Separable functors for the category of Doi-Hopf modules II

S. Caenepeel  
University of Brussels, VUB  
Faculty of Applied Sciences,  
Pleinlaan 2  
B-1050 Brussels, Belgium

Bogdan Ion*  
Department of Mathematics,  
Princeton University,  
Fine Hall, Washington Road  
Princeton, NJ 08544-1000, USA

G. Militaru†  
University of Bucharest  
Faculty of Mathematics  
Str. Academiei 14  
RO-70109 Bucharest 1, Romania

Shenglin Zhu‡  
Institute of Mathematics  
Fudan University  
Shanghai 200433, China

Abstract

We give necessary and sufficient conditions for an induction functor (and its adjoint) between categories of Doi-Hopf modules to be separable. As a consequence, we find new versions of Maschke’s Theorem, and we recover some existing ones. In particular, we apply our results to the functor forgetting the action, and to Hopf Galois extensions.

0 Introduction

Separable functors have been introduced by Năstăsescu, Van den Bergh and Van Oystaeyen in [22], and it turns out that they provide the categorical explanation for the various versions of Maschke’s Theorem that exist in the literature. Roughly stated, a functor $F$ that is separable in the sense of [22] reflects split exact sequences: if an exact sequence becomes split after we apply the functor $F$, then it is already split itself. The name comes from the fact that a ring extension is separable if and only if the restriction of scalars functor is separable. Thus every Maschke Theorem appearing in the literature can be restated by saying that a certain functor (usually a forgetful functor) is separable.

In [14], Doi introduced a new type of module unifying many existing module structures, such as ordinary modules, graded modules, modules graded by $G$-sets, (relative) Hopf modules, and Yetter-Drinfel’d modules. The idea is to consider modules with an action by an $H$-comodule algebra $A$ and a coaction by an $H$-module coalgebra $C$, with $H$ a bialgebra or a Hopf algebra, and such that a certain compatibility relation holds. $(H, A, C)$ will be called a Doi-Hopf datum. Doi-Hopf data form a category, and, associated to every morphism in this category, we have a pair of adjoint functors $(F, G)$ (cf. [9]). $F$ can be viewed as a generalized induction functor. The aim of this paper is to study when the functors $F$ and $G$ are separable. In [5], we have done this in the particular

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situation where $F$ is the functor forgetting the $C$-coaction. The crucial argument in [5] (and in the present paper) is the fact that a functor $F: \mathcal{C} \to \mathcal{D}$ having a right adjoint $G$ is separable if and only if the unit natural transformation of the adjunction splits. In a dual way, $G$ is separable if and only if the counit splits (see [23], [24]). In the case where $\mathcal{C}$ and $\mathcal{D}$ are categories of Doi-Hopf modules, we have to study natural transformations $GF \to 1_C$ and $1_D \to FG$. We will see that these natural transformations form a $k$-module (even a $k$-algebra), and the splittings of the unit or counit are one-sided units in this algebra. Thus they are idempotents, and one can view them as generalized separability idempotents.

It is known that $C \otimes A$ can be made into a Doi-Hopf module, and it turned out in previous publications (see e.g. [9], [7],[7],...) that $C \otimes A$ often plays a special role. One would expect that $C \otimes A$ generates the category, but this is not the case. In Section 2, we will see that it has a weaker property: $C \otimes A$ is a so-called $T$-generator of the category of Doi-Hopf modules. Another important property of $C \otimes A$ is a two-sided Doi-Hopf module (see Definition 1.1).

Now take an induction functor $F: \mathcal{C} \to \mathcal{D}$ between Doi-Hopf modules categories, and let $G$ be the right adjoint of $F$. Obviously, if $\nu: GF \to 1_C$ is a natural transformation, then $\nu_{C \otimes A}$ is a morphism in $\mathcal{C}$, this means that $\nu_{C \otimes A}$ is left $C$-colinear, and right $A$-linear. But there is more: $\nu_{C \otimes A}$ is a morphism in the category of two-sided Doi-Hopf modules, it is also right $C$-colinear, and left $A$-linear. Moreover, if $C$ is $C'$-coflat, then $\nu$ is completely determined by $\nu_{C \otimes A}$, and we have an isomorphism between the above mentioned $k$-modules consisting of natural transformations $\nu$, and the two-sided linear and colinear maps $\tilde{\nu}: GF(C \otimes A) \to C \otimes A$. If $\tilde{\nu}$ satisfies a certain normalizing condition, then $\nu$ splits $\rho$, and $F$ is separable (see Theorem 3.6). Similar arguments can be applied to the functor $G$, and this will be discussed in Section 4. If we apply our results to the functor forgetting the $C$-coaction, we recover the main result of [5]. In Section 5, we discuss the other forgetful functor, forgetting the $A$-action. The natural transformations are now described by maps $\lambda: C \otimes A \otimes A$ satisfying certain conditions, and, if $C = k$, then $\lambda(1)$ is just a separability idempotent in the classical sense (see [10]). If $A = H$, then the map $\phi = (\varepsilon_H \otimes I_H) \circ \lambda$ is right $H$-linear, and can be viewed as a dual version of Doi’s total integrals. In Section 5.2, we discuss when the existence of such a total integral $\phi$ implies the separability of the forgetful functor. In the situation where $A = H$, we find classical right integrals, and the Larson-Sweedler of Maschke’s Theorem. In Section 6, we apply our results to Hopf Galois extensions.

In [2] and [4], entwining structures have been introduced as generalizations of Doi-Hopf data. Separability between functors of entwined modules has been studied recently by Brzeziński in [3].

1 Preliminary results

1.1 Doi-Hopf modules

Let $k$ be a commutative ring with unit, and $C$ a $k$-coalgebra, with comultiplication $\Delta_C$ and counit $\varepsilon_C$. We will use the Heyneman-Sweedler notation

$$\Delta_C(c) = \sum c_{(1)} \otimes c_{(2)}$$

for $c \in C$. If $\rho^l_M: M \to M \otimes C$ defines a left coaction of $C$ on a $k$-module $M$ (that is, $M$ is a left $C$-comodule), then we write

$$\rho^l_M(m) = \sum m_{(-1)} \otimes m_{(0)}$$

for any $m \in M$. In a similar way, if $N$ is a right $C$-comodule, then we write

$$\rho^r_N(n) = \sum n_{(0)} \otimes n_{(1)}$$
The indices will be omitted if no confusion is possible. \( C^\mathcal{M} \) and \( \mathcal{M}^C \) will denote the categories of left and right \( C \)-comodules. The cotensor product \( N \square C M \) is defined as the coequalizer

\[
0 \longrightarrow N \square C M \longrightarrow N \otimes M \xrightarrow{\rho_N \otimes I_M} N \otimes C \otimes M \xrightarrow{I_N \otimes \rho_M}
\]

Fix a left \( C \)-comodule \( M \). If \( M \) is flat as a \( k \)-module (e.g. if \( k \) is a field), then the functor

\[
\bullet \square C M : C^\mathcal{M} \rightarrow k\text{-mod} : N \mapsto N \square C M
\]

is left exact, but not necessarily exact. If \( \bullet \square C M \) is exact, then we call \( M \) left \( C \)-coflat. In a similar way, we introduce right \( C \)-coflatness.

Now let \( H \) be a Hopf algebra, \( A \) a left \( H \)-comodule algebra, and \( C \) a right \( H \)-module coalgebra. For technical reasons, we will always assume that \( C \) is flat as a \( k \)-module. Following [6], we will call \( \mathcal{H} = (H, A, C) \) a (left-right) Doi-Hopf datum. A morphism

\[
h : \mathcal{H} = (H, A, C) \rightarrow \mathcal{H}' = (H', A', C')
\]

between two Doi-Hopf data consists of a threetuple \( h = (h, \alpha, \gamma) \), where

\[
h : H \rightarrow H', \quad \alpha : A \rightarrow A', \quad \gamma : C \rightarrow C'
\]

are respectively a Hopf algebra map, an algebra map, and a coalgebra map satisfying the compatibility relations

\[
\rho_{A'}(\alpha(a)) = \sum h(a_{(-1)}) \otimes \alpha(a_{(0)}) \quad (1)
\]

\[
\gamma(ch) = \gamma(c)h(h) \quad (2)
\]

for all \( a \in A \), \( c \in C \) and \( h \in H \). \( \mathcal{D}H(k) \) will be the category of Doi-Hopf data over \( k \).

A right-left Doi-Hopf \((H, A, C)\)-module is a \( k \)-module \( M \), that is at once a right \( A \)-module and a left \( C \)-comodule, in such a way that

\[
\rho_M^r(ma) = \sum m_{(-1)}a_{(1)} \otimes m_{(0)}a_{(0)} \quad (3)
\]

for all \( m \in M \) and \( a \in A \). \( \mathcal{C} = C^\mathcal{M}(H)_A \) will denote the category of right-left Doi-Hopf modules and \( A \)-linear, \( C \)-colinear maps. Doi-Hopf modules were first introduced by Doi in [14], and, independently, by Koppenen in [16].

A left-right Doi-Hopf \((H, A, C)\)-module is a \( k \)-module \( M \), which is at once a left \( A \)-module and a right \( C \)-comodule, but now with a different compatibility relation, namely

\[
\rho_M^l(am) = \sum a_{(0)}m_{(0)} \otimes a_{(0)}S(m_{(-1)}) \quad (4)
\]

The category of left-right Doi-Hopf modules will be denoted by \( _A\mathcal{M}(H)^C \).

**Definition 1.1** Consider two Doi-Hopf data \( \mathcal{H} = (H, A, C) \) and \( \mathcal{H}' = (H', A', C') \). We call \( M \) a Doi-Hopf \( \mathcal{H}-\mathcal{H}' \)-bimodule if

1) \( M \in C^\mathcal{M}(H)_A \);
2) \( M \in A^\mathcal{M}(H')^C \);
3) \( M \) is a \((C, C')\)-bicommodule;
4) \( M \) is an \((A', A)\)-bimodule;
5) the right \( A \)-action on \( M \) is right \( C' \)-colinear;
6) the left \( A' \)-action on \( M \) is left \( C \)-colinear.
4) and 5) are equivalent to the following formulas:

\[
\sum (ma)(0) \otimes (ma)(1) = \sum m(0)a \otimes m(1)
\]

(5)

\[
\sum i(a'm)(-1) \otimes (a'm)(0) = \sum m(-1) \otimes a'm(m)
\]

(6)

for all \( m \in M, \ a \in A \) and \( a' \in A' \). The category of Doi-Hopf \( \mathcal{H} \)-\( \mathcal{H}' \)-bimodules and two-sided linear and colinear maps will be denoted by \( \mathcal{C}_A \mathcal{M}(H, H') \). This definition appeared first in [9], but with a different left-right convention.

Let \( \mathcal{H} \rightarrow \mathcal{H}' \) be a morphism in \( \mathcal{D}\mathcal{H}(k) \). We have a functor (the induction functor)

\[
F : \mathcal{C} = \mathcal{C}_A \mathcal{M}(H) \rightarrow \mathcal{C}' = \mathcal{C}'_A \mathcal{M}(H')
\]

defined as follows (see [9] for details):

\[
F(M) = M \otimes_A A'
\]

where \( A' \) is a left \( A \)-module via \( \alpha \). The \( A' \)-action and \( C' \)-coaction on \( F(M) \) are given by

\[
(m \otimes b')a' = m \otimes b'a'
\]

(7)

\[
\rho'(m \otimes b') = \sum \gamma(m_{(-1)}b'_{(-1)} \otimes m(0) \otimes b'(0)
\]

(8)

for all \( a', b' \in A' \) and \( m \in M \). If \( C \) is flat as a \( k \)-module, then we also have a functor

\[
G : \mathcal{C}' \rightarrow \mathcal{C}
\]

given by

\[
G(M') = C \square_{C'} M'
\]

where \( C \) is a right \( C' \)-comodule via \( \gamma \). Now the \( A \)-action and \( C \)-coaction are given by

\[
\left( \sum_i c_i \otimes m'_i \right)a = \sum_i c_i a(-i) \otimes m'_i \alpha(a(0))
\]

(9)

\[
\rho' \left( \sum_i c_i \otimes m'_i \right) = \sum_i c_i(1) \otimes c_i(2) \otimes m'_i
\]

(10)

for all \( a \in A \) and \( \sum_i c_i \otimes \otimes C' \).

