Group Gradings on Full Matrix Rings

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Abstract

We study $G$-gradings of the matrix ring $M_n(k)$, $k$ a field, and give a complete description of the gradings where all the elements $e_{i,j}$ are homogeneous, called good gradings. Among these, we determine the ones that are strong gradings or crossed products. If $G$ is a finite cyclic group and $k$ contains a primitive $|G|^{th}$ root of 1, we show how all $G$-gradings of $M_n(k)$ can be produced. In particular we give a precise description of all $C_2$-gradings of $M_2(k)$, and show that for algebraically closed $k$, any such grading is isomorphic to one of the two good gradings.

Introduction and Preliminaries

Let $G$ be a group, $k$ a field, and $V$ a finite dimensional $G$-graded $k$-vector space. If $n = dim(V)$, then $\text{End}(V) \simeq M_n(k)$ has a structure of a $G$-graded $k$-algebra. Conversely, given a matrix ring $M_n(k)$ and a group $G$, it is a challenging problem to describe the algebra $G$-gradings of $M_n(k)$. An interesting type of grading is one for which all the matrices $e_{i,j}$ are homogeneous elements, where $e_{i,j}$ is the matrix with 1 on the $(i,j)$-position, and 0 elsewhere. We call these good gradings, and it turns out that a grading is good if and only if it can be obtained from an $n$-dimensional $G$-graded vector space as above. We give several characterizations for gradings isomorphic to good gradings, in particular we show that any grading of $M_n(k)$ by a torsion-free group is isomorphic to a good grading. Good gradings have appeared before in another setting, in the work of E. L. Green and E. N. Marcos (see
viewing $M_n(k)$ as a quotient of the path algebra of the quiver $\Gamma$, where $\Gamma$ is the complete graph on $n$ points. Then good gradings arise from weight functions on $\Gamma$. Our approach is different, and the only overlap with the results contained in the cited papers is Proposition 2.1. We describe in Section 2 all good gradings on $M_n(k)$, showing that there are $|G|^{n-1}$ such structures. We also determine the good gradings which are strong gradings, respectively crossed products. This produces many examples of strongly graded rings that are not crossed products.

It seems to be much harder to find all $G$-gradings of the $k$-algebra $M_n(k)$. In Section 3 we give a method for doing this in the case where $G = C_m$, a cyclic group of order $m$, and $k$ contains a primitive $m^{th}$ root of 1 (in particular $\text{char}(k)$ does not divide $m$). If $\text{char}(k)$ divides $m$, the method does not work. However, we are able to produce a $C_p$-grading of $M_p(k)$ which is not good, for any field $k$ of characteristic $p > 0$. In Section 4 we give a precise description of all $C_2$-gradings of the algebra $M_2(k)$. The methods are essentially different when $\text{char}(k) = 2$ or $\text{char}(k) \neq 2$. If $\text{char}(k) \neq 2$, we use the results of Section 3. If $\text{char}(k) = 2$, we give a different approach. We characterize all such gradings isomorphic to a good grading. In particular, if $k$ is algebraically closed, any $C_2$-grading of $M_2(k)$ is isomorphic to a good grading, no matter what the characteristic of $k$ is. We mention that in [5] it is presented a general open problem, posed by E. Zelmanov, asking to describe all semigroup gradings of a full matrix ring.

If $G$ is a multiplicative group with identity element 1, we say that a $k$-algebra $R$ is $G$-graded if $R = \bigoplus_{g \in G} R_g$ is a direct sum of $k$-subspaces such that $R_g R_h \subseteq R_{gh}$ for any $g, h \in G$. In this case $R$ is called strongly graded if $R_g R_h = R_{gh}$ for any $g, h \in G$ (or equivalently $R_g R_{g^{-1}} = R_1$ for any $g \in G$), and $R$ is called a crossed product if every $R_g$ contains an invertible element. A non-zero element $r \in R_g$ is called homogeneous of degree $g$, we write $\text{deg}(r) = g$. If any non-zero homogeneous element is invertible, $R$ is called a graded division ring. A right $R$-module $M$ is a graded module if $M = \bigoplus_{g \in G} M_g$, a direct sum of $k$-subspaces, such that $M_h R_g \subseteq M_{hg}$ for any $g, h \in G$. We can define the category $gr - R$ of all $G$-graded right $R$-modules, where a morphism between two objects $M$ and $N$ in this category is a morphism $f : M \to N$ of $R$-modules, such that $f(M_g) \subseteq N_g$ for any $g \in G$. Similarly we can introduce the category $R - gr$ of $G$-graded left $R$-modules. We refer to [6] for all definitions and basic properties of graded rings and modules.

1 Gradings from endomorphism rings

Let $R$ be a $G$-graded ring, and $V$ a right $G$-graded $R$-module. For any $\sigma \in G$ let

$$\text{END}(V)_\sigma = \{ f \in \text{End}(V) \mid f(V_g) \subseteq V_{\sigma g} \text{ for any } g \in G \}$$

which is an additive subgroup of $\text{End}_R(V)$. Note that $\text{END}(V)_\sigma = \text{Hom}_{R-gr}(V, (\sigma)V)$, where $(\sigma)V$ is the $G$-graded right $R$-module which is just $V$ as an $R$-module, and has the shifted
grading $(\sigma)V_g = V_{\sigma g}$ for any $g \in G$. Then the sum $\sum_{\sigma \in G} \text{END}(V)_\sigma$ is direct, and we denote by $\text{END}_R(V) = \oplus_{\sigma \in G} \text{END}(V)_\sigma$, which is a $G$-graded ring. A similar construction can be performed for left graded modules. If $R = k$ with the trivial $G$-grading, then a right graded $R$-module is just a vector space $V$ with a $G$-grading, i.e. $V = \oplus_{g \in G} V_g$ for some subspaces $(V_g)_{g \in G}$. In this situation we denote by $\text{END}(V) = \text{END}_k(V)$ and $\text{End}(V) = \text{End}_k(V)$.

If $V$ has finite dimension $n$, then $\text{END}(V) = \text{End}(V) \cong M_n(k)$, and this induces a $G$-grading on $M_n(k)$. If $(v_i)_{1 \leq i \leq n}$ is a basis of homogeneous elements of $V$, say $\text{deg}(v_i) = g_i$ for any $1 \leq i \leq n$, let $(E_{i,j})_{1 \leq i,j \leq n}$ be the basis of End(V) defined by $E_{i,j}(v_t) = \delta_{t,j}v_i$ for $1 \leq i, j, t \leq n$. Clearly $\text{deg}(E_{i,j}) = g_ig_j^{-1}$. We have an algebra isomorphism between End(V) and $M_n(k)$ by taking $E_{i,j}$ to $e_{i,j}$ for $i, j$, and in this way $M_n(k)$ is endowed with a good $G$-grading. In fact any good $G$-grading can be produced as above. To see this, we first need the following.

