SEVERAL EXAMPLES OF NONCOMMUTATIVE
NONCOCOMMUTATIVE BIALGEBRAS
ARISING FROM THE PENTAGONAL EQUATION

BOGDAN ION AND MONA STĂNCIULESCU

In this paper we shall construct new examples of noncommutative noncocommutative bialgebras arising from a FRT type theorem for the Hopf equation $R^{12}R^{23} = R^{23}R^{13}R^{12}$. As the relations through which we factor are not all homogenous, all our examples are different from the ones which appear in quantum group theory. Several examples of noncommutative noncocommutative bialgebras of prime dimension are given.

INTRODUCTION

Let $M$ be a vector space over a field $k$ and $R \in \text{End}_k(M \otimes M)$. We say that $R$ is a solution for the quantum Yang-Baxter equation if

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}.$$  

The quantum Yang-Baxter equation has been thoroughly studied in the last few years and it had a great impact to the Hopf algebras theory. Recently, a new equation was involved in the study of the Hopf modules category. This is the Hopf equation

$$R^{12}R^{23} = R^{23}R^{13}R^{12}.$$  

Although it appeared from computations, the Hopf equation proved to have many links with both algebra and analysis. For example if $H$ is a Hopf algebra then the application

$$R : H \otimes H \to H \otimes H, \quad R(g \otimes h) = \sum h_{(1)}g \otimes h_{(2)}$$

is a bijective solution for the Hopf equation. This operator was defined first by Takesaki for Hopf-von Neumann algebras and it is linked with the theory of duality for von Neumann algebras. The operator $W = \tau R\tau$ is called by S. Majid in [5] the evolution operator for a Hopf algebra and it is involved in the description of the Markov transition operator for the quantum random walks. The evolution operator is a solution for the pentagonal equation

$$R^{23}R^{12} = R^{12}R^{13}R^{23}.$$  

In fact the Hopf and the pentagonal equations are equivalent: $R$ is a solution for the Hopf equation if and only if $\tau R\tau$ is a solution for the pentagonal equation; if $R$ is a bijective solution of the Hopf equation then $R^{-1}$ is a solution of the pentagonal equation.

Another reason to study the solutions of the Hopf equation comes from analysis and it is given by S. Baaj and G. Skandalis in [1]. Let $H$ be a Hilbert space. An unitary operator $V \in \mathcal{L}(H \otimes H)$ is said to be multiplicative if it satisfies the pentagonal equation. In papers on operator algebras with duality a multiplicative unitary plays a fundamental role.
This paper presents solutions for the Hopf equation and, as application, new examples of noncommutative and noncocommutative bialgebras of both finite and infinite dimension.

The starting point for us is a FRT type theorem for the Hopf equation given in [6]: if $M$ is a finite dimensional vector space and $R \in \text{End}_k(M \otimes M)$ is a solution for the Hopf equation, then there exists a bialgebra $B(R)$ such that $M$ is a $B(R)$-Hopf module and $R$ is the natural application associated to $M$.

A description of new bialgebras arising from geometrical objects is given in section 3. In [10] S. Zhu proved that any Hopf algebra of prime dimension is isomorphic to a groupal algebra $k[Z_p]$ if the field $k$ is algebraically closed and it has $\text{char}(k) = 0$; there are also results that describe completely Hopf algebras of small dimension. In section 4 we give several examples of bialgebras of any dimension bigger than 4 which are both noncommutative and noncocommutative. $M_2(k)$ and $GL_2(k)$ are presented from a different point of view in section 5, where they appear as quotient bialgebras. The last section presents a general operator for the Hopf equation and some more applications.

1. PRELIMINARIES

Throughout this paper, $k$ will be a field. All vector spaces, algebras, coalgebras and bialgebras that we consider are over $k$. $\otimes$ and $\text{Hom}$ will mean $\otimes_k$ and $\text{Hom}_k$. For a coalgebra $C$, we will use Sweedler’s $\Sigma$-notation, that is, $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$, $(I \otimes \Delta)\Delta(c) = \sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$. We will also use Sweedler’s notation for right $C$-comodules: $\rho_M(m) = \sum m_{<\Delta>} \otimes m_{<\Delta>}$, for any $m \in M$ if $(M, \rho_M)$ is a right $C$-comodule. $\mathcal{M}^C$ will be the category of right $C$-comodules and $C$-colinear maps and $A \mathcal{M}$ will be the category of left $A$-modules and $A$-linear maps, if $A$ is a $k$-algebra. If $H$ is a bialgebra then a left $H$-module $(M, \cdot)$ which is also a right $H$-comodule $(M, \rho)$ such that

$$\rho(h \cdot m) = \sum h_{(1)} m_{<\Delta>} \otimes h_{(2)} m_{<\Delta>}$$

is called a (left-right) $H$-Hopf module; the category of $H$-Hopf modules is denoted by $\mathcal{H}\mathcal{M}^H$.

Let $H$ be a bialgebra and $(M, \cdot, \rho)$ a left $H$-module and a right $H$-comodule. The natural map associated with $M$ is

$$R_{(M, \cdot, \rho)} : M \otimes M \to M \otimes M, \