**Theorem 1.2** ([9]) Let \( \mathcal{H} \rightarrow \mathcal{H}' \) be a morphism in \( \mathcal{D}\mathcal{H}(k) \), and assume that \( C \) is flat as a \( k \)-module. Then the functor \( G \) is a right adjoint of \( F \).

For further use, we give an explicit description of the unit \( \eta : 1_C \rightarrow GF \) and counit \( \delta : FG \rightarrow 1_{C'} \) of this adjunction. For \( M \in \mathcal{C} \) and \( M' \in \mathcal{C}' \),

\[
\eta_M : M 
\rightarrow C \square_{C'} (M \otimes_A A') \quad \text{and} \quad \delta_{M'} : (C \square_{C'} M') \otimes_A A' \rightarrow M'
\]

are given by the formulas

\[
\eta_M(m) = \sum m_{(-1)} \otimes (m(0) \otimes 1_A)
\]

(11)

\[
\delta_{M'} \left( \sum_i c_i \otimes m'_i \otimes a' \right) = \sum_i \epsilon_C(c_i) m'_i \otimes a'
\]

(12)

¿From [9, Lemma 2.3 and 2.6], we deduce
Proposition 1.3 Let $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{H}_0$ and $\mathcal{H}_1$ be Doi-Hopf data, and let $h : \mathcal{H} \rightarrow \mathcal{H}'$ be a morphism of Doi-Hopf data. If $M$ is a Doi-Hopf $(\mathcal{H}, \mathcal{H}_0)$-bimodule, then $F(M)$ is a Doi-Hopf $(\mathcal{H}', \mathcal{H}_0)$-bimodule. If $M'$ is a Doi-Hopf $(\mathcal{H}', \mathcal{H}_1)$-bimodule, then $F(M)$ is a Doi-Hopf $(\mathcal{H}, \mathcal{H}_1)$-bimodule.

Proof As we already pointed out, a different left-right convention is used in [9]. We give the $\mathcal{H}_0$-structure on $F(M)$, and the $\mathcal{H}_1$-structure on $G(M')$, leaving all the other straightforward verifications to the reader. For $a_0 \in A_0$, $m \in M$ and $a' \in A'$, we put

$$a_0(m \otimes a') = a_0m \otimes a'$$
(13)

$$\rho^f(\delta \otimes a') = \sum m_i(0) \otimes a' \otimes m_i(1)$$
(14)

In a similar way, for $a_1 \in A_1$ and $\sum_i c_i \otimes n_i \in C \otimes M'$, we let

$$a_1(\sum_i c_i \otimes m_i') = c_i \otimes a_1 m_i'$$
(15)

$$\rho^f(\sum_i c_i \otimes m_i') = \sum_i c_i \otimes m_i'(0) \otimes m_i'(1)$$
(16)

Recall from [9] that $C \otimes A$ is a Doi-Hopf bimodule. The structure is given by the following formulas:

$$(c \otimes b)a = c \otimes ba$$
(17)

$$\rho^f(c \otimes b) = \sum c_i(1) b_{(-1)}(0) \otimes c_i(2) \otimes b_{(0)}$$
(18)

$$a(c \otimes b) = \sum cS(a_{(-1)}(0) \otimes a_{(0)}(0) b$$
(19)

$$\rho^f(c \otimes b) = \sum c(1) \otimes b \otimes c(2)$$
(20)

Using Proposition 1.3, we find that $F(C \otimes A)$ is a Doi-Hopf $(\mathcal{H}', \mathcal{H})$-bimodule, and $G(C \otimes A)$ is again a Doi-Hopf $(\mathcal{H}, \mathcal{H})$-bimodule. Using (17), we see that

$$F(C \otimes A) = (C \otimes A) \otimes_A A' \cong C \otimes A'$$
(21)

and

$$G(F(C \otimes A)) \cong C \otimes G(C \otimes A'$$
(22)

The structure maps on $C \otimes A'$ are given by the formulas

$$(c \otimes b')a' = c \otimes b' a'$$
(23)

$$\rho^f(c \otimes b') = \sum \gamma(c_{(1)}(1)) b_{(-1)}(0) \otimes c_{(2)}(1) \otimes b'_{(0)}$$
(24)

$$a(c \otimes b') = \sum cS(a_{(-1)}(0) \otimes a_{(0)} b'$$
(25)

$$\rho^f(c \otimes b') = \sum c_{(1)}(1) \otimes b' \otimes c_{(2)}$$
(26)

On $G(F(C \otimes A)) \cong C \otimes G(C \otimes A')$, we find

$$\left(\sum_i c_i \otimes (d_i \otimes a'_i)\right) a = \sum_i c_i a_{(-1)}(0) \otimes (d_i \otimes a'_i \alpha(a_{(0)}))$$
(27)

$$\rho^f(\sum_i c_i \otimes (d_i \otimes a'_i)) = \sum_i c_i(1) \otimes (c_i(2) \otimes c_i(3) \otimes d_i \otimes a'_i)$$
(28)

$$a(\sum_i c_i \otimes (d_i \otimes a'_i)) = \sum_i c_i \otimes (d_i S(a_{(-1)}(0) \otimes a_{(0)} a'_i)$$
(29)

$$\rho^f(\sum_i c_i \otimes (d_i \otimes a'_i)) = \sum_i (c_i \otimes (d_i(1) \otimes a'_i)) \otimes d_i(2)$$
(30)
\(A \otimes C\) is also a Doi-Hopf bimodule; the structure maps are
\[
(b \otimes c)a = \sum ba_{(0)} \otimes ca_{(-1)}
\]
(31)
\[
\rho^I(b \otimes c) = \sum c_{(1)} \otimes b \otimes c_{(2)}
\]
(32)
\[
a(b \otimes c) = ab \otimes c
\]
(33)
\[
\rho^S(b \otimes c) = \sum b_{(0)} \otimes c_{(1)} \otimes c_{(2)}S(b_{(-1)})
\]
(34)
The map \(f : A \otimes C \to C \otimes A\) is given by
\[
f(b \otimes c) = \sum cS(b_{(-1)}) \otimes b_{(0)}
\]
is an isomorphism of Doi-Hopf bimodules, with inverse
\[
f^{-1}(c \otimes b) = \sum b_{(0)} \otimes cb_{(-1)}
\]
We will need \(C \otimes A\) in Section 3, where we study the separability of the induction functor \(F\). \(A' \otimes C'\) will appear in Section 4, where we investigate the separability of \(G\).
In view of (32), we have
\[
G(A' \otimes C') = \text{C}(-\text{C},(A' \otimes C')) \cong A' \otimes C \in C_A^C, \mathcal{M}(H', H')_{A'}^C
\]
(35)
and
\[
FG(A' \otimes C') \cong (A' \otimes C) \otimes_A A' \in C_{A'}, \mathcal{M}(H', H')_A^C
\]
(36)
A straightforward computation gives us all the structure maps. On \(A' \otimes C\), they are
\[
(b' \otimes c)a' = \sum b'c(a_{(0)} \otimes ca_{(-1)})
\]
(37)
\[
\rho^I(b' \otimes c) = \sum c_{(1)} \otimes b' \otimes c_{(2)}
\]
(38)
\[
a'(b' \otimes c) = a'b' \otimes c
\]
(39)
\[
\rho^S(b' \otimes c) = \sum b'_{(0)} \otimes c_{(1)} \otimes c_{(2)}S(b'_{(-1)})
\]
(40)
and on \((A' \otimes C) \otimes_A A'\):
\[
((b' \otimes c) \otimes b'')a' = (b' \otimes c) \otimes b'a'
\]
(41)
\[
\rho^I((b' \otimes c) \otimes b'') = \sum \gamma(c_{(1)})b''_{(-1)} \otimes (b' \otimes c_{(2)}) \otimes b''_{(0)}
\]
(42)
\[
a'((b' \otimes c) \otimes b'') = (a'b' \otimes c) \otimes b''
\]
(43)
\[
\rho^S((b' \otimes c) \otimes b'') = \sum b'_{(0)} \otimes c_{(1)} \otimes b'' \otimes \gamma(c_{(2)})S(b_{(-1)})
\]
(44)
for all \(a', b', b'' \in A'\) and \(c \in C\).
Now consider the maps
\[
\eta_{C \otimes A} : C \oplus A \to GF(C \otimes A) = C\text{-}\text{C}(-\text{C},(C \otimes A)) = C_{A'}^C, \mathcal{M}(H', H')_{A'}^C
\]
\[
\delta_{A' \otimes C'} : FG(A' \otimes C') = (A' \otimes C) \otimes_A A' \to A' \otimes C'
\]
which are given by (cf. (11-12)): \(\not\exists\)
\[
\eta_{C \otimes A}(c \otimes b) = \sum c_{(1)}b_{(-1)} \otimes (c_{(2)} \otimes \beta(b_{(0)}))
\]
(45)
\[
\delta_{A' \otimes C'}((b' \otimes c) \otimes b'') = \sum b'b''_{(0)} \otimes \gamma(c)b''_{(-1)}
\]
(46)
We already know that \(\eta_{C \otimes A} \in C_M(H)_A\) and \(\delta_{A' \otimes C'} \in C_M(H', H')_{A'}\). A direct verification shows

**Proposition 1.4**

\(\eta_{C \otimes A} \in C_A^C, \mathcal{M}(H, H)_A^C\) and \(\delta_{A' \otimes C'} \in C_{A'}^C, \mathcal{M}(H', H')_{A'}^C\)
1.2 Separable functors

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and $F : \mathcal{C} \to \mathcal{D}$ a covariant functor. Observe that we have two covariant functors

$$\text{Hom}_\mathcal{C}(\bullet, \bullet) : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Sets} \quad \text{and} \quad \text{Hom}_\mathcal{D}(F(\bullet), F(\bullet)) : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Sets}$$

and a natural transformation

$$\mathcal{F} : \text{Hom}_\mathcal{C}(\bullet, \bullet) \to \text{Hom}_\mathcal{D}(F(\bullet), F(\bullet))$$

Recall from [22] that $F$ is called separable if $\mathcal{F}$ splits, this means that we have a natural transformation $\mathcal{P} : \text{Hom}_\mathcal{D}(F(\bullet), F(\bullet)) \to \text{Hom}_\mathcal{C}(\bullet, \bullet)$ such that $\mathcal{P} \circ \mathcal{F}$ is the identity natural transformation of $\text{Hom}_\mathcal{C}(\bullet, \bullet)$.

The terminology is motivated by the fact that a ring extension $R \to S$ is separable (in the sense of [10]) or right semisimple (in the sense of [15]) if and only if the restriction of scalars functor $R\mathcal{M} \to S\mathcal{M}$ is separable.

If the functor $F$ is separable, then we have the following version of Maschke's Theorem (cf. [22, Prop. 1.2]): if $\alpha : M \to N$ in $\mathcal{C}$ is such that $F(\alpha)$ splits or co-splits in $\mathcal{D}$, then $f$ splits or co-splits in $\mathcal{C}$.

Now suppose that $F : \mathcal{C} \to \mathcal{D}$ has a right adjoint $G$, and write $\rho : 1_\mathcal{C} \to GF$ and $\delta : FG \to 1_\mathcal{D}$ for the unit and counit of this adjunction. Then we have the following result (see [23] and [24]):

**Theorem 1.5** Let $G : \mathcal{D} \to \mathcal{C}$ be a right adjoint of $F : \mathcal{C} \to \mathcal{D}$.

1) $F$ is separable if and only if $\rho$ splits, this means that there is a natural transformation $\nu : GF \to 1_\mathcal{C}$ such that $\nu \circ \rho$ is the identity natural transformation of $\mathcal{C}$, or $\nu_M \circ \rho_M = 1_M$ for all $M \in \mathcal{C}$.