**Lemma 1.1** Let us consider a good $G$-grading on $M_n(k)$. Then $\text{deg}(e_{i,i}) = 1$, $\text{deg}(e_{i,j}) = \text{deg}(e_{i+1,i+1}) \text{deg}(e_{i+1,i+2}) \cdots \text{deg}(e_{j-1,j})$ for $i < j$ and $\text{deg}(e_{i,j}) = \text{deg}(e_{i-1,i})^{-1} \text{deg}(e_{i-2,i-1})^{-1} \cdots \text{deg}(e_{j,j+1})^{-1}$ for $i > j$.

**Proof.** Since $e_{i,i}$ is a homogeneous idempotent, we see that $\text{deg}(e_{i,i}) = 1$. The second relation follows from $e_{i,j} = e_{i+1,i+1}e_{i+1,i+2} \cdots e_{j-1,j}$ for any $i < j$. The third relation follows then from $e_{i,j}e_{j,i} = e_{i,i}$, which implies $\text{deg}(e_{i,j}) = \text{deg}(e_{i,i})^{-1}$.

**Proposition 1.2** Let us consider a good $G$-grading on $M_n(k)$. Then there exists a $G$-graded vector space $V$, such that the isomorphism $\text{End}(V) \cong M_n(k)$ with respect to a homogeneous basis of $V$, is an isomorphism of $G$-graded algebras.

**Proof.** We must find some $g_1, \ldots, g_n \in G$ such that $\text{deg}(e_{i,j}) = g_ig_j^{-1}$ for any $i, j$. Lemma 1.1 shows that it is enough to check this for the pairs $(i, j) \in \{(1, 2), (2, 3), \ldots, (n-1, n)\}$, i.e. $g_ig_j^{-1} = \text{deg}(e_{i,i+1})$ for any $1 \leq i \leq n - 1$. But clearly $g_n = 1$, $g_i = \text{deg}(e_{i,i+1})\text{deg}(e_{i+1,i+2}) \cdots \text{deg}(e_{n-1,n})$ for any $1 \leq i \leq n - 1$, is such a set of group elements.

A $G$-grading of the $k$-algebra $M_n(k)$ is good if all $e_{i,j}$’s are homogeneous elements. However there exists gradings which are not good, but are isomorphic to good gradings, as the following example shows.

**Example 1.3** Let $R = S = M_2(k)$ with the $C_2 = \{1, g\}$-gradings defined by

$$
R_1 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, \quad R_g = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix},
$$

$$
S_1 = \left\{ \begin{pmatrix} a & b-a \\ 0 & b \end{pmatrix} \middle| a, b \in k \right\}, \quad S_g = \left\{ \begin{pmatrix} d & c \\ d & -d \end{pmatrix} \middle| c, d \in k \right\}.
$$
Then the map
\[ f : R \rightarrow S, \quad f \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} a + c & b + d - a - c \\ c & d - c \end{pmatrix}, \]
is an isomorphism of $C_2$-graded algebras. The grading of $S$ is not good, since $e_{1,1}$ is not homogeneous, but $S$ is isomorphic to a graded algebra to $R$, which has a good grading. \( \square \)

We see that in the previous example, although the grading of $S$ is not good, the element $e_{1,2}$ is homogeneous. A corollary of the following result shows that in general a grading of the algebra $M_n(k)$ is isomorphic to a good grading whenever one of the $e_{i,j}$’s is a homogeneous element.

**Theorem 1.4** Let $R$ be the algebra $M_n(k)$ endowed with a $G$-grading such that there exists $V \in R - gr$ which is simple as an $R$-module. Then there exists an isomorphism of graded algebras $R \cong S$, where $S$ is $M_n(k)$ endowed with a certain good grading.

**Proof.** As a simple $M_n(k)$-module, $V$ must have dimension $n$. Let $\Delta = \text{END}_R(V)$, as a $G$-graded algebra with multiplication the inverse map composition, hence $V$ is a $G$-graded right $\Delta$-module, so we may consider $\text{BIEND}_R(V) = \text{END}_\Delta(V)$, a $G$-graded algebra with map composition as multiplication. Since $V$ is a simple $R$-module and $\Delta = \text{END}_R(V) = \text{End}_R(V) \cong k$, so $\Delta$ is isomorphic to $k$ with the trivial grading. This shows that $\text{BIEND}_R(V)$ is the endomorphism algebra of a $G$-graded $k$-vector space of dimension $n$, thus it is isomorphic to $M_n(k)$ with a certain good $G$-grading. On the other hand, the graded version of the Density Theorem [2, Proposition 2.4] shows that the map $\varphi : R \rightarrow \text{BIEND}_R(V)$, $\varphi(r)(v) = rv$ for $r \in R$, $v \in V$, is a surjective morphism of $G$-graded algebras. As $\text{Ann}_R(V) = 0$, we see that $\varphi$ is injective, hence an isomorphism, which ends the proof. \( \square \)

**Corollary 1.5** If $G$ is a torsion-free group, then a $G$-grading of $M_n(k)$ is isomorphic to a good grading.

**Proof.** Let $V$ be a graded simple module (with respect to the given $G$-grading of $M_n(k)$). Then [1, Theorem 3.2] shows that $V$ is a simple $R$-module, and the result follows from Theorem 1.4. \( \square \)

**Corollary 1.6** Let $R$ be the algebra $M_n(k)$ endowed with a $G$-grading such that the element $e_{i,j}$ is homogeneous for some $i, j \in \{1, \ldots, n\}$. Then there exists an isomorphism of graded algebras $R \cong S$, where $S$ is $M_n(k)$ endowed with a certain good grading.

