
A Useful Identity

Corollary 1: With $n \geq 0$, 
\[ \sum_{k=0}^{n} \binom{n}{k} = 2^n. \]

Proof (using binomial theorem): With $x = 1$ and $y = 1$, from the binomial theorem we see that:
\[ 2^n = (1 + 1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}. \]

Proof (combinatorial): Consider the subsets of a set with $n$ elements. There are $\binom{n}{0}$ subsets with zero elements, $\binom{n}{1}$ with one element, $\binom{n}{2}$ with two elements, ..., and $\binom{n}{n}$ with $n$ elements. Therefore the total is
\[ \sum_{k=0}^{n} \binom{n}{k}. \]

Since, we know that a set with $n$ elements has $2^n$ subsets, we conclude:
\[ \sum_{k=0}^{n} \binom{n}{k} = 2^n. \]
Generalized Permutations and Combinations

Section 6.5
Section Summary

- Permutations with Repetition
- Combinations with Repetition
- Permutations with Indistinguishable Objects
- Distributing Objects into Boxes
Permutations with Repetition

**Theorem 1:** The number of $r$-permutations of a set of $n$ objects with repetition allowed is $n^r$.

**Proof:** There are $n$ ways to select an element of the set for each of the $r$ positions in the $r$-permutation when repetition is allowed. Hence, by the product rule there are $n^r$ $r$-permutations with repetition.

**Example:** How many strings of length $r$ can be formed from the uppercase letters of the English alphabet?

**Solution:** The number of such strings is $26^r$, which is the number of $r$-permutations of a set with 26 elements.
Combinations with Repetition

**Example:** How many ways are there to select five bills from a box containing at least five of each of the following denominations: $1, $2, $5, $10, $20, $50, and $100?

**Solution:** Place the selected bills in the appropriate position of a cash box illustrated below:

![Cash Box Illustration](continued →)
Combinations with Repetition

- Some possible ways of placing the five bills:

- The number of ways to select five bills corresponds to the number of ways to arrange six bars and five stars in a row.
- This is the number of unordered selections of 5 objects from a set of 11. Hence, there are

\[ C(11, 5) = \frac{11!}{5!6!} = 462 \]

ways to choose five bills with seven types of bills.
Combinations with Repetition

**Theorem 2:** The number of \( r \)-combinations from a set with \( n \) elements when repetition of elements is allowed is
\[
C(n + r - 1, r) = C(n + r - 1, n - 1).
\]

**Proof:** Each \( r \)-combination of a set with \( n \) elements with repetition allowed can be represented by a list of \( n - 1 \) bars and \( r \) stars. The bars mark the \( n \) cells containing a star for each time the \( i \)th element of the set occurs in the combination.

The number of such lists is \( C(n + r - 1, r) \), because each list is a choice of the \( r \) positions to place the stars, from the total of \( n + r - 1 \) positions to place the stars and the bars. This is also equal to \( C(n + r - 1, n - 1) \), which is the number of ways to place the \( n - 1 \) bars.
Example: How many solutions does the equation
\[ x_1 + x_2 + x_3 = 11 \]
have, where \( x_1 \), \( x_2 \) and \( x_3 \) are nonnegative integers?

Solution: Each solution corresponds to a way to select 11 items from a set with three elements; \( x_1 \) elements of type one, \( x_2 \) of type two, and \( x_3 \) of type three.

By Theorem 2 it follows that there are
\[
C(3 + 11 - 1, 11) = C(13, 11) = C(13, 2) = \frac{13! \cdot 12}{1! \cdot 2!} = 78
\]
solutions.
**Example**: Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?

**Solution**: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. By Theorem 2

\[ C(9, 6) = C(9, 3) = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84 \]

is the number of ways to choose six cookies from the four kinds.
Summarizing the Formulas for Counting Permutations and Combinations with and without Repetition

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<tr>
<th>Type</th>
<th>Repetition Allowed?</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$-permutations</td>
<td>No</td>
<td>$\frac{n!}{(n-r)!}$</td>
</tr>
<tr>
<td>$r$-combinations</td>
<td>No</td>
<td>$\frac{n!}{r!(n-r)!}$</td>
</tr>
<tr>
<td>$r$-permutations</td>
<td>Yes</td>
<td>$n^r$</td>
</tr>
<tr>
<td>$r$-combinations</td>
<td>Yes</td>
<td>$\frac{(n+r-1)!}{r!(n-1)!}$</td>
</tr>
</tbody>
</table>
Permutations with Indistinguishable Objects