2) $G$ is separable if and only if $\delta$ co-splits, this means that there is a natural transformation $\theta : FG \to 1_\mathcal{D}$ such that $\delta \circ \theta$ is the identity natural transformation of $\mathcal{D}$.

In [5], we studied the separability of the forgetful functor $F : C\mathcal{M}(H)_A \to \mathcal{M}_A$ and its adjoint $G$.

In the present paper, we generalize our results to the induction functor $F$ and its adjoint $G$ from Theorem 1.2.

2 Generators for the category of Doi-Hopf modules

$C \otimes A \cong A \otimes C$ is a special object in the category of Doi-Hopf modules, although it is not a generator. In this Section, we will see that it satisfies a weaker property.

**Definition 2.1** Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant functor. We call $D \in \mathcal{D}$ an $F$-generator if $F$ generates all $F(C)$, with $C \in \mathcal{C}$.

Consider the morphism $(\varepsilon_H, I_A, \varepsilon_C) : (H, A, C) \to (k, A, k)$ in $\mathbb{D}H(k)$, and we write $T = GF$, where $(F, G)$ is the adjoint pair of functors associated to this morphism. With this notation, we have

**Lemma 2.2** $C \otimes A \cong A \otimes C$ is a $T$-generator of $C\mathcal{M}(H)_A$.

**Proof** It is well-known that $\mathcal{C} = C\mathcal{M}(H)_A$ is an abelian category. Assume that $f : T(M) = C \otimes M \to N$ is a morphism in $\mathcal{C}$ such that $f \circ g = 0$ for all $g : C \otimes A \to C \otimes M$ in $\mathcal{C}$. Take $m \in M$, and consider the map

$$g_m : C \otimes A \to C \otimes M : c \otimes a \mapsto c \otimes ma$$
$g_m$ is left $C$-colinear and right $A$-linear, hence
\[ 0 = (f \circ g_m)(c \otimes 1_A) = f(c \otimes m) \]
for all $c \in C$, and this shows that $f = 0$. \hfill \Box

**Proposition 2.3** Let $F, F' : C \to D$ be additive covariant functors between abelian categories $C$ and $D$, $T : C \to C$ a functor, and $\rho : 1_C \to T$ a natural transformation. If
1) $C$ is a $T$-generator for $C$;
2) $F$ preserves epimorphisms;
3) $F'(\rho_M)$ is monic, for all $M \in C$
then any natural transformation $\nu : F \to F'$ is completely determined by $\nu_C : F(C) \to F(C')$.

**Proof** Let $\nu, \nu' : F \to F'$ be two natural transformations such that
\[ \nu_C = \nu'_C \]
We will show that $\nu_M = \nu'_M$, for all $M \in C$. $C$ is a $T$-generator, so we have an epimorphism

\[ g : C(I) \to T(M) \]
in $C$, for some index set $I$. By the naturality of $\nu$ and $\nu'$, we have commutative diagrams
\[
\begin{array}{ccc}
F(C(I)) & \xrightarrow{\nu_C(I)} & F'(C(I)) \\
F(g) \downarrow & & \downarrow F(g) \\
F(T(M)) & \xrightarrow{\nu_T(M)} & F'(T(M))
\end{array}
\]
\[
\begin{array}{ccc}
F(C(I)) & \xrightarrow{\nu'_C(I)} & F'(C(I)) \\
F(g) \downarrow & & \downarrow F(g) \\
F(T(M)) & \xrightarrow{\nu'_T(M)} & F'(T(M))
\end{array}
\]
in $C$. Now $F$ and $F'$ are additive functors, and therefore
\[ \nu_{C(I)} = (\nu_C)(I) = (\nu'_C)(I) = \nu'_{C(I)} \]
and it follows from (47) that
\[ \nu'_{T(M)} \circ F(g) = \nu_{T(M)} \circ F(g) \]
Now $g$ is an epimorphism, and $F$ preserves epimorphisms, so $F(g)$ is an epimorphism, and
\[ \nu'_{T(M)} = \nu_{T(M)} \]
for every $M \in C$. Now consider $\rho_M : M \to T(M)$ in $C$, and the following commutative diagrams arising from the naturality of $\nu$ and $\nu'$:
\[
\begin{array}{ccc}
F(M) & \xrightarrow{\nu_M} & F'(M) \\
F(\rho_M) \downarrow & & \downarrow F(\rho_M) \\
F(T(M)) & \xrightarrow{\nu_T(M)} & F'(T(M))
\end{array}
\]
\[
\begin{array}{ccc}
F(M) & \xrightarrow{\nu'_M} & F'(M) \\
F(\rho_M) \downarrow & & \downarrow F(\rho_M) \\
F(T(M)) & \xrightarrow{\nu'_T(M)} & F'(T(M))
\end{array}
\]
Since $\nu_{T(M)} = \nu'_{T(M)}$, we have
\[ F'(\rho_M) \circ \nu'_M = F'(\rho_M) \circ \nu_M \]
and $\nu_M = \nu'_M$ since $F'(\rho_M)$ is monic. \hfill \Box

**Corollary 2.4** With assumptions as in Proposition 2.3, the natural transformations $\nu : F \to F'$ form a set, actually it is a subset of $\text{Hom}_D(F(C), F(C'))$.  


3 Separability of the induction functor

As before, $(\mathcal{H}, \alpha, \gamma): \mathcal{H} = (H, A, C) \rightarrow \mathcal{H}' = (H', A', C')$ is a morphism in $\text{DH}(k)$. As in Section 1.1, we write $\mathcal{C} = C \mathcal{M}(H)_A$, $\mathcal{C}' = C' \mathcal{M}(H')_{A'}$, and $F: \mathcal{C} \rightarrow \mathcal{C}'$, $G: \mathcal{C}' \rightarrow \mathcal{C}$ for the associated pair of adjoint functors. We will also consider the functors (see Lemma 2.2)

\[
T : \mathcal{C} \rightarrow \mathcal{C} : M \mapsto C \otimes M \\
T' : \mathcal{C}' \rightarrow \mathcal{C}' : M' \mapsto C' \otimes M'
\]

and the natural transformations

\[
\rho : 1_{\mathcal{C}} \rightarrow T \quad \text{and} \quad \rho' : 1_{\mathcal{C}'} \rightarrow T'
\]

In fact $\rho$ and $\rho'$ are deformed by the $C$ and $C'$-coaction defined on any $M \in \mathcal{C}$ and $M' \in \mathcal{C}'$. In order to investigate the separability of $F$, we have to study the natural transformations (see Theorem 1.5)

\[
\nu : GF \rightarrow 1_{\mathcal{C}}
\]

Our first aim is to show that $\nu$ is completely determined by $\nu_{C \otimes A}$, using Proposition 2.3.

**Proposition 3.1** Let $\nu : GF \rightarrow 1_{\mathcal{C}}$ be a natural transformation, and assume that $C$ is right $C'$-coflat. Then $\nu$ is completely determined by $\nu_{C \otimes A}$.

**Proof** We apply Proposition 2.3, with $\mathcal{C} = \mathcal{D} = C \mathcal{M}(H)_A$, $F$ replaced by $GF$ and $G$ by $1_{\mathcal{C}}$, $T$ and $\rho$ as above, and $C$ replaced by $C \otimes A$.

We have seen in Lemma 2.2 that $C \otimes A$ is a $T$-generator, and obviously $1_{\mathcal{C}}(\rho_M) = \rho_M$ is monic for all $M \in C \mathcal{M}(H)_A$. The only thing we have to show is that $GF$ preserves epimorphisms. $F = \bullet \otimes_A A'$ preserves epimorphisms, since the tensor product is always right exact, and $G = C \mathcal{M}(H')_{A'} \otimes \bullet$ preserves epimorphisms since we assumed that $C$ is right $C'$-coflat.

**Lemma 3.2** Let $\nu : GF \rightarrow 1_{\mathcal{C}}$ be a natural transformation, $M \in \mathcal{C}$ and a flat $k$-module. Consider $M \otimes N \in \mathcal{C}$ with structure maps induced by the structure maps on $M$:

\[
\rho_{M \otimes N} = \rho_M \otimes I_N \quad \text{and} \quad a(m \otimes n) = am \otimes n
\]

Then

\[
\nu_{M \otimes N} = \nu_M \otimes I_N
\]

**Proof** For any $n \in N$, we define $f_n : M \rightarrow M \otimes N$ by

\[
f_n(m) = m \otimes n
\]

It is clear that $f_n \in \mathcal{C}$, and we have the following commutative diagram, by the naturality of $\nu$:

\[
\begin{array}{ccc}
GF(M) & \xrightarrow{\nu_M} & M \\
\downarrow{GF(f_n)} & & \downarrow{f_n} \\
GF(M \otimes N) & \xrightarrow{\nu_{M \otimes N}} & M \otimes N
\end{array}
\]

Now $GF(M \otimes N) = C \mathcal{M}(M \otimes_A A') \otimes N = GF(M) \otimes N$ since $N$ is $k$-flat (the flatness is needed to make the cotensor product associative). Furthermore $GF(f_n)(x) = x \otimes n$, for all $x \in GF(M)$, and the result follows after we evaluate the diagram at $x$. \(\blacksquare\)
Proposition 3.3 Let \( \nu : \text{GF} \to 1_C \) be a natural transformation. For any Doi-Hopf \((\mathcal{H}, \mathcal{H})\)-bimodule \( M \), \( \nu_M \) is left and right \( A \)-linear and \( C \)-colinear:

\[
\nu_M \in \mathcal{C}(\mathcal{M}(H)_A^C)
\]

In particular \( \nu_{C \otimes A} \in \mathcal{C}(\mathcal{M}(H)_A^C) \).

Proof We already know that \( \nu_M \) is left \( C \)-colinear and right \( A \)-linear. Let us first prove that \( \nu_M \) is also left \( A \)-linear.

For any \( a \in A \), we consider the map \( f_a : M \to M \) mapping \( m \in M \) to \( am \). Then \( f_a \in \mathcal{C}(\mathcal{M}(H)_A) \), by conditions 4) and 6) in Definition 1.1, and we have a commutative diagram

\[
\begin{array}{ccc}
GF(M) & \xrightarrow{\nu_M} & M \\
\downarrow{GF(f_a)} & & \downarrow{f_a} \\
GF(M) & \xrightarrow{\nu_M} & M \\
\end{array}
\]

Take \( x = \sum_i c_i \otimes (m_i \otimes a'_i) \in GF(M) = \mathcal{C}(\mathcal{M}(M \otimes_A A')) \). Then

\[
GF(f_a)(x) = (I_C \otimes (f_a \otimes I_A'))(x) = \sum_i c_i \otimes (am_i \otimes a'_i) = ax
\]

Evaluating the diagram at \( x \), we find

\[
\nu_M(ax) = a\nu_M(x)
\]

as needed.