**Proof.** Since $e_{i,j}$ is a homogeneous element, then $V = Re_{i,j}$ is a $G$-graded $R$-submodule of $R$. Clearly $V$ is the set of the matrices with zero entries outside the $j^{th}$ column, so $V$ is a simple $R$-module, and we apply Theorem 1.4. \( \square \)
Example 1.7  There exist gradings isomorphic to good gradings, but where no \( e_{i,j} \) is homogeneous. Let \( R \) be the \( C_2 \)-graded algebra from Example 1.3 and \( S = M_n(k) \) with the grading

\[
S_1 = \left\{ \begin{pmatrix} 2a - b & -2a + 2b \\ a - b & -a + 2b \end{pmatrix} \right\} \quad a, b \in k, \quad S_2 = \left\{ \begin{pmatrix} a - 2b & -a + 4b \\ a - b & -a + 2b \end{pmatrix} \right\} \quad a, b \in k.
\]

Then the map

\[
f : R \to S, \quad f \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} 2a + c - 2b - d & -2a - c + 4b + 2d \\ a + c - b - d & -a - c + 2b + 2d \end{pmatrix}.
\]

is an isomorphism of \( C_2 \)-graded algebras. However, none of the elements \( e_{i,j} \) is homogeneous in \( S \). \( \square \)

2  Good gradings of \( M_n(k) \)

In this section we will study the good \( G \)-gradings of the algebra \( M_n(k) \). The algebra \( M_n(k) \) endowed with a good \( G \)-grading will be freely regarded (up to an isomorphism) as a graded algebra \( \text{END} (V) = \text{End} (V) \) for a \( G \)-graded vector space \( V \), as described in Section 1. We start by counting these gradings.

Proposition 2.1  There is a bijective correspondence between the set of all good \( G \)-gradings on \( M_n(k) \), and the set of all maps \( f : \{1, 2, \ldots, n-1\} \to G \), such that to a good \( G \)-grading we associate the map defined by \( f(i) = \deg(e_{i,i+1}) \) for any \( 1 \leq i \leq n-1 \).

Proof. Lemma 1.1 shows that in order to define a good \( G \)-grading on \( M_n(k) \), it is enough to assign some degrees to the elements \( e_{1,2}, e_{2,3}, \ldots, e_{n-1,n} \). The inverse of the correspondence mentioned in the statement takes a map \( f : \{1, 2, \ldots, n-1\} \to G \) to the \( G \)-grading of \( M_n(k) \) such that

\[
\deg(e_{i,i}) = 1,
\]

\[
\deg(e_{i,j}) = f(i)f(i+1) \cdots f(j-1), \quad \text{and}
\]

\[
\deg(e_{j,i}) = f(j-1)^{-1}f(j-2)^{-1} \cdots f(i)^{-1}
\]

for any \( 1 \leq i < j \leq n \). \( \square \)

Corollary 2.2  There exist \( |G|^{n-1} \) good gradings on \( M_n(k) \).

We will describe now the good \( G \)-gradings making \( M_n(k) \) a strongly graded (respectively a crossed-product) algebra.
Proposition 2.3 Let us consider the algebra \( \text{End}(V) \cong M_n(k) \) with a good \( G \)-grading such that \( \deg(e_{i,i+1}) = h_i \) for \( 1 \leq i \leq n-1 \), where \( V = \bigoplus_{g \in G} V_g \) is a graded vector space. The following assertions are equivalent:

\begin{itemize}
  \item[i)] \( M_n(k) \) is a strongly graded algebra
  \item[ii)] \( V_g \neq 0 \) for any \( g \in G \)
  \item[iii)] All the elements of \( G \) appear in the sequence \( h_1, h_1h_2, \ldots, h_1h_2 \cdots h_{n-1} \).
\end{itemize}

**Proof.** We recall that \( V \) is an object of \( gr-k \), where \( k \) is regarded as a \( G \)-graded algebra with the trivial grading. Then \( \text{END}(V) = \text{End}(V) \) is strongly graded if and only if \( V \) weakly divides \( (\sigma)\ V \) for any \( \sigma \in G \) (by [6]). This means that \( V \) is isomorphic to a graded submodule of a finite direct sum of copies of \( (\sigma)\ V \) in \( gr-k \), and it is clearly equivalent to \( V_g \neq 0 \) for any \( g \in G \). If \( g_1, \ldots, g_n \) are the degrees of the elements of the basis of \( V \) which induces the isomorphism \( \text{End}(V) \cong M_n(k) \), then

\[
\begin{align*}
  h_1 &= g_1g_2^{-1}, h_2 = g_2g_3^{-1}, \ldots, h_{n-1} = g_{n-1}g_n^{-1}, \text{ thus } \\
  g_2 &= h_1^{-1}g_1, g_3 = h_2^{-1}h_1^{-1}g_1, \ldots, g_n = h_{n-1}^{-1}h_{n-2}^{-1} \cdots h_1^{-1}g_1.
\end{align*}
\]

Then \( V_g \neq 0 \) for any \( g \in G \) if and only if all the elements of \( G \) appear in the sequence \( g_1, g_2, \ldots, g_n \). Since

\[
g_2 = h_1^{-1}g_1, g_3 = h_2^{-1}h_1^{-1}g_1, \ldots, g_n = h_{n-1}^{-1}h_{n-2}^{-1} \cdots h_1^{-1}g_1
\]

this is equivalent to iii) in the statement. \( \square \)

Corollary 2.4 If \( M_n(k) \) has a good \( G \)-grading making it a strongly graded algebra, then \( |G| \leq n \).

Corollary 2.5 Let \( |G| = m \leq n \). Then the number of good gradings on \( M_n(k) \) making it a strongly graded algebra is

\[
m^{n-1}+(m-1)^{n-1} - \sum_{i=1, m-1} (-1)^{i+1} \binom{m}{i} (m-i)^{n-1} - \sum_{i=1, m-2} (-1)^{i+1} \binom{m-1}{i} (m-i-1)^{n-1}.
\]

**Proof.** Let \( x_1 = h_1, x_2 = h_1h_2, \ldots, x_{n-1} = h_1h_2 \cdots h_{n-1} \). Clearly the \( (n-1) \)-tuples \( (h_1, h_2, \ldots, h_{n-1}) \) and \( (x_1, x_2, \ldots, x_{n-1}) \) uniquely determine each other, so we have to count the number of maps \( f : \{1, 2, \ldots, n-1\} \to G \) such that \( G - \{1\} \subseteq \text{Im}(f) \). This is \( N_1 + N_2 \), where \( N_1 \) (respectively \( N_2 \)) is the number of surjective maps \( f : \{1, 2, \ldots, n-1\} \to G - \{1\} \) (respectively \( f : \{1, 2, \ldots, n-1\} \to G \)). A classical combinatorial fact shows that

\[
N_1 = m^{n-1} - \sum_{i=1, m-1} (-1)^{i+1} \binom{m}{i}, \text{ and } \\
N_2 = (m-1)^{n-1} - \sum_{i=1, m-2} (-1)^{i+1} \binom{m-1}{i} (m-i-1)^{n-1},
\]

which ends the proof. \( \square \)
Proposition 2.6 Let $\text{End} (V) \cong M_n(k)$ with a good $G$-grading, where $V = \bigoplus_{g \in G} V_g$ is a graded vector space. The following assertions are equivalent:

1) $M_n(k)$ is a crossed product
2) $\dim(V) = |G| \cdot \dim(V_g)$ for any $g \in G$
3) $\dim(M_n(k)_1) \cdot |G| = n^2$.