**Example:** How many different strings can be made by reordering the letters of the word *SUCCESS*.

**Solution:** There are seven possible positions for the three Ss, two Cs, one U, and one E.

- The three Ss can be placed in $C(7,3)$ different ways, leaving four positions free.
- The two Cs can be placed in $C(4,2)$ different ways, leaving two positions free.
- The U can be placed in $C(2,1)$ different ways, leaving one position free.
- The E can be placed in $C(1,1)$ way.

By the product rule, the number of different strings is:

$$C(7, 3)C(4, 2)C(2, 1)C(1, 1) = \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!} = \frac{7!}{3!2!1!1!} = 420.$$

The reasoning can be generalized to the following theorem.
Permutations with Indistinguishable Objects

**Theorem 3:** The number of different permutations of \( n \) objects, where there are \( n_1 \) indistinguishable objects of type 1, \( n_2 \) indistinguishable objects of type 2, \ldots, and \( n_k \) indistinguishable objects of type \( k \), is:

\[
\frac{n!}{n_1! n_2! \cdots n_k!}.
\]

**Proof:** By the product rule the total number of permutations is:

\[
\binom{n}{n_1} \binom{n-n_1}{n_2} \cdots \binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}
\]

since:

- The \( n_1 \) objects of type one can be placed in the \( n \) positions in \( \binom{n}{n_1} \) ways, leaving \( n - n_1 \) positions.
- Then the \( n_2 \) objects of type two can be placed in the \( n - n_1 \) positions in \( \binom{n-n_1}{n_2} \) ways, leaving \( n - n_1 - n_2 \) positions.
- Continue in this fashion, until \( n_k \) objects of type \( k \) are placed in \( \binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k} \) ways.

The product can be manipulated into the desired result as follows:

\[
\frac{n!}{n_1!(n-n_1)!} \cdot \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \cdots \frac{(n-n_1-\cdots-n_{k-1})!}{n_k!0!} = \frac{n!}{n_1!n_2!\cdots n_k!}.
\]
Many counting problems can be solved by counting the ways objects can be placed in boxes.

- The objects may be either different from each other (distinguishable) or identical (indistinguishable).
- The boxes may be labeled (distinguishable) or unlabeled (indistinguishable).
Distributing Objects into Boxes

- **Distinguishable objects and distinguishable boxes.**
  - There are \( n!/(n_1!n_2! \cdots n_k!) \) ways to distribute \( n \) distinguishable objects into \( k \) distinguishable boxes.
  - (See Exercises 47 and 48 for two different proofs.)
  - Example: There are \( 52!/(5!5!5!5!32!) \) ways to distribute hands of 5 cards each to four players.

- **Indistinguishable objects and distinguishable boxes.**
  - There are \( \binom{n + r - 1}{n - 1} \) ways to place \( r \) indistinguishable objects into \( n \) distinguishable boxes.
  - Proof based on one-to-one correspondence between \( n \)-combinations from a set with \( k \)-elements when repetition is allowed and the ways to place \( n \) indistinguishable objects into \( k \) distinguishable boxes.
  - Example: There are \( \binom{8 + 10 - 1}{10} = \binom{17}{10} = 19,448 \) ways to place 10 indistinguishable objects into 8 distinguishable boxes.
Distributing Objects into Boxes

- **Distinguishable objects and indistinguishable boxes.**
  - Example: There are 14 ways to put four employees into three indistinguishable offices (see Example 10).
  - There is no simple closed formula for the number of ways to distribute \( n \) distinguishable objects into \( j \) indistinguishable boxes.
  - See the text for a formula involving *Stirling numbers of the second kind*.

- **Indistinguishable objects and indistinguishable boxes.**
  - Example: There are 9 ways to pack six copies of the same book into four identical boxes (see Example 11).
  - The number of ways of distributing \( n \) indistinguishable objects into \( k \) indistinguishable boxes equals \( p_k(n) \), the number of ways to write \( n \) as the sum of at most \( k \) positive integers in increasing order.
  - No simple closed formula exists for this number.