To prove that \( \nu_M \) is right \( C \)-colinear, we consider the right \( C \)-coaction \( \rho^r : M \to M \otimes C \). We make \( M \otimes C \) into a Doi-Hopf module as in Lemma 3.2 (using the fact that \( C = N \) is \( k \)-flat by assumption). Then \( \rho^r \) is left \( C \)-colinear because \( M \) is a \((C, C)\)-bicomodule (see 3) in Definition 1.1) and right \( A \)-linear since

\[
\rho^r(ma) = \sum (ma)_0 \otimes (ma)_1 \\
= \sum m_0 a \otimes m_1 \\
= \rho^r(m)a
\]

(we used 5) in Definition 1.1). We now have a commutative diagram

\[
\begin{array}{ccc}
GF(M) & \xrightarrow{\nu_M} & M \\
\downarrow{GF(\rho^r)} & & \downarrow{\rho^r} \\
GF(M \otimes C) & \xrightarrow{\nu_{M \otimes C}} & M \otimes C \\
\end{array}
\]

From Lemma 3.2, we know that \( GF(M \otimes C) = GF(M) \otimes C \) and \( \nu_{M \otimes C} = \nu_M \otimes \nu_C \). For \( x = \sum_i c_i \otimes (m_i \otimes a'_i) \in GF(M) = \mathcal{C}(\mathcal{M}(M \otimes_A A')) \), we have

\[
GF(\rho^r)(x) = \sum_i c_i \otimes (m_i(0) \otimes a'_i) \otimes m_i(1)
\]
(use the definition of $F$ and $G$ on the morphisms). This means that $GF(\rho')$ is the right $C$-coaction on $GF(M)$. The commutativity of the diagram then tells us that
\[
\rho' \circ \nu_M = (\nu_M \otimes I_C) \circ GF(\rho')
\]
which means in fact that $\nu_M$ is right $C$-colinear. □

Define
\[
V = \{ \nu : GF \to 1_C \mid \nu \text{ is a natural transformation} \}
\]
We know from Proposition 3.1 that $V$ is a set. Actually $V$ is a $k$-algebra, under the following operations:
\[
(x\nu)_M = x\nu_M ; \quad (\nu + \nu')_M = \nu_M + \nu'_M
\]
\[
\nu \cdot \nu' = \nu' \circ \eta \circ \nu
\]
for all $x \in k, \nu, \nu' \in V$. If $\nu$ is a splitting of $\eta$ (which means that $\nu \circ \eta$ is the identity natural transformation, then $\nu$ is a right unit in $V$ (and therefore an idempotent; we will see further on that this generalizes in a sense the notion of separability idempotent). Now define
\[
V_1 = \{ \tilde{\nu} : GF(C \otimes A) \to C \otimes A \mid \tilde{\nu} \in C \mathcal{M}(H, H) \}
\]
$V_1$ is a $k$-submodule of $\text{Hom}_C(GF(C \otimes A), C \otimes A)$. Our main result is now the following.

**Theorem 3.4** Let $\mathcal{H} : \mathcal{H} \to \mathcal{H}'$ be a morphism of Doi-Hopf data, and assume that $C$ is right $C'$-coflat. Then the map $f : V \to V_1$ mapping $\nu$ to $\nu_{C \otimes A}$ is an isomorphism of $k$-modules.

**Proof** It follows from Proposition 3.1 and Proposition 3.3 that $f$ is a well-defined monomorphism. If for any $\tilde{\nu} \in V_1$ we can construct a natural transformation $\nu : GF \to 1_C$ such that
\[
\nu_{C \otimes A} = \tilde{\nu}
\]
then it follows that $f$ is surjective, and the proof is finished.

For any $M \in C \mathcal{M}(H)_A$, we define $\phi : M \otimes_A A' \to M \otimes_A (C \otimes A')$ by
\[
\phi(m \otimes a') = \sum m_{(0)} \otimes (m_{(-1)} \otimes a')
\]
($C \otimes A'$ is a left $A$-module via (25)). $\phi$ is well-defined since
\[
\phi(ma \otimes a') = \sum m_{(0)}a_{(0)} \otimes (m_{(-1)}a_{(-1)} \otimes a')
\]
\[
(25) = \sum m_{(0)} \otimes (m_{(-1)}a(-2)S(a(-1)) \otimes \alpha(a(0))a')
\]
\[
= \sum m_{(0)} \otimes (m_{(-1)} \otimes \alpha(a)\alpha a')
\]
\[
= \phi(m \otimes \alpha(a)\alpha a')
\]
The left $C'$-coaction (24) on $C \otimes A'$ induces a left $C'$-coaction on $M \otimes_A (C \otimes A')$. Let us check that this coaction is well-defined
\[
\rho'(m \otimes a(c \otimes a')) = \rho'(m \otimes (cS(a(-1)) \otimes \alpha(a(0))a'))
\]
\[
= \sum \gamma(c_{(1)}S(a(-2))h(a(-1))a'_{(-1)} \otimes (m \otimes (c_{(2)}S(a_{(-3)}) \otimes \alpha(a(0))a'_{(0)}))
\]
\[
= \sum \gamma(c_{(1)})a'_{(-1)} \otimes (m \otimes (c_{(2)}S(a_{(-1)}) \otimes \alpha(a(0))a'_{(0)}))
\]
\[
= \sum \gamma(c_{(1)})a'_{(-1)} \otimes (m \otimes a(c_{(2)} \otimes a'_{(0)}))
\]
\[
= \sum \gamma(c_{(1)})a'_{(-1)} \otimes (ma \otimes (c_{(2)} \otimes a'_{(0)}))
\]
\[
= \rho'(ma \otimes (c \otimes a'))
\]
It is also clear that \( \phi \) is left \( C' \)-colinear. Now \( C \) is \( C' \)-coflat, so

\[
C \otimes (M \otimes_A (C \otimes A')) \simeq M \otimes_A (C \otimes (C \otimes A'))
\]

(see for example [9]), and we have a well-defined map

\[
I_C \otimes \phi : C \otimes (M \otimes_A A') \rightarrow M \otimes_A (C \otimes (C \otimes A'))
\]

We define \( \nu_M \) as the composition

\[
\nu_M = (I_M \otimes ((\varepsilon_C \otimes I_A)\tilde{\nu})) \circ (I_C \otimes \phi) : C \otimes (M \otimes_A A') \rightarrow M \otimes_A A \simeq M
\]

that is,

\[
\nu_M\left(\sum_i c_i \otimes (m_i \otimes a'_i)\right) = \sum m_i(0)(\varepsilon_C \otimes I_A)\nu(\sum_i c_i \otimes (m_i(1) \otimes a'_i))
\]  

(49)

a) \( \nu_M \) is right \( A \)-linear.

Using the fact that \( \varepsilon_C \otimes I_A \) and \( \nu \) are right \( A \)-linear, we obtain

\[
\nu_M((\sum_i c_i \otimes (m_i \otimes a'_i)a) = \nu_M\left(\sum_i c_i a(1) \otimes (m_i \otimes a'_i)(a(0))\right)
\]

\[
= \sum m_i(0)(\varepsilon_C \otimes I_A)\nu(\sum_i c_i \otimes (m_i \otimes a'_i)a)
\]

b) \( \nu_M \) is left \( C \)-colinear.

This part of the proof is somewhat more technical, and uses the fact that \( \tilde{\nu} \) is left and right \( C \)-colinear. Take

\[
x = \sum_i c_i \otimes (d_i \otimes a'_i) \in C \otimes (C \otimes A')
\]

and write

\[
\tilde{x} = \sum_j e_j \otimes b_j \in C \otimes A
\]

\( \tilde{\nu} \) is left \( C \)-colinear, hence

\[
\sum c_i(1) \otimes \tilde{\nu}(c_i(2) \otimes d_i \otimes a'_i) = \sum e_j(1) b_j(-1) \otimes e_j(2) \otimes b_j(0)
\]

Applying \( I_C \otimes \varepsilon_C \otimes I_A \) to both sides, we obtain

\[
\sum c_i(1) \otimes (\varepsilon_C \otimes I_A)\tilde{\nu}(c_i(2) \otimes d_i \otimes a'_i) = \sum e_j b_j(-1) \otimes b_j(0)
\]

(50)
Now \( \tilde{\nu} \) is also right \( C \)-colinear, so
\[
\sum_i \tilde{\nu}(c_i \otimes (d_{(1)} \otimes a_i')) \otimes d_{(2)} = \sum_j (e_{j(1)} \otimes b_j) \otimes e_{j(2)}
\]
and, applying first \((\varepsilon_C \otimes I_A) \otimes I_C\), and then the switch map, we find
\[
\sum_j e_j \otimes b_j = \sum_i d_{(2)} \otimes (\varepsilon_C \otimes I_A) \tilde{\nu}(c_i \otimes (d_{(1)} \otimes a_i'))
\]
and, using (50)
\[
\sum_i d_{(2)}((\varepsilon_C \otimes I_A)(\tilde{\nu}(c_i \otimes (d_{(1)} \otimes a_i'))))(-1) \otimes ((\varepsilon_C \otimes I_A)(\tilde{\nu}(c_i \otimes (d_{(1)} \otimes a_i'))))_0
\]
\[
= \sum_j e_j b_j(-1) \otimes b_j(0)
\]
\[
= \sum_i \tilde{\nu}(c_i \otimes (d_{(1)} \otimes a_i')) \otimes d_{(2)}
\]
(51)

Now take \( x = \sum_i c_i \otimes (m_i \otimes a_i') \in C\boxtimes^\gamma (M \otimes A' A') \). Then
\[
\rho'(\nu_M(x)) = \rho'(\sum_i m_i(0)(\varepsilon_C \otimes I_A)\tilde{\nu}(\sum_i c_i \otimes (m_i(1) \otimes a_i')))
\]
\[
= \sum_i m_i(1)(\varepsilon_C \otimes I_A)\tilde{\nu}(\sum_i c_i \otimes (m_i(2) \otimes a_i'))(-1)
\]
\[
\otimes m_i(0)((\varepsilon_C \otimes I_A)\tilde{\nu}(\sum_i c_i \otimes (m_i(-2) \otimes a_i')))_0
\]
(51) \[
= \sum_i c_i(1) \otimes m_i(0)(\varepsilon_C \otimes I_A)\tilde{\nu}(c_i(2) \otimes (m_i(-1) \otimes a_i'))
\]
\[
= (I_C \otimes \nu_M)\rho'(x)
\]

c) We now prove the naturality condition. For any morphism \( f : M \to N \) in \( \mathcal{C} \), we have to prove that the diagram
\[
\begin{array}{ccc}
C\boxtimes^\gamma (M \otimes A' A') & \overset{\nu_M}{\longrightarrow} & M \\
I_C \otimes (f \otimes I_A') \downarrow & & \downarrow f \\
C\boxtimes^\gamma (N \otimes A' A') & \overset{\nu_N}{\longrightarrow} & N
\end{array}
\]
commutes. Take \( x = \sum_i c_i \otimes (m_i \otimes a_i') \in C\boxtimes^\gamma (M \otimes A' A') \). Using the fact that \( f \) is \( C \)-colinear and \( A \)-linear, we find
\[
\nu_N(\sum_i c_i \otimes (f(m_i) \otimes a_i')) = \sum_i f(m_i(0))(\varepsilon_C \otimes I_A)\tilde{\nu}(\sum_i c_i \otimes (f(m_i)(-1) \otimes a_i'))
\]
\[
= \sum_i f(m_i(0))(\varepsilon_C \otimes I_A)\tilde{\nu}(\sum_i c_i \otimes (m_i(-1) \otimes a_i'))
\]
\[
= f(\sum_i c_i \otimes (m_i \otimes a_i'))
\]
\[
= f(\nu_N(\sum_i c_i \otimes (m_i \otimes a_i')))\]
This proves that \( \nu \) is a natural transformation. The proof will be finished if we can show that

\[
\nu_{C \otimes A} = \tilde{\nu}
\]

This is a straightforward verification: for \( x = \sum_i c_i \otimes (d_i \otimes a'_i) \in C \square_{C'} (C \otimes A') \) we compute, using the right \( C \)-collinearity of \( \tilde{\nu} \),

\[
\nu_{C \otimes A}(x) = \sum_i (d_i(2) \otimes 1_A)(\varepsilon_C \otimes I_A)\tilde{\nu}\left(\sum_i c_i \otimes (d_i(1) \otimes a'_i)\right)
\]

(30) \[= \sum_i (x(1) \otimes 1_A)(\varepsilon_C \otimes I_A)\tilde{\nu}(x(0))\]

\[= \sum_i (\tilde{\nu}(x)(1) \otimes 1_A)(\varepsilon_C \otimes I_A)\tilde{\nu}(x)(0)\]

\[= \tilde{\nu}(x)\]

At the last step, we used the following argument: for any \( c \in C \) and \( a \in A \), we have, using (20)

\[
\sum_i ((c \otimes a)(1) \otimes 1_A)(\varepsilon_C \otimes I_A)(c \otimes a)(0) = \sum_i (c(2) \otimes 1_A)(\varepsilon_C \otimes I_A)(c(1) \otimes a)
\]

\[= c \otimes a\]

\[\square\]

As before, we let \( \eta : 1_C \rightarrow GF \) be the unit of the adjoint pair of functors. We have seen in Theorem 1.5 that \( F \) is separable if and only if there exists a \( \nu \in V \) such that \( \nu \circ \eta \) is the identity natural transformation. In this situation,

\[
\nu_{C \otimes A} \circ \eta_{C \otimes A} = I_{C \otimes A}
\]

which is equivalent to

\[
\tilde{\nu}\left(\sum c(1)b(-1) \otimes (c(2) \otimes \beta(b(0)))\right) = c \otimes b
\]

(52)

for all \( b \in B \) and \( c \in C \).