Proof. $\text{END} (V) = \text{End} (V)$ is a crossed product if and only if $\text{End} (V)_\sigma = \text{Hom}_{k-\text{gr}}(V, (\sigma)V)$ contains an invertible element for any $\sigma \in G$, which means that $V \cong (\sigma)V$ as $k$-graded modules. This is equivalent to $\dim(V_g) = \dim(V_{\sigma g})$ for all $\sigma, g \in G$, which is just ii). Thus i) $\Leftrightarrow$ ii).

Clearly i)$\Rightarrow$iii). Suppose now that iii) holds. As $\text{End} (V)_1 = \text{End}_{k-\text{gr}}(V) = \bigoplus_{g \in G} \text{End} (V_g)$, we find $\dim(\text{End} (V)_1) = \sum_{g \in G} (\dim(V_g))^2$. Then

$$|G| \sum_{g \in G} (\dim(V_g))^2 = n^2 = \left( \sum_{g \in G} \dim(V_g) \right)^2,$$

and the Cauchy-Schwarz inequality shows that all $\dim(V_g)$, $g \in G$ must be equal. Thus $\dim(V) = |G| \cdot \dim(V_g)$ for any $g \in G$. $\square$

Corollary 2.7 Any two crossed product structures on $M_n(k)$ which are good $G$-gradings, are isomorphic as graded algebras.

Proof. The two graded algebras are isomorphic to $\text{End} (V)$, respectively $\text{End} (W)$, where $\dim(V_g) = \dim(W_g) = \frac{n}{|g|}$ for any $g \in G$. Therefore $V \cong W$ as $k$-graded modules, and this shows that $\text{End} (V) \cong \text{End} (W)$ as graded algebras. $\square$

As examples, we give a description of all good $C_2$-gradings on $M_2(k)$ and all good $C_2$-gradings on $M_3(k)$.

Example 2.8 Let $R = M_2(k)$, $k$ an arbitrary field. Then a good $C_2$-grading of $R$ is of one of the following two types:

1. The trivial grading, $R_1 = M_2(k), R_g = 0$;
2. $R_1 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, $R_g = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$. $\square$

Example 2.9 Let $R = M_3(k)$, $k$ an arbitrary field. Then a good $C_2$-grading of $R$ is of one of the following types:

1. The trivial grading, $R_1 = M_2(k), R_g = 0$;
(ii) \( R_1 = \begin{pmatrix} k & k & 0 \\ k & k & 0 \\ 0 & 0 & k \end{pmatrix}, \quad R_g = \begin{pmatrix} 0 & 0 & k \\ 0 & 0 & k \\ k & k & 0 \end{pmatrix} \);

(iii) \( R_1 = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & k & k \end{pmatrix}, \quad R_g = \begin{pmatrix} 0 & k & k \\ k & 0 & 0 \\ k & 0 & 0 \end{pmatrix} \);

(iv) \( R_1 = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ k & 0 & k \end{pmatrix}, \quad R_g = \begin{pmatrix} 0 & k & 0 \\ k & 0 & k \\ 0 & k & 0 \end{pmatrix} \).

Using Proposition 2.3 and Proposition 2.6, we see that the examples (ii), (iii) and (iv) are strongly graded rings, but they are not crossed products. Example (ii), due to E. Dade, was known as a strongly graded ring that is not a crossed product (see [7]). The same example also appears in [8, page 131]. □

3 Gradings over cyclic groups

Let \( m \) be a positive integer and \( C_m = \langle g \rangle \) the cyclic group of order \( m \). We assume that a primitive \( m^{th} \) root of unity \( \xi \) exists in \( k \) (in particular this implies that the characteristic of \( k \) does not divide \( m \)).

**Theorem 3.1** Let \( R = M_n(k) \), where \( k \) is a field containing a primitive \( m^{th} \) root of unity \( \xi \). Then a \( C_m \)-grading of \( R \) is of the form

\[
R_{g^i} = \{ A + \xi^{-i} XAX^{-1} + \xi^{-2i} X^2AX^{-2} + \cdots + \xi^{-(m-1)i} X^{m-1}AX^{-m+1} \mid A \in M_n(k) \}
\]

where \( X \in GL_n(k) \) is such that \( X^m \in kI_n \).

**Proof:** Let \( R = \oplus_{g^i} R_{g^i} \) be a \( C_m \)-grading of \( R = M_n(k) \). Define the map \( \Psi : R \to R \) by \( \Psi(A) = \sum_{g^i} \xi^i A_{g^i} \) for any \( A \in R \). We obviously have that \( \Psi \) is a linear map. Moreover, for any \( A, B \in R \) we have that

\[
\Psi(A)\Psi(B) = (\sum_{i \in \mathbb{Z}_m} \xi^i A_{g^i})(\sum_{j \in \mathbb{Z}_m} \xi^j B_{g^j}) = \sum_{i, j \in \mathbb{Z}_m} \xi^{i+j} A_{g^i}B_{g^j} = \sum_{s \in \mathbb{Z}_m} \sum_{i+j=s} \xi^s A_{g^i}B_{g^j} = \sum_{s \in \mathbb{Z}_m} \xi^s (AB)_{g^s} = \Psi(AB)
\]
showing that $\Psi$ is an algebra morphism. Moreover, for any $j$ we have that $\Psi^j(A) = \sum_{i \in \mathbb{Z}_m} \xi^i A_{g^i}$ for any $A \in R$. In particular $\Psi^m = Id$, and $\Psi$ is an algebra automorphism of $R$. By the Skolem-Noether Theorem, an algebra automorphism $\Psi$ of $M_n(k)$ is of the form $\Psi(A) = XAX^{-1}$ for any $A \in M_n(k)$, where $X \in GL_n(k)$. In order to have $\Psi^m = Id$, we must require the condition $X^m \in Z(M_n(k)) = kI_n$, $I_n$ the identity matrix.