**Definition 3.5** A separability idempotent map for the induction functor

\[
F : C\mathcal{M}(H)_A \rightarrow C'\mathcal{M}(H')_{A'}
\]

is a map

\[
\tilde{\nu} : C\square_{C'} (C \otimes A') = GF(C \otimes A) \rightarrow C \otimes A \in \mathcal{M}(H)_{A'}^C
\]

satisfying (52)

**Theorem 3.6 (Maschke’s Theorem for the induction functor)** Assume that \( C \) is left \( C' \)-coflat. Then the induction functor \( F \) is separable if and only if there exists a separability idempotent map for \( F \). In this situation, Maschke’s Theorem holds for the induction functor \( F \).

**Proof** One implication has been done above. Conversely, assume that \( \tilde{\nu} \) is a separability idempotent map, and let \( \nu \) be the corresponding natural transformation, constructed in the proof of Theorem 3.4. Then \( \nu \circ \eta \) and the identity natural transformation take the same value at \( C \otimes A \), hence they are equal, by Proposition 2.3 (take \( F = F' = 1_C \)). \[\square\]
4 Separability of the adjoint of the induction functor

We keep the same notation as in the previous Section.

**Lemma 4.1** Let $\zeta : 1_{C'} \to FG$ be a natural transformation, $M' \in C' \mathcal{M}(H')_{A'}$ and $N \in k$-mod. We consider $M' \otimes N$ as a Doi-Hopf module, with structure induced by the structure on $M'$. If $N$ is $k$-flat, then $FG(M' \otimes N) \cong FG(M') \otimes N$ and $\zeta_{M' \otimes N} = \zeta_{M'} \otimes I_N$.

**Proof** The proof is similar to the proof of Lemma 3.2. The first statement follows immediately from the $k$-flatness of $N$. For any $n \in N$, we consider the map $f_n : M' \to M' \otimes N$ mapping $m'$ to $m' \otimes n$. Then $f_n \in C'$, and, by the naturality of $\zeta$, we have a commutative diagram

\[
\begin{array}{ccc}
M' & \xrightarrow{\zeta_{M'}} & FG(M') \\
f_n \downarrow & & \downarrow FG(f_n) \\
M' \otimes N & \xrightarrow{\zeta_{M' \otimes N}} & FG(M' \otimes N)
\end{array}
\]

and the second statement follows also.

**Lemma 4.2** Let $\zeta : 1_{C'} \to FG$ be a natural transformation, and assume that $FG(\rho_{M'})$ is injective, for all $M' \in C'$. Then $\zeta$ is completely determined by $\zeta_{A' \otimes C'}$.

**Proof** In Proposition 2.3, we replace $C$ and $D$ by $C'$. We take for $\rho$ the natural transformation $1_{C'} \to T'$ given by the right $C'$-coaction, and we replace $F$ by $1_{C'}$ and $F'$ by $FG$. All the conditions of Proposition 2.3 are satisfied, and the result follows.

**Proposition 4.3** Let $\zeta : 1_{C'} \to FG$ be a natural transformation. For any Doi-Hopf $(H', H')$-bimodule $M'$, the map $\zeta_{M'} : M' \to FG(M')$ is left and right $A'$-linear and $C'$-colinear.

**Proof** a) $\zeta_{M'}$ is left $A'$-linear.

For any $a' \in A'$, we consider the map $f_{d'} : M' \to M'$ by

\[
f_{d'}(m') = a'm'
\]

It is clear that $f_{d'}$ is right $A'$-linear and left $C'$-colinear, so we have a commutative diagram

\[
\begin{array}{ccc}
M' & \xrightarrow{\zeta_{M'}} & FG(M') = (C \boxtimes_{C'} M') \otimes_A A' \\
f_{d'} \downarrow & & \downarrow FG(f_{d'}) \\
M' & \xrightarrow{\zeta_{M'}} & FG(M') = (C \boxtimes_{C'} M') \otimes_A A'
\end{array}
\]

Take $(\sum_i c_i \otimes b') \in (C \boxtimes_{C'} H') \otimes_A A'$ and $a' \in A'$. Then

\[
a'(\left(\sum_i c_i \otimes m'_i \otimes b'\right)) = \left(\sum_i c_i \otimes a'm'_i \otimes b'\right) = FG(f_{d'})(\left(\sum_i c_i \otimes m'_i \otimes b'\right))
\]

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and the commutative diagram tells us that
\[ \zeta_{M'}(a'm') = a'\zeta_M(m') \]

b) \( \zeta_{M'} \) is right \( C' \)-colinear.
\( M' \otimes C' \in C'_I \mathcal{M}(H'), \) and \( \rho' : M' \rightarrow M' \otimes C' \) is left \( C' \)-colinear and right \( A' \)-linear (see the proof of Proposition 3.3). Thus we have a commutative diagram

\[ \begin{array}{ccc}
M' & \xrightarrow{\zeta_{M'}} & FG(M') \\
\rho' \downarrow & & \downarrow \rho' \\
M' \otimes C' & \xrightarrow{\zeta_{M'} \otimes C'} & FG(M' \otimes C')
\end{array} \]

\( \zeta_M \) From Lemma 4.1, using that \( C' \) is \( k \)-flat by assumption, we know that \( FG(M' \otimes C') = FG(M') \otimes C' \)
and \( \zeta_{M'} \otimes C' = \zeta_{M'} \otimes I_{C'} \), and the diagram tells us that
\[ (\zeta_{M'} \otimes I_{C'})(\rho'(m)) = FG(\rho')(\zeta_M(m')) \]

Now the right \( C' \)-coaction on \( FG(M') \) is given by \( FG(\rho') \) and we find the right \( C' \)-colinearity. \( \square \)

Now we define
\[ W = \{ \zeta : 1_{C'} \rightarrow FG \mid \zeta \text{ is a natural transformation} \} \]

\( W \) is a set, by Lemma 4.2, and a \( k \)-algebra. Addition and scalar multiplication are defined in the obvious way, and the multiplication is given by the formula
\[ \zeta \cdot \zeta' = \zeta' \circ \zeta \]

\( \zeta \) cosplits \( \delta' \) (i.e. \( \delta' \circ \zeta \)) is the identity natural transformation) if and only if \( \delta' \) is a left unit in \( W \).
Now let
\[ W_1 = \{ \tilde{\zeta} : A' \otimes C' \rightarrow FG(A' \otimes C') \mid \tilde{\zeta} \in C'_I \mathcal{M}(H', H')_{A'} \} \]
which is a submodule of \( \text{Hom}_C(A' \otimes C', FG(A' \otimes C')) \).

**Theorem 4.4** Let \( \mathcal{H} : \mathcal{H} \rightarrow \mathcal{H}' \) be a morphism of \( \text{Doi-Hopf} \) data, and assume that \( A' \) is flat as a left \( A \)-module, and that \( FG(\rho_{M'}) \) is injective, for all \( M' \in C' \). Then the map
\[ f : W \rightarrow W_1 : \zeta \mapsto \zeta_{A' \otimes C'} \]
is an isomorphism of \( k \)-modules.

**Proof** It is clear that \( f \) is a well-defined homomorphism, and it follows from Lemma 4.2 that \( f \) is injective.
Take \( \tilde{\zeta} \in W_1 \). The proof will be finished if we can construct a natural transformation
\[ \zeta : 1_{C'} \rightarrow FG \]
such that \( \zeta_{A' \otimes C'} = \tilde{\zeta} \). Take \( M' \in C' \), and consider the following objects in \( C' \):
a) \( C' \otimes M' = T'(M') \). We recall that the structure is given by the formulas
\[ (c \otimes m')_{a'} = \sum d a'_{(-1)} \otimes m' a'_{(0)} \]
\[ \rho'(c' \otimes m') = \sum c'_{(1)} \otimes c'_{(2)} \otimes m' \]
b) Recall that $A' \otimes C'$ is a two-sided Doi-Hopf module. The left $C'$-coaction and right $A'$-action induce a left $C'$-coaction and a right $A'$-action on $M' \otimes_{A'} (A' \otimes C')$, making $M' \otimes_{A'} (A' \otimes C') \in C'$.  

c) $A' \otimes C \in {\mathcal{M}(H, H')}_{A'}^{C'}$ (see (37-40)), hence

$$F(A' \otimes C) = (A' \otimes C) \otimes_{A} A' \in {\mathcal{M}(H, H')}_{A'}^{C'}$$

As in b), we obtain that $M' \otimes_{A'} (A' \otimes C) \otimes_{A} A' \in C'$.

Now we have an isomorphism of $k$-modules

$$f_1 : M' \otimes_{A'} (A' \otimes C) \otimes_{A} A' \rightarrow (C \otimes M') \otimes_{A} A'$$

given by $f_1 (m' \otimes (a' \otimes c) \otimes b') = (c \otimes m'a') \otimes b'$. The Doi-Hopf structure on $M' \otimes_{A'} (A' \otimes C) \otimes_{A} A'$ induces a Doi-Hopf structure on $A' \rightarrow (C \otimes M') \otimes_{A} A'$, namely

$$\rho^l ((c \otimes m') \otimes a') = \sum \gamma (c_{(1)}) a'_{(-1)} \otimes (c_{(2)} \otimes m') \otimes a'_{(0)}$$

and then $f_1$ is a morphism in $\mathcal{C}$. Another direct verification shows that the $k$-module isomorphism

$$f_2 : M' \otimes_{A'} (A' \otimes C') \rightarrow C' \otimes M'$$

given by $f_2 (m' \otimes (b' \otimes c')) = c' \otimes m'b'$ is an isomorphism in $C'$. Furthermore the maps

$$\rho^l : M' \rightarrow C' \otimes M'$$

$$\tilde{\zeta} : A' \otimes C' \rightarrow (A' \otimes C) \otimes_{A} A'$$

$$I_{M'} \otimes \tilde{\zeta} : M' \otimes_{A'} (A' \otimes C') \rightarrow M' \otimes_{A'} (A' \otimes C) \otimes_{A} A'$$

are morphisms in $\mathcal{C}$, and we consider the composition

$$\zeta_{M'} : M' \xrightarrow{\rho^l} C' \otimes M' \xrightarrow{f_2^{-1}} M' \otimes_{A'} (A' \otimes C') \xrightarrow{I_{M'} \otimes \tilde{\zeta}} M' \otimes_{A'} (A' \otimes C) \otimes_{A} A' \xrightarrow{f_1} (C \otimes M') \otimes_{A} A'$$

which is a morphism in $\mathcal{C}$. Now consider the map

$$FG(\rho^l) : FG(M') = (C \otimes_{C'} M') \otimes_{A} A' \rightarrow FG(C' \otimes M') = (C \otimes_{C'} (C' \otimes M')) \otimes_{A} A'$$

$FG(\rho^l)$ is injective, by assumption. It is easy to verify that the natural $k$-module isomorphism

$$(C \otimes_{C'} (C' \otimes M')) \otimes_{A} A' \cong (C \otimes M') \otimes_{A} A'$$

is an isomorphism in $\mathcal{C}$, so we can view $FG(M') = (C \otimes_{C'} M') \otimes_{A} A'$ as a submodule of $(C \otimes M') \otimes_{A} A'$ in $\mathcal{C}$. With this identification, we claim that

$$\text{Im} \left( \zeta_{M'} \right) \subset FG(M')$$

We introduce the following notation, for $c' \in C'$:

$$\tilde{\zeta} (1_{A'} \otimes c') = \sum (a^{(c')} \otimes c^{(c')}) \otimes b^{(c')} \in (A' \otimes C) \otimes A'$$

(54)

The definition of $\zeta_{M'}$ can then be rewritten as

$$\zeta_{M'} (m) = \sum \left( c^{(m_{(1)})} \otimes m_{(0)} a^{(m_{(1)})} \right) \otimes b^{(m_{(-1)})}$$

(55)
for all $m \in M$. From the assumption that $A'$ is flat as a left $A$-module, we deduce that $FG(M')$ is the coequalizer of the maps

$$(C \otimes M') \otimes_A A' \rightarrow \rightarrow (C \otimes C' \otimes M') \otimes_A A'$$

and it suffices to prove that

$$\sum c^{(m(-2))} \otimes m(-1)(a^{(m(-2))}(-1)) \otimes m(0)(a^{(m(-2))}(0)) \otimes b^{(m(-2))}$$

$$= \sum (c^{(m(-1))})_1 \otimes \gamma ((c^{(m(-1))})_2) \otimes m(0)a^{(m(-1))} \otimes b^{(m(-1))}$$

(56)

in $(C \otimes C' \otimes M') \otimes_A A'$, for any $m \in M'$. From the right $C'$-linearity of $\tilde{\zeta}$, we deduce

$$\sum \tilde{\zeta}(1_{A'} \otimes c'(1)) \otimes c'(2) = \rho'(\tilde{\zeta}(1_{A'} \otimes c'))$$

or

$$\sum \left((a^{(c'(1))} \otimes c'(2)) \otimes c'(2) \right) \otimes c'(2)$$

$$(57)$$

in $((A' \otimes C) \otimes_A A') \otimes C'$. We apply $\rho_A$ to the first factor, and then we let the first factor act from the right on the last one. Then we obtain

$$\sum \left(\left((a^{(c'(1))}(0) \otimes c'(2)) \otimes c'(2)\right)(a^{(c'(1))})_1 \otimes b^{(c'(1))}\right)$$

$$= \sum \left(\left((a^{(c'(1))} \otimes c'(2)) \otimes c'(2)\right)(a^{(c'(1))}_1 \otimes b^{(c'(1))}\right)$$

(57)

Taking $c' = m(-1)$ in (56), we find (55). This shows that we have a map $\zeta_{M'} : M' \rightarrow FG(M')$ in $C$. Let us show that this defines a natural transformation $\zeta : 1_C \rightarrow FG$. Let $f : M' \rightarrow N'$ be a morphism in $C$. For all $m \in M'$, we have

$$\zeta_{N'}(f(m)) = \sum (I_C \otimes f(m)(0) \otimes I_{A'})(1_{A'} \otimes f(m)(-1))$$

$$= \sum (I_C \otimes f(m(0)) \otimes I_{A'})(1_{A'} \otimes m(-1))$$

$$= \sum c^{(m(-1))} \otimes f(m(0)) a^{(m(-1))} \otimes b^{(m(-1))}$$

$$= \sum c^{(m(-1))} \otimes f(m(0)) a^{(m(-1))} \otimes b^{(m(-1))}$$

$$= \sum c^{(m(-1))} \otimes a^{(m(-1))} \otimes b^{(m(-1))}$$

$$= \sum c^{(m(-1))} \otimes a^{(m(-1))} \otimes b^{(m(-1))}$$

$$= FG(f) \zeta_{M'}(m)$$

Let us finally prove that $\zeta_{A' \otimes C'} = \tilde{\zeta}$. For all $c' \in C'$, we have

$$\zeta_{A' \otimes C'}(1_{A'} \otimes c') = \sum c^{(c'(1))} \otimes (1_{A'} \otimes c'(2)) a^{(c'(1))} \otimes b^{(c'(1))}$$

(31)

$$= \sum c^{(c'(1))} \otimes \left((a^{(c'(1))}(0) \otimes c'(2))(a^{(c'(1))}(0) \otimes c'(1)) \right)$$

(57)

$$= \sum c^{(c'(1))} \otimes a^{(c'(1))} \otimes \gamma((c^{(c'(1))}(0) \otimes b^{(c'(1))})$$

$$= \sum c^{(c'(1))} \otimes a^{(c'(1))} \otimes b^{(c'(1))} = \tilde{\zeta}(1_{A'} \otimes c')$$

where we used the identification

$$FG(A' \otimes c') = (C \otimes C')(A' \otimes C') \otimes_A A' = (C \otimes A') \otimes_A A'$$

The statement now follows from the right $A'$-linearity of $\zeta_{A' \otimes C'}$ and $\tilde{\zeta}$. $\square$
Remarks 4.5 1) If \( k \) is field, then \( F \) is exact (since \( A' \) is \( A \)-flat), and \( G \) is left exact (since the cotensor product is left exact), so \( FG \) is left exact, and the condition that \( FG(\rho_{M'}) \) has to be injective is automatically fulfilled.
2) If \( C' = k \), then \( (C' \otimes M') \otimes_A A' = (C \otimes M') \otimes_A A' \), and (53) always holds. Thus we do not need the assumption that \( A' \) is flat as a left \( A \)-module.

In Theorem 1.5, we have seen that \( G \) is separable if and only if there exists \( \zeta \in W \) such that \( \delta \circ \zeta = 1_{C'} \). In this situation

\[
\delta_{A' \otimes C'} \circ \zeta_{A' \otimes C'} = I_{A' \otimes C'}
\]

which is equivalent to

\[
\sum a^{(c')} (b^{(c')})_{(0)} \otimes \gamma(d^{(c')}) (b^{(c')})_{(-1)} = 1_{A'} \otimes c'
\]

for all \( c' \in C' \). Here we use the notation (54) and the right \( A' \)-linearity of both \( \delta_{A' \otimes C'} \) and \( \zeta_{A' \otimes C'} \). This motivates the next definition.

Definition 4.6 A separability idempotent for the functor \( G : C' \to C \) is a map \( \tilde{\zeta} : A' \otimes C' \to (A' \otimes C) \otimes_A A' \) in \( C_{\text{sep}} G(H', H') C' \) such that the normalizing condition (58) holds.

Theorem 4.7 (Maschke’s Theorem for the adjoint of the induction functor)

Let \( \tilde{b} : (H, A, C) \to (H', A', C') \) be a morphism of Doi-Hopf data. Assume that \( A' \) is \( A \)-flat and \( FG(\rho_{M'}) \) is injective, for all \( M' \in C' \). Then the functor \( G \) is separable if and only if there exists a separability idempotent \( \tilde{\zeta} \) for \( G \). In this situation, Maschke’s Theorem holds for the functor \( G \).

Proof We have already done one implication. The proof of the other one is similar to the proof of Theorem 3.6.

5 Application to the forgetful functor

In [5], we have discussed the separability of the forgetful functor \( C \mathcal{M}(H)_A \to \mathcal{M}_A \), and its (right) adjoint. These results can also be derived from the results in the two previous Sections. In these Section, we will apply our results to discuss the separability of the other forgetful functor

\[
C \mathcal{M}(H)_A \to C \mathcal{M}
\]

and its (left) adjoint.

5.1 The functor forgetting the \( A \)-action

Take a Doi-Hopf datum \( (H, A, C) \) and consider the morphism \( (\eta_H, \eta_A, I_C) : (k, k, C) \to (H, A, C) \). Then the right adjoint of the associated induction functor \( F : C \mathcal{M} \to C \mathcal{M}(H)_A \) is the functor \( G \) forgetting the \( A \)-action. For convenience, we recall the Doi-Hopf bimodule structure on \( A \otimes C \) and \( FG(A \otimes C) = A \otimes C \otimes A \). For all \( a, b, b' \in A \) and \( c \in C \), we have

\[
(b \otimes c) a = \sum b a_{(0)} \otimes a_{(-1)}
\]

(59)

\[
\rho' (b \otimes c) = \sum c_{(1)} \otimes b \otimes c_{(2)}
\]

(60)

\[
a (b \otimes c) = ab \otimes c
\]

(61)

\[
\rho'(b \otimes c) = \sum b_{(0)} c_{(1)} c_{(2)} S(b_{(-1)})
\]

(62)
and

\[
(b \otimes c \otimes b')a = \sum b \otimes c \otimes b' a
\]  
\[
\rho'(b \otimes c \otimes b') = \sum c(1) b'(-1) \otimes b \otimes c(2) \otimes b'_{(0)}
\]  
\[
a(b \otimes c \otimes b') = ab \otimes c \otimes b'
\]  
\[
\rho'(b \otimes c \otimes b') = \sum b(0) \otimes c(1) \otimes b' \otimes c(2) S(b_{(-1)})
\]  

Take a map \( \tilde{\zeta} : A \otimes C \to A \otimes C \otimes A \) in \( W_1 \). Recall that this is nothing else then a left and right \( A \)-linear and \( C \)-colinear map. Consider \( \theta = \tilde{\zeta} \circ (\eta_A \otimes I_C) : C \to A \otimes C \otimes A \), and write, as in Section 4

\[
\theta(c) = \sum a^{(c)} \otimes c^{(c)} \otimes b^{(c)} \in A \otimes C \otimes A
\]  

匮From the left \( A \)-linearity of \( \tilde{\zeta} \), it follows that

\[
\tilde{\zeta}(a \otimes c) = a \theta(c)
\]  

so \( \tilde{\zeta} \) is completely determined by \( \theta \). From the fact that \( \eta_A \otimes I_C : C \to A \otimes C \) is left and right \( C \)-colinear, we deduce that \( \theta \) is also left and right \( C \)-colinear. Furthermore, the right and left \( A \)-linearity of \( \tilde{\zeta} \) imply

\[
\sum a(0) \theta(ca_{(-1)}) = \sum \tilde{\zeta}(a(0) \otimes ca_{(-1)}) = \theta(c)a
\]  

We now define

\[
W_2 = \{ \theta : C \to A \otimes C \otimes A \mid \theta \text{ is left and right } C\text{-colinear and satisfies (69)} \}
\]  

and we have

**Proposition 5.1** We have an isomorphism of \( k \)-modules \( f_1 : W_1 \to W_2 \), given by \( f_1(\tilde{\zeta}) = \theta = \tilde{\zeta} \circ (\eta_A \otimes I_C) \).

**Proof** Given \( \theta \in W_2 \), we define \( \tilde{\zeta} \) by (69). It is straightforward to show that \( \tilde{\zeta} \in W_1 \), and that we have an isomorphism of \( k \)-modules. \( \square \)

Take \( \theta \in W_2 \), and consider \( \lambda = (I_A \otimes \varepsilon_C \otimes I_A) \circ \theta : C \to A \otimes A \). We introduce the following notation:

\[
\lambda(c) = \sum a^{[c]} \otimes b^{[c]}
\]  

We claim that \( \lambda \) is determined by \( \theta \). From the fact that \( \theta \) is right \( C \)-colinear, we deduce

\[
\sum a^{(c_{(1)})} \otimes c^{(c_{(1)})} \otimes b^{(c_{(1)})} \otimes c^{(2)} = \sum \left(a^{(c)}\right)_{(0)} \otimes \left(c^{(c)}\right)_{(1)} \otimes b^{(c)} \otimes \left(c^{(c)}\right)_{(2)} S\left(a^{(c)}_{(-1)}\right)
\]  

We apply \( \rho_A \) to the first factor, and then we let the first factor act from the right on the last one. This gives

\[
\sum \left(a^{(c_{(1)})}\right)_{(0)} \otimes c^{(c_{(1)})} \otimes b^{(c_{(1)})} \otimes c^{(2)} \left(a^{(c_{(1)})}_{(-1)}\right) = \sum a^{(c)} \otimes \left(c^{(c)}\right)_{(1)} \otimes b^{(c)} \otimes \left(c^{(c)}\right)_{(2)}
\]  