It is possible to recover the grading from the automorphism $\Psi$. Indeed, let $j \in \mathbb{Z}_m$, and $A \in R$. Multiply the equations

$$A = \sum_{i \in \mathbb{Z}_m} A_{g^i}, \quad \Psi(A) = \sum_{i \in \mathbb{Z}_m} \xi^i A_{g^i}, \quad \ldots, \quad \Psi^{m-1}(A) = \sum_{i \in \mathbb{Z}_m} \xi^{(m-1)i} A_{g^i}$$

by $1, \xi^{-j}, \xi^{-2j}, \ldots, \xi^{-(m-1)j}$ respectively, and then add the obtained equations. We find that

$$A + \xi^{-j} \Psi(A) + \ldots + \xi^{-(m-1)j} \Psi^{m-1}(A) = mA_{g^j}$$

therefore

$$A_{g^j} = \frac{1}{m} (A + \xi^{-j} \Psi(A) + \ldots + \xi^{-(m-1)j} \Psi^{m-1}(A))$$

$$= \frac{1}{m} (A + \xi^{-i} XAX^{-1} + \xi^{-2i} X^2 AX^{-2} + \ldots + \xi^{-(m-1)i} X^{m-1} AX^{-m+1})$$

which ends the proof. □

**Remark 3.2** The proof of Theorem 3.1 shows that for any $A \in M_n(k)$, the homogeneous components of $A$ in the grading defined by the matrix $X$ as in the statement are

$$A_{g^i} = \frac{1}{m} (A + \xi^{-i} XAX^{-1} + \xi^{-2i} X^2 AX^{-2} + \ldots + \xi^{-(m-1)i} X^{m-1} AX^{-m+1})$$

for any $0 \leq i \leq m - 1$.

If the characteristic of $k$ divides $m$, we can not proceed in the same way for describing $C_m$-gradings of $M_n(k)$ since $m$ is not invertible in $R$. Nevertheless, we are able to produce an example of a $C_p$-grading of $M_p(\mathbb{Z}_p)$ which is not a good grading.

**Proposition 3.3** Let $p \geq 3$ be a prime number, $R = M_p(\mathbb{Z}_p)$ and $a, b \in R$, $a = (a_{i,j})_{1 \leq i, j \leq p}$, $b = (b_{i,j})_{1 \leq i, j \leq p}$, where $a_{i,j} = \delta_{i+1,j} + \delta_{i,j}(-\delta_{j,1} + \delta_{j,2})$, forall $1 \leq i, j \leq p$.

$$b_{i,j} = \left( \begin{array}{c} i - 1 \\ j - 1 \end{array} \right), \quad \text{forall } 1 \leq i, j \leq p$$

($\delta_{i,j}$ denotes Kronecker’s delta). Then $K = \mathbb{Z}_p[a]$ is a field, the sum $\sum_{i=0}^{p-1} Kb^i$ is direct and

$$R = K \oplus Kb \oplus \ldots \oplus Kb^{p-1}$$

is a $C_p$-graded division ring structure on $R$. In particular, this grading is not isomorphic to a good grading.
Proof. Let $P(X) = X^p - X + 1 \in \mathbb{Z}_p[X]$. It is well known that $P$ is irreducible and it is the minimal polynomial of $a$, which is written in the Jordan canonical form. So, $K = \mathbb{Z}_p[a]$ is a field and it has $p^p$ elements. The matrix $b$ has zero entries above the diagonal and $b_{i,i} = 1$, for $i = 1, p$, thus the minimal polynomial of $b$ is $(X - 1)^p = X^p - 1$, and $b$ is invertible. An easy computation shows that

$$ab = b(a + I_p),$$

so $Kb = bK$ and $b \notin K$. Now everything follows if we show that the sum

$$K + Kb + \cdots + Kb^{p-1}$$

is direct. We prove by induction that for any $0 \leq j \leq p - 1$, $(a_i)_{i=0,j} \subseteq K$ such that $\sum_{0 \leq i \leq j} a_i(2b)^i = 0$, we have $a_i = 0$, for every $0 \leq i \leq j$.

If $j = 0$, there is nothing to prove.

If $j = 1$ and

$$a_0 + a_1(2b) = 0$$

then $a_1(2b) = -a_0$. If $a_1 \neq 0$, then $a_1$ is invertible and $b = -2^{-1}a_1^{-1}a_0$. That means that $b \in K$ a contradiction. So $a_1 = 0$ implying $a_0 = 0$.

If $1 < j < p - 1$ and

$$\sum_{i=0}^{j+1} a_i(2b)^i = 0,$$ (1)

then multiplying this relation by $2b, (2b)^2, \ldots, (2b)^{p-1}$ and adding them, we obtain

$$\left(\sum_{i=0}^{j+1} a_i\right) \left( I_p + (2b) + \cdots + (2b)^{p-1} \right) = 0.$$

Because $1 + 2b + \cdots + (2b)^{p-1} \neq 0$ and $\sum_{i=0}^{j+1} a_i \in K$ we obtain

$$\sum_{i=0}^{j+1} a_i = 0.$$ (2)

Subtracting (2) from (1) we get

$$\sum_{i=0}^{j+1} a_i((2b)^i - I_p) = 0$$

and since $2b - 1$ is invertible

$$\sum_{i=0}^{j+1} a_i((2b)^{i-1} + \cdots + I_p) = 0,$$

which means that

$$\sum_{i=1}^{j+1} a_i + \left(\sum_{i=2}^{j+1} a_i\right) (2b) + \cdots + a_{j+1}(2b)^j = 0$$

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and \( \alpha_{j+1} = 0 \) by the induction hypothesis \( \alpha_{j+1} = 0 \). Now (1) shows that \( \alpha_i = 0 \) for every \( i = 0, j + 1 \).

Finally we remark that a good \( C_p \)-grading on \( M_p(k) \) can not be a graded division ring, since the elements \( e_{i,j} \) are homogeneous, but not invertible. This shows that our grading is not isomorphic to a good grading. \( \Box \)

**Remark 3.4** We can also produce a \( C_2 \)-graded division ring structure on \( M_2(\mathbb{Z}_2) \). Indeed, let \( R = M_2(\mathbb{Z}_2) \), and the matrices \( A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \) and \( B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). Then \( R_1 = \{0, I_2, A, A^2\} \) and \( R_g = \{0, B, AB, BA\} \) define a \( C_2 \)-grading on \( R \). Moreover, this is clearly a graded division ring structure.

If \( k \) is a field of characteristic \( p > 0 \), since \( M_p(k) \cong k \otimes_{\mathbb{Z}_p} M_p(\mathbb{Z}_p) \), the \( C_p \)-grading of \( M_p(\mathbb{Z}_p) \) given in Proposition 3.3 and Remark 3.4 extends to a \( C_p \)-grading of \( M_p(k) \), which is not a good grading.

**Remark 3.5** We can obtain other graded division ring structures on matrix rings in the following way. Let \( R = M_n(k) \), and \( S \in R - \text{mod the (unique) type of simple } R\text{-module. Assume that } R \text{ has a } G\text{-grading such that } S \text{ is not gradable. Then let } \Sigma \text{ be a graded simple } R\text{-module. Since } R \text{ is simple artinian and } \Sigma \text{ is finite dimensional, we have } \Sigma \cong S^t \text{ as } R\text{-modules, for some integer } t. \text{ Since } S \text{ is not gradable, we have } t > 1. \text{ Then } M_t(k) \cong \text{End}_R(S^t) \cong \text{End}_R(\Sigma) = \text{END}_R(\Sigma). \text{ Since } \Sigma \text{ is a simple object in the category } R - \text{gr}, \text{ then so is } \Sigma(\sigma) \text{ for any } \sigma \in G. \text{ Thus any element of } \text{END}_R(\Sigma)_\sigma \text{ is either zero or invertible, showing that } \text{END}_R(\Sigma) \text{ is a graded division ring. This transfers to a graded division ring structure on } M_t(k).}

### 4 \( C_2 \)-gradings of \( M_2(k) \)

The purpose of this section is to describe \( C_2 \)-gradings of \( M_2(k) \). We start with the situation where \( \text{char}(k) \neq 2 \), when we use the method developed in Section 3. We are able to give a very precise description in this case.

**Theorem 4.1** Let \( k \) be a field with \( \text{char}(k) \neq 2 \), and \( R = M_2(k) \). Then a \( C_2 \)-grading of the \( k \)-algebra \( R \) is of one of the following three types:

(i) \( R_1 = \left\{ \begin{pmatrix} au + v & bu \\ cu & -au + v \end{pmatrix} \middle| u, v \in k \right\}, \quad R_g = \left\{ \begin{pmatrix} -\frac{c}{2a} \delta - \frac{b}{2a} \gamma & \delta \\ \gamma & \frac{a}{2a} \delta + \frac{b}{2a} \gamma \end{pmatrix} \right\} \quad \gamma, \delta \in k \right\} \),
where \( a, b, c \in k \), \( a \neq 0 \) and \( a^2 + bc \neq 0 \);

(ii) 
\[
R_1 = \left\{ \begin{pmatrix} v & bu \\ cu & v \end{pmatrix} \bigg| u, v \in k \right\}, \quad R_g = \left\{ \begin{pmatrix} \gamma & b\delta \\ -c\delta & -\gamma \end{pmatrix} \bigg| \gamma, \delta \in k \right\},
\]
where \( b, c \in k \setminus \{0\} \);

(iii) The trivial grading, \( R_1 = M_2(k), R_g = 0 \).

**Proof.** We start by finding all matrices \( X \in M_2(k), X \neq 0 \) such that \( X^2 \in kI_2 \). Let
\[
X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]
Then \( X^2 = \alpha I_2 \) for some \( \alpha \in k \) if and only if
\[
a^2 + bc = \alpha, \quad d^2 + bc = \alpha, \quad b(a + d) = 0, \quad c(a + d) = 0.
\]
(3)

If \( a + d \neq 0 \), then \( b = c = 0 \), and \( a = d \). If \( a + d = 0 \), then \( d = -a \) and \( a^2 + bc \neq 0 \). Thus there are two types of matrix solutions: \( X = \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} \), with \( \beta \in k \setminus \{0\} \), and \( X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \), with \( a, b, c \in k \), \( a^2 + bc \neq 0 \). For the first type, we obtain the trivial grading, since
\[
R_1 = \{ A + XAX^{-1} | A \in M_2(k) \} = \{ 2A | A \in M_2(k) \} = M_2(k).
\]

Let now \( X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \), with \( a^2 + bc \neq 0 \). If \( A = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in M_2(k) \), then the homogeneous components of \( A \) in the \( C_2 \)-grading associated to \( X \) are
\[
A_1 = \frac{1}{2} (A + XAX^{-1})
\]
\[
= \frac{1}{2(a^2 + bc)} \begin{pmatrix} 2a^2 + bc & x + cy + abz + bct & abx + bcy + b^2z - abt \\ axc + c^2y + bcz - act & bcx - acy - abz + (2a^2 + bc)t \end{pmatrix}
\]

\[
A_g = \frac{1}{2} (A - XAX^{-1})
\]
\[
= \frac{1}{2(a^2 + bc)} \begin{pmatrix} bxc - acy - abz - bct & -abx + (2a^2 + bc)y - b^2z + abt \\ -acz - c^2y + (2a^2 + bc)z + act & -bcx + acy + abz + bct \end{pmatrix}.
\]

We distinguish two possibilities. If \( a \neq 0 \), denote
\[
u = (ax + cy + bz - at)/2(a^2 + bc), \quad v = (x + t)/2.
\]
Then \( A_1 = \begin{pmatrix} au + v & bu \\ cu & -au + v \end{pmatrix} \), and \( u, v \) can take any values in \( k \), since the matrix
\[
\begin{pmatrix} a & c & b & -a \\ 1 & 0 & 0 & 1 \end{pmatrix}
\]
has rank 2. Thus
\[
R_1 = \left\{ \begin{pmatrix} au + v & bu \\ cu & -au + v \end{pmatrix} \bigg| u, v \in k \right\}.
\]
On the other hand, denoting
\[
\gamma = (-acx - c^2y + (2a^2 + bc)z + act)/2(a^2 + bc), \quad \delta = (-abx + (2a^2 + bc)y - b^2z + abt)/2(a^2 + bc)
\]
we have
\[
A_g = \left( \begin{array}{ccc} -\frac{\delta}{2a} & -\frac{b}{2a} & \delta \\ \gamma & \frac{\delta}{2a} & + \frac{b}{2a} \gamma \end{array} \right)
\]
and the matrix
\[
\left( \begin{array}{ccc} -ac & -c^2 & (2a^2 + bc) \\ -ab & (2a^2 + bc) & ac \\ -b^2 & ab & \end{array} \right)
\]
has rank 2, so
\[
R_g = \left\{ \left( \begin{array}{ccc} -\frac{\delta}{2a} & -\frac{b}{2a} & \delta \\ \gamma & \frac{\delta}{2a} & + \frac{b}{2a} \gamma \end{array} \right) \mid \gamma, \delta \in k \right\}
\]
showing that the grading is of type (i).
If \( a = 0 \), then \( bc \neq 0 \). In this case \( A_1 = \left( \begin{array}{ccc} v & bu \\ cu & v \end{array} \right) \), where \( u = (cy + bz)/2bc \) and \( v = (x + t)/2 \) run through the elements of \( k \). Then denoting \( \gamma = (x - t)/2, \delta = (cy - bz)/2bc \), we see that
\[
A_g = \left\{ \left( \begin{array}{ccc} \gamma & b\delta \\ -c\delta & -\gamma \end{array} \right) \mid \gamma, \delta \in k \right\},
\]
and this gives a grading of type (ii). \( \Box \)