Now apply \( \varepsilon_C \) to the second factor, and then switch the two last factors. Then we obtain

\[
\theta(c) = \sum a^{(c)} \otimes c^{(c)} \otimes b^{(c)}
\]  

\[
= \sum \left(a^{(c_{(1)})}\right)_{(0)} \otimes c^{(2)} \left(a^{(c_{(1)})}_{(-1)} \otimes \varepsilon(c^{(c_{(1)})})b^{(c_{(1)})}\right)
\]  

\[
= \sum \left(a^{(c_{(1)})}\right)_{(0)} \otimes c^{(2)} \left(a^{(c_{(1)})}_{(-1)} \otimes b^{(c_{(1)})}\right)
\]  

\[
(71)
\]
and it follows that $\theta$ can be described in terms of $\lambda$. Let us investigate the other properties of $\lambda$. From the fact that $\theta$ satisfies (69), and taking (63) and (65) into account, we see that
\[
\sum a_{(0)} \lambda(c a_{(-1)}) = \lambda(c) a
\]
where $A \otimes A$ is an $A$-bimodule in the classical way: $a(b \otimes b') a' = ab \otimes b' a'$. Now $\theta$ is also left $C$-colinear, so
\[
\sum c_{(1)} \otimes \theta(c_{(2)}) = \rho^i(\theta(c)) = \sum \left( c_{(1)} \otimes a_{(c)} \otimes \left( \begin{array}{c} b_{(c)} \\ b_{(0)} \end{array} \right)_{(0)} \right)
\]
Applying $I_C \otimes I_A \otimes \varepsilon_C \otimes I_A$ to both sides, this gives
\[
\sum c_{(1)} \otimes \lambda(c_{(2)}) = \sum c_{(2)} \left( a^{[c_{(1)}]} \otimes b^{[c_{(1)}]} \right)_{(-1)} \otimes \left( a^{[c_{(1)}]} \otimes b^{[c_{(1)}]} \right)_{(0)}
\]
(71) $= \sum c_{(2)} \left( a^{[c_{(1)}]} \otimes b^{[c_{(1)}]} \right)_{(-1)} \otimes \left( a^{[c_{(1)}]} \otimes b^{[c_{(1)}]} \right)_{(0)}$ (73)

Now we define
\[W_3 = \{ \lambda : C \to A \otimes A \mid \lambda \text{ satisfies (72-73)} \}\]

**Proposition 5.2** We have an isomorphism of $k$-modules $f_2 : W_2 \to W_3$, given by
\[f_2(\theta) = \lambda = (I_A \otimes \varepsilon_C \otimes I_A) \circ \theta\]

**Proof** We have proved already that $f_2$ is a well-defined homomorphism. Given $\lambda \in W_3$, we define $\theta : C \to A \otimes C \otimes A$ by (71). We have to show that $\theta \in W_2$.

a) $\theta$ is left $C$-colinear. From (73), it follows that
\[
\sum c_{(1)} \otimes a^{[c_{(1)}]} \otimes b^{[c_{(1)}]} \otimes c_{(3)} = \sum c_{(2)} \left( a^{[c_{(1)}]} \otimes b^{[c_{(1)}]} \right)_{(-1)} \otimes \left( a^{[c_{(1)}]} \otimes b^{[c_{(1)}]} \right)_{(0)} \otimes c_{(3)}
\]
Apply $\rho_A$ to the second factor, and then apply the following permutation on the five tensor factors:
\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & 4 & 2 & 5 & 3
\end{pmatrix}
\]
Finally let the fourth factor act on the third one. This gives
\[
\sum c_{(1)} \otimes \left( a^{[c_{(2)}]} \right)_{(0)} \otimes c_{(3)} \left( a^{[c_{(2)}]} \otimes b^{[c_{(2)}]} \right)
\]
\[
= \sum c_{(2)} \left( a^{[c_{(1)}]} \otimes b^{[c_{(1)}]} \right)_{(-1)} \otimes \left( a^{[c_{(1)}]} \otimes b^{[c_{(1)}]} \right)_{(0)} \otimes c_{(3)} \left( a^{[c_{(1)}]} \otimes b^{[c_{(1)}]} \right)_{(0)}
\]
and this proves that $\theta$ is left $C$-colinear (cf. (60) and (64)).

b) $\theta$ is right $C$-colinear. Using (66), we find
\[
\rho^r(\theta(c)) = \sum \left( a^{[c_{(1)}]} \otimes c_{(2)} \right) \left( a^{[c_{(1)}]} \otimes b^{[c_{(1)}]} \otimes c_{(3)} \right) S \left( \left( a^{[c_{(1)}]} \right)_{(-1)} \right)
\]
\[
= \sum \left( a^{[c_{(1)}]} \otimes c_{(2)} \right) \left( a^{[c_{(1)}]} \otimes b^{[c_{(1)}]} \otimes c_{(3)} \right)
\]
\[
= \sum \theta(c_{(1)}) \otimes c_{(2)}
\]
We leave it to the reader to check that (72) implies that
\[
\sum a_{(0)} \theta(c a_{(-1)}) = \theta(c) a
\]
for all $c \in C$ and $a \in A$. This shows that $\theta \in W_2$, and it is easy to prove that the map $W_3 \to W_2$ mapping $\lambda$ to $\theta$ is the inverse of $f_2$. □
Proposition 5.3 Take \( \tilde{\zeta} \in W_1 \), and let \( \lambda = f_2 f_1(\tilde{\zeta}) \) be the corresponding map in \( W_3 \). Then \( \tilde{\zeta} \) satisfies the normalizing condition (58) if and only if
\[
\sum a^{[c]} b^{[c]} = \varepsilon_C(c) 1_A
\]
for all \( c \in C \).

Proof The normalizing condition (58) takes the form
\[
\sum a^{(c)}(b^{(c)})(0) \otimes c^{(c)}(b^{(c)})(-1) = 1_A \otimes c
\]
Applying \( I_A \otimes \varepsilon_C \) to both sides, we obtain (74). Conversely, if \( \lambda \) satisfies (74), then we find, using (71) that
\[
\sum a^{(c)}(b^{(c)})(0) \otimes c^{(c)}(b^{(c)})(-1) = \sum (a^{[c]}(1))(b^{[c]}(1))(0) \otimes c^{(2)}(a^{[c]}(1))(-1)(b^{[c]}(1))(-1) = 1_A \otimes c
\]
\( \square \)

In view of Remark 4.5, we conclude from the above observations and Theorem 4.7:

Theorem 5.4 (Maschke’s Theorem for the forgetful functor)
The forgetful functor \( C \mathcal{M}(H)_A \to C \mathcal{M} \) is separable if and only if there exists a map \( \lambda : C \to A \otimes A \) in \( W_3 \) satisfying the normalizing condition (74). 

Example 5.5 If \( C = k \), then \( \lambda : k \to A \otimes A \) is completely determined by the image of \( \lambda(1) = \sum a^{[1]} \otimes b^{[1]} \). Condition (73) is always fulfilled, and (72) amounts to
\[
\sum a a^{[1]} \otimes b^{[1]} = \sum a^{[1]} \otimes b^{[1]} a
\]
for all \( a \in A \). The normalizing condition (74) takes the form
\[
\sum a^{[1]} b^{[1]} = 1
\]
In fact this means that a normalized map in \( V_3 \) is a separability idempotent in the classical sense (see [10]).

5.2 Total integrals

We now consider the particular situation where \( A = H \), and \( \rho_A = \Delta_H \). Then a Doi-Hopf module is a \([C, H]\)-module in the sense of [11]. We call a right \( H \)-linear map \( \phi : C \to H \) an integral. An integral \( \phi \) such that \( \varepsilon_H \circ \phi = \varepsilon_C \) is called a total integral.

If \( C = k \), then our notion of integral coincides with the notion of integral in \( H \) in the classical sense (see [25]). Our notion is the dual of the total integrals \( H \to A \) introduced by Doi (cf. [12]).

Proposition 5.6 For \( \lambda \in W_3 \), the map \( \phi = (\varepsilon_H \otimes I_H) \circ \lambda \) is right \( H \)-linear. If \( \lambda \) is normalized, then \( \phi \) is a total integral.
Proof Using the notation of Section 5.1, we have
\[ \phi(c) = \sum \varepsilon_H(a^{[c]}b^{[\ell]}) \]
Applying \( \varepsilon_H \otimes I_H \) to (72), we find, for all \( c \in C \) and \( h \in H \),
\[ \sum \varepsilon_H(a^{[\ell]}b^{[\ell]}) = \sum \varepsilon_H(a^{[\ell]}b^{[\ell]}h) \]
and
\[ \phi(\lambda a) = \phi(c)a \]
as needed. If \( \lambda \) is normalized, then \( \sum a^{[c]}b^{[\ell]} = \varepsilon_C(c)1_H \), for all \( c \in C \), and
\[ \varepsilon_H(\phi(c)) = \sum \varepsilon_H(a^{[c]})\varepsilon_H(b^{[\ell]}) = \varepsilon_C(c) \]

Conversely, take a right \( H \)-linear map \( \phi : C \rightarrow H \), and define \( \lambda^\phi : C \rightarrow H \otimes H \) by
\[ \lambda^\phi(c) = \sum S(\phi(c)_{(1)}) \otimes \phi(c)_{(2)} \]

Theorem 5.7 Let \( \phi : C \rightarrow H \) be right \( H \)-linear. Then \( \lambda^\phi \in W_3 \) if and only if the following two conditions hold:
\[ \sum c_{(1)} \otimes \phi(c_{(2)}) = \sum c_{(2)} \otimes \phi(c_{(1)}) \]  \hspace{1cm} (75)
\[ \sum h_{(2)} \otimes \phi(ch_{(1)}) = \sum h_{(1)} \otimes \phi(ch_{(2)}) \]  \hspace{1cm} (76)
for all \( c \in C \) and \( h \in H \). In this situation, if \( \phi \) is a total integral, then \( \lambda^\phi \in W_3 \) is normalized, i.e., it is a separability idempotent.

Proof Let \( A_1 \) and \( A_2 \) be the left and right handside of (72):
\[ A_1 = \sum c_{(1)} \otimes \lambda^\phi(c_{(2)}) \]
\[ = \sum c_{(1)} \otimes S(\phi(c_{(2)})_{(1)}) \otimes \phi(c_{(2)})_{(2)} \]
\[ A_2 = \sum c_{(2)} \left(S(\phi(c_{(1)})_{(1)})_{(1)} \right)^{(\phi(c_{(1)})_{(2)})_{(1)}} \otimes \sum c_{(2)} \left(S(\phi(c_{(1)})_{(1)})_{(2)} \right)^{(\phi(c_{(1)})_{(2)})_{(2)}} \]
\[ = \sum c_{(2)} \otimes S(\phi(c_{(1)})_{(1)}) \otimes \phi(c_{(2)})_{(2)} \]
(72) is equivalent to \( A_1 = A_2 \). If we apply \( I_C \otimes \varepsilon_H \otimes I_C \) to \( A_1 = A_2 \), we find (75). Conversely, (75) implies that \( A_1 = A_2 \).

Now let \( B_1 \) and \( B_2 \) be the left and right handside of (73):
\[ B_1 = \sum h_{(2)} \lambda^\phi(ch_{(1)}) \]
\[ = \sum h_{(2)} S(\phi(ch_{(1)})_{(1)}) \otimes \phi(ch_{(1)})_{(2)} \]
\[ B_2 = \lambda(c)h \]
\[ = \sum S(\phi(c)_{(1)}) \otimes \phi(c)_{(2)})h \]
(73) is equivalent to \( B_1 = B_2 \). Assume that \( B_1 = B_2 \). Applying \( \Delta_H \) to the second factor, and then letting the second factor act on the first one, we obtain
\[ \sum h_{(2)} \otimes \phi(ch_{(1)}) = \sum h_{(1)} \otimes \phi(c)h_{(2)} \]
\[ = \sum h_{(1)} \otimes \phi(ch_{(2)}) \]
and (76) follows. Conversely, if (76) holds, then

\[ B_1 = \sum h(2)S(\phi(ch(1))_1) \otimes \phi(ch(1))_2 \]
\[ = \sum h(1)S(\phi(ch(2))_1) \otimes \phi(ch(2))_2 \]
\[ = \sum h(1)S(h(2))S(\phi(c)_1) \otimes \phi(c)_2h(3) = B_2 \]

The last statement is obvious.

\[ \square \]

**Corollary 5.8** If \( \phi : C \to H \) is a total integral such that (75-76) hold, then the forgetful functor \( C.M(H)_A \to C.M \) is separable.