**Corollary 4.2** A \( C_2 \)-algebra grading of \( M_2(k) \), \( \text{char}(k) \neq 2 \), different from the trivial grading, is a crossed product.

**Proof:** It is enough to show that for any grading of type (i) or (ii) \( R_g \) contains an invertible element. But this clearly follows from the fact that \( m\delta^2 + n\delta \gamma + p\gamma^2 = 0 \) for any \( \gamma, \delta \in k \) if and only if \( m = n = p = 0 \). \( \Box \)

In the next two propositions we describe which of the \( C_2 \)-gradings of \( M_2(k) \) is isomorphic to a good grading.

**Proposition 4.3** Let \( b, c \in k - \{0\} \). Then the grading \( R_1 = \left\{ \left( \begin{array}{ccc} v & bu \\ cu & v \end{array} \right) \mid u, v \in k \right\} \),
\[
R_g = \left\{ \left( \begin{array}{ccc} \gamma & b\delta \\ -c\delta & -\gamma \end{array} \right) \mid \gamma, \delta \in k \right\}
\]
of \( M_2(k) \) is isomorphic to a good grading if and only if \( bc \) is a square in \( k \).
Proof. Let \( S = M_2(k) \) with the trivial \( C_2 \)-grading \( S_1 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, S_2 = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} \). If \( f : S \to R \) is an isomorphism of graded algebras, then there exists \( Y \in GL_2(k) \) with \( f(A) = YAY^{-1} \) for any \( A \in S \). Let \( Y = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \). Then for \( A = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \in S_y \) we have

\[
 f(A) = YAY^{-1} = \frac{1}{ps - qr} \begin{pmatrix} qs y - pr x & -q^2 y + p^2 x \\ s^2 y - r^2 x & -qs y + pr x \end{pmatrix} \in R_y
\]

and this shows that \( b(s^2 y - r^2 x) + c(-q^2 y + p^2 x) = 0 \) for any \( x, y \in k \). Thus \( br^2 = cp^2 \) and \( bs^2 = cq^2 \). We obtain that \( bc = \left( \frac{cp}{r} \right)^2 \), a square in \( k \).

Conversely, suppose that \( bc = d^2 \) for some \( d \in k \). Let

\[
 Y = \begin{pmatrix} 1 & \frac{d}{b} \\ -\frac{d}{b} & \frac{2}{b} \end{pmatrix} \in GL_2(k).
\]

Then for \( A = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in R_1 \) we have

\[
 YAY^{-1} = \begin{pmatrix} (x + y)/2 & d(y - x)/2c \\ d(y - x)/2b & (x + y)/2 \end{pmatrix}
\]

and for \( A = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \in R_y \) we have

\[
 YAY^{-1} = \begin{pmatrix} (by + 4cx)/4d & (-by + 4cx)/4c \\ (by - 4cx)/4b & -(by + 4cx)/4d \end{pmatrix}.
\]

These show that the map \( f : S \to R, f(A) = YAY^{-1} \), is an isomorphism of graded algebras. \( \square \)

Proposition 4.4 Let \( a, b, c \in k \) such that \( a \neq 0, a^2 + bc \neq 0 \). The the grading \( R_1 = \left\{ \begin{pmatrix} au + v & bu \\ cu & -au + v \end{pmatrix} \mid u, v \in k \right\} , R_y = \left\{ \begin{pmatrix} -\frac{c}{2a} & -\frac{b}{2a} \gamma \\ \gamma & \frac{c}{2a} \delta + \frac{b}{2a} \gamma \end{pmatrix} \mid \gamma, \delta \in k \right\} \) is isomorphic to a good grading if and only if \( a^2 + bc \) is a square in \( k \).

Proof. Keeping the notation from the proof of Proposition 4.3, suppose that \( f : S \to R, f(A) = YAY^{-1} \) is an isomorphism of graded algebras. Since \( f \left( \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \right) \in R_y \), we find that

\[
 qsy - pr x = -\frac{c}{2a}(-q^2 y + p^2 x) - \frac{b}{2a}(s^2 y - r^2 x) = \left( \frac{c}{2a}q^2 - \frac{b}{2a}s^2 \right)y - \left( \frac{c}{2a}p^2 - \frac{b}{2a}r^2 \right)x
\]
for any $x, y \in k$. In particular
\[
\frac{c}{2a}q^2 - \frac{b}{2a}s^2 = qs \quad \text{or} \quad c\left(\frac{q}{s}\right)^2 - 2a\left(\frac{q}{s}\right) - b = 0.
\]

If $bc = 0$, then clearly $a^2 + bc$ is a square in $k$. If $bc \neq 0$, then in order to have roots in $k$ for the equation $ct^2 - 2at - b = 0$, we need $a^2 + bc$ to be a square.

Conversely, suppose that $a^2 + bc$ is a square in $k$. We first consider the case where $bc \neq 0$. Let $t_1, t_2$ be the (distinct) roots of the equation $ct^2 - 2at - b = 0$, and $X = \begin{pmatrix} t_2 & t_1 \\ 1 & 1 \end{pmatrix}$. If $f : R \to S, f(A) = XAX^{-1}$, is the algebra isomorphism induced by $X$, then
\[
f \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} = \frac{1}{t_1 - t_2} \begin{pmatrix} t_1 y - t_2 x & -t_1^2 y + t_2^2 x \\ y - x & -t_1 y + t_2 x \end{pmatrix} \in S_g
\]
and
\[
f \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \frac{1}{t_1 - t_2} \begin{pmatrix} t_2 x - t_1 y & b(x - y) \\ x - y & -t_1 x + t_2 y \end{pmatrix} \in S_1
\]
showing that $f$ is an isomorphism of graded algebras. If $b = c = 0$, then $R = S$ as graded algebras. If $c = 0$ and $b \neq 0$, then $X = \begin{pmatrix} 1 & 1 \\ 0 & -\frac{2a}{b} \end{pmatrix}$ induces in a similar way a graded isomorphism between $R$ and $S$. Similarly for $c \neq 0, b = 0$, and this ends the proof. □

**Corollary 4.5** If $k$ is an algebraically closed field of characteristic not 2, then any $C_2$-grading of the algebra $M_2(k)$ is isomorphic either to $R_1 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, $R_g = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$ or to $R_1 = M_2(k), R_g = 0$.

We turn now to the characteristic 2 case.

**Theorem 4.6** Let $R = M_2(k)$, $k$ a field of characteristic 2. Then a $C_2$-grading of $R$ is of one of the following two types:

(i) The trivial grading, $R_1 = M_2(k), R_g = 0$;

(ii) $R_1 = \left\{ \begin{pmatrix} x & \beta(x + y) \\ \alpha(x + y) & y \end{pmatrix} \middle| x, y \in k \right\}$, $R_g = \left\{ \begin{pmatrix} \alpha x + \beta y & x \\ y & \alpha x + \beta y \end{pmatrix} \middle| x, y \in k \right\}$

for some $\alpha, \beta \in k$.

**Proof.** Let us consider a $C_2$-grading of $R$. Then for any $A, B \in R$ we have
\[
(AB)_1 = A_1B_1 + A_gB_g
= A_1B_1 + (A - A_1)(B - B_1)
= AB + AB_1 + A_1B.
\]

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Let $\varphi : R \to R$, $\varphi(A) = A_1$. A straightforward (but tedious) computation shows that the matrix of $\varphi$ in the basis $e_{11}, e_{12}, e_{21}, e_{22}$ is of the form

$$X = \begin{pmatrix} 1 & \alpha & \beta & 0 \\ \beta & \gamma & 0 & \beta \\ \alpha & 0 & \gamma & \alpha \\ 0 & \alpha & \beta & 1 \end{pmatrix}$$

for some $\alpha, \beta, \gamma \in k$ (to see this we let $A$ and $B$ run through elements of the basis in the previously displayed formula). Since $\varphi^2 = \varphi$, we must have $X^2 = X$, implying that either $\gamma = 1, \alpha = \beta = 0$ or $\gamma = 0$. In the first case $X = I_4$, thus $\varphi = Id$, and we find the trivial grading. Let now

$$X = \begin{pmatrix} 1 & \alpha & \beta & 0 \\ \beta & 0 & 0 & \beta \\ \alpha & 0 & 0 & \alpha \\ 0 & \alpha & \beta & 1 \end{pmatrix}$$

for $\alpha, \beta \in k$. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(k)$, then

$$A_1 = \varphi(A) = \begin{pmatrix} a + b\alpha + c\beta & \beta(a+d) \\ \alpha(a+d) & d + b\alpha + c\beta \end{pmatrix} = \begin{pmatrix} x & \beta(x+y) \\ \alpha(x+y) & y \end{pmatrix}$$

where $x = a + b\alpha + c\beta$, $y = d + b\alpha + c\beta$. Also $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(k)$, then

$$A_g = A - A_1 = \begin{pmatrix} b\alpha + c\beta & b + a\beta + d\beta \\ c + a\alpha + d\alpha & b\alpha + c\beta \end{pmatrix} = \begin{pmatrix} \alpha x + \beta y & x \\ y & \alpha x + \beta y \end{pmatrix}$$

where $x = b + a\beta + d\beta$, $y = c + a\alpha + d\alpha$. □

Theorem 4.6 now allows us obtain the same result for the $\text{char}(k) = 2$ case as for the $\text{char}(k) \neq 2$ case (cf. Corollary 4.2), albeit by completely different methods.

**Corollary 4.7** If $\text{char}(k) = 2$, then any non-trivial $C_2$-grading of $M_2(k)$ is a crossed product.

**Proof:** If $\alpha = \beta = 0$, then clearly $R_g$ contains invertible elements. If at least one of $\alpha$ and $\beta$, say $\alpha$, is non-zero, then $\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$ is an invertible element of $R_g$. □

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Proposition 4.8 Let \( \text{char}(k) = 2 \), and \( R = M_2(k) \) with the grading

\[
R_1 = \left\{ \begin{pmatrix} x & \beta(x + y) \\ \alpha(x + y) & y \end{pmatrix} \right\}, \quad x, y \in k,
R_g = \left\{ \begin{pmatrix} \alpha x + \beta y & x \\ y & \alpha x + \beta y \end{pmatrix} \right\}, \quad x, y \in k.
\]

Then this grading is isomorphic to a good grading if and only if there exists \( t \in k \) such that \( \alpha t^2 + t + \beta = 0 \).

Proof. We proceed as in the proof of Proposition 4.3. If \( X = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \) is an invertible matrix inducing an isomorphism of graded algebras between \( R \) and \( M_2(k) \) with the only non-trivial \( C_2 \)-grading, then

\[
\begin{align*}
\beta(ps + rq) &= pq \\
\alpha(ps + qr) &= rs \\
\alpha p^2 + \beta r^2 &= pr \\
\alpha q^2 + \beta s^2 &= qs.
\end{align*}
\]

If \( \alpha, \beta \neq 0 \), then \( p, q, r, s \neq 0 \) (since \( ps + qr = \text{det}(X) \neq 0 \)). Then \( \alpha \left( \frac{p}{r} \right)^2 + \frac{p}{r} + \beta = 0 \), and we take \( t = \frac{p}{r} \). If \( \alpha = 0 \) or \( \beta = 0 \), then clearly there is \( t \in k \) such that \( \alpha t^2 + t + \beta = 0 \).

Conversely, suppose that \( \alpha t^2 + t + \beta = 0 \) for some \( t \in k \). If \( \alpha \neq 0 \), let \( t_1, t_2 \in k \) be the (distinct) roots of this equation, and then the matrix \( X = \begin{pmatrix} t_1 & t_2 \\ 1 & 1 \end{pmatrix} \) produces the required isomorphism. If \( \alpha = 0 \), we take \( X = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \), which also induces an isomorphism as wanted. \( \square \)

Corollary 4.9 If \( \text{char}(k) = 2 \) and \( k \) is algebraically closed, then any \( C_2 \)-grading of \( M_2(k) \) is isomorphic to a good grading.

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References