**Remark 5.9** If \( C = k \), then \( \phi : C \to H \) is an integral if and only if \( \phi(1) = t \) is a right integral in \( H \) in the classical sense: \( t \in \int^r_H \). Conditions (75-76) are then automatically fulfilled, and we obtain the classical Larson-Sweedler version of Maschke’s Theorem: \( H \) is a separable Hopf algebra (over a field \( k \): a semisimple Hopf algebra) if and only if there exists a right integral \( t \in H \) such that \( \epsilon(t) = 1 \). In this case, we have a projection

\[ p : W_3 \to \int^r_H \]

given by \( p(\lambda) = \sum \epsilon(a^{[1]}b^{[1]} \), where we wrote \( \lambda(c) = \sum a^{[c]} \otimes b^{[c]} \), for all \( c \in C = k \), as before. The left inverse \( i \) of \( p \) is given by

\[ i(t)(1) = \sum S(t(1)) \otimes t(2) \]

The image of \( i \) in \( W_3 \) consists of those \( \lambda \in W_3 \) for which \( \lambda(1) = \sum a^{[1]} \otimes b^{[1]} \) satisfies

\[ \sum a^{[1]} \otimes b^{[1]} = (i \circ p)(\sum a^{[1]} \otimes b^{[1]} ) = \sum \epsilon(a^{[1]})S((b^{[1]})(1)) \otimes (b^{[1]})(2) \]

Applying \( \rho^i_{H \otimes H} \) to both sides, we find, after a small computation:

\[ \sum (a^{[1]})(1)(b^{[1]})_1 \otimes (a^{[1]})(2) \otimes (b^{[1]})(2) = \sum \epsilon(a^{[1]})1_H \otimes S((b^{[1]})(1)) \otimes (b^{[1]})(2) \]
\[ = \sum 1_H \otimes a^{[1]} \otimes b^{[1]} \]

which means that \( \lambda : k \to H \otimes H \) is left \( H \)-colinear (compare to [5, Sec. 3.3]). Conversely, if (78) holds, then \( \lambda \in \text{Im}(i) \). Indeed, applying \( I_H \otimes \epsilon_H \otimes (S_H \otimes I_H) \circ \Delta_H \) to both sides of (78), we find

\[ \sum a^{[1]}(b^{[1]})(1) \otimes S((b^{[1]})(2)) \otimes (b^{[1]})(3) = \sum \epsilon(a^{[1]})1_H \otimes S((b^{[1]})(1)) \otimes (b^{[1]})(1) \]

Multiplying the first two factors, we find (77).

**5.3 The adjoint of the forgetful functor**

We now consider the left adjoint functor \( F : C.M \to C.M(H)_A \) of the forgetful functor \( G \). Recall that \( GF(C) = C \otimes A \), with

\[ \rho^l(c \otimes a) = \sum c(1)a(-1) \otimes c(2) \otimes a(0) \]
\[ \rho^r(c \otimes a) = \sum c(1) \otimes a \otimes c(2) \]

and

\[ V_1 = \{ \tilde{\nu} : C \otimes A \to C \mid \text{\( \tilde{\nu} \) is left and right \( C \)-colinear} \} \]
Now we define $V_2$ as the submodule of $(C \otimes A)^*$ consisting of maps $\theta : C \otimes A \to k$ satisfying
\[
\sum \theta(c(2) \otimes a(0))c(1)a(-1) = \sum \theta(c(1) \otimes a)c(2)
\] (78)

Then we have

**Theorem 5.10** The map $f_1 : V_1 \to V_2$ mapping $\tilde{\nu}$ to $\theta = \varepsilon_C \circ \tilde{\nu}$ is an isomorphism of $k$-modules. $F$ is separable if and only if there exists $\theta \in V_2$ satisfying the normalizing condition
\[
\theta(c \otimes 1_A) = \varepsilon_C(c)
\]
for all $c \in C$.

**Proof** Take $\tilde{\nu} \in V_1$, and let $\theta = \varepsilon_C \circ \tilde{\nu}$. From the fact that $\tilde{\nu}$ is left and right $C$-colinear, we deduce
\[
\sum(\tilde{\nu}(c \otimes a))(1) \otimes (\tilde{\nu}(c \otimes a))(2) = \sum \tilde{\nu}(c(1) \otimes a) \otimes c(2)
\] (79)
\[
= c(1)a(-1) \otimes \tilde{\nu}(c(2) \otimes a(0))
\] (80)

Applying $\varepsilon_C \otimes I_C$ to (79), we find
\[
\tilde{\nu}(c \otimes a) = \sum \theta(c(1) \otimes a)c(2)
\] (81)

Applying $I_C \otimes \varepsilon_C$ to (80), and using (81), we find that $\theta \in V_2$ as needed. Thus $f_1$ is well-defined. For $\theta \in V_2$, we define $\tilde{\nu} = f^{-1}(\theta)$ by (81). We leave it to the reader to show that $f^{-1}$ is well-defined, and an inverse for $f$.

$\tilde{\nu} \in V_1$ is normalized if $\tilde{\nu}(c \otimes 1_A) = c$, for all $c \in C$. This is equivalent to $\theta(c \otimes 1_A) = \varepsilon_C(c)$, for all $c \in C$, and the second statement follows immediately from Theorem 3.6.

**Remarks 5.11** 1) Theorem 5.10 is a dual version of [5, Theorem 2.14].
2) If $C = k$, then we find that $F : \mathcal{M}_k \to \mathcal{M}_A$ is separable if and only if $\eta_A : k \to A$ splits as a $k$-module map. This also follows from part 2) of [22, Prop. 1.3].

### 6 Application to Hopf Galois extensions

Let $A$ be a left $H$-comodule algebra. Assume that $H$ is flat as a $k$-module, and put
\[
B = A^{\text{co}H} = \{a \in A \mid \rho(a) = a \otimes 1\}
\]

Consider the morphism
\[
(\eta_H, \alpha, \eta_H) : (k, B, k) \to (H, A, H)
\]
in $\mathcal{DH}(k)$, with $\alpha : B \to A$ the natural inclusion. We will examine the separability of the induction functor $F = \bullet \otimes_B A : \mathcal{M}_B \to \mathcal{M}(H)_A$ and its adjoint $G = (\bullet)^{\text{co}H}$.

**Proposition 6.1** If $k$ is right $H$-coflat, then the functor $F$ is separable.

**Proof** We have $k \otimes B = B$, $GF(k \otimes B) = A^{\text{co}H} = B$. The identity $\tilde{\nu} : B \to B$ is a $B$-bimodule map obviously satisfying (52), and the result follows from Theorem 3.6.

In the next Proposition, we give a sufficient condition for the right $H$-coflatness of $k$: 
**Proposition 6.2** If $H$ is coseparable as a $k$-coalgebra (see [17]), then $k$ is right $H$-coflat.

**Proof** $H$ is coseparable if and only if we have a right integral $\varphi : H \to k$ such that $\varphi(1_H) = 1$. The fact that $\varphi$ is a right integral means that

$$\varphi(h)1_H = \sum \varphi(h_{(1)})h_{(2)}$$

For any left $H$-comodule $M$, we define $\varphi_M : M \to M$ by

$$\varphi_M(m) = \sum \varphi(m_{(-1)})m_{(0)}$$

Then

$$\rho_M(\varphi_M(m)) = \sum \varphi(m_{(-2)})m_{(-2)} \otimes m_{(0)}$$

$$= \sum 1_H \otimes \varphi(m_{(-1)})m_{(0)}$$

$$= \sum 1_H \otimes \varphi_M(m)$$

so $\varphi_M(m) \in \text{co}^H_M$. If $m \in \text{co}^H_M$, then $\varphi_M(m) \in \text{co}^H_M$, so we have a projection $\varphi_M : M \to \text{co}^H_M$. Now let $f : M \to N$ be a surjective map between two left $H$-comodules. Take $n \in N^\text{co}H$. We know that there exists $m \in M$ such that $f(m) = n$. Then $\varphi_M(m) \in \text{co}^H_M$ and

$$f(\varphi_M(m)) = \sum \varphi(f(m)_{(-1)})f(m_{(0)})$$

$$= \sum \varphi(f(m)_{(-1)})f(m_{(0)})$$

$$= \varphi_N(f(m)) = n$$

This shows that $\mathbf{(\bullet ^{\text{co}H})} = k \boxtimes_H \mathbf{\bullet}$ is right exact. It is obvious that this functor is also left exact. □

Recall that $A/B$ is called an $H$-Galois extension if the canonical map

$$\beta : A \otimes_B A \to A \otimes H$$

given by

$$\beta(a \otimes b) = \sum ab_{(0)} \otimes b_{(-1)}$$

is bijective. According to (46), $\beta = \delta_{A \otimes H}$.

**Proposition 6.3** If $A$ is left $B$-flat, and $A/B$ is an $H$-Galois extension, then the functor $G = (\mathbf{\bullet ^{\text{co}H}})$ is separable.

**Proof** $F$ is exact since $A$ is left $B$-flat. As we have seen above, $G$ is left exact, so $FG$ is left exact, and this implies that $FG(\rho_M)$ is injective, for all $M \in _H^H A$. Put $\tilde{\zeta} = \beta^{-1}$. Since $\beta = \delta_{A \otimes H}$ is an $A$-bimodule and $H$-bicomodule map, the same assertion holds for $\tilde{\zeta}$. Thus $\tilde{\zeta}$ is a separability idempotent for $G$, and the assertion follows from Theorem 4.7. □

We will now give a converse of Proposition 6.3.

**Theorem 6.4** Assume that $k$ is right $H$-coflat, and $A$ is left $B$-flat. Then $A/B$ is $H$-Galois if and only if $G = (\mathbf{\bullet ^{\text{co}H}})$ is a separable functor.
**Proof** One direction follows from Proposition 6.3. Conversely, if \( G \) is separable, then there exists a natural transformation \( \zeta : 1_C \to FG \) cosplitting the counit of the adjunction (we wrote \( C = \mathcal{H}_A \)). In particular

\[
\delta_{A \otimes H} \circ \zeta_{A \otimes H} = I_{A \otimes H}
\]

and this implies that \( \delta_{A \otimes H} = \beta \) is surjective.

Now observe that \( A \) itself is also a Doi-Hopf module, with the obvious structure. We claim that \( A \) is a \( T \)-generator for \( C \) (with the same notation as in Section 2). We know from Lemma 2.2 that \( A \otimes H \) is a \( T \)-generator, so it suffices to show that \( A \) generates \( A \otimes H \). For every \( a \in A \), we consider \( \psi_a : A \to A \otimes H \) given by

\[
\psi_a(b) = \beta(a \otimes b) = \sum ab(0) \otimes b(-1)
\]

A straightforward verification shows that \( \psi_a \in \mathcal{H}_A \). Take \( x \in A \otimes H \). Since \( \beta \) is surjective, we can find \( y = \sum_i a_i \otimes b_i \in A \otimes_B A \) such that

\[
x = \beta\left( \sum_i a_i \otimes b_i \right) = \sum_i \psi_{a_i}(b_i)
\]

and this proves that \( A \) generates \( A \otimes H \).

Now observe that

\[
\delta_A : FG(A) = A^{coH} \otimes_B A = A \to A
\]

is the identity map. As \( \delta_A \circ \zeta_A = I_A \), it follows also that \( \zeta_A = I_A \), and

\[
\zeta_A \circ \delta_A = I_A = I_{FG(A)}
\]

Thus the natural transformations \( \zeta \circ \delta \) and \( 1_{FG} \) coincide in \( A \). By assumption, \( F \) and \( G \) are exact, so \( FG(\rho_M) \) is monic, for every \( M \in \mathcal{H}_A \). Thus we can apply Proposition 2.3, with \( F \) replaced by the identity functor, \( C \) by \( A \), and \( F' \) by \( FG \). We conclude that \( \zeta \circ \delta = 1_{FG} \), and, in particular,

\[
\zeta_{H \otimes A} \circ \delta_{H \otimes A} = I_{A \otimes_B A}
\]

proving that \( \delta_{B \otimes A} = \beta \) is injective.

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**References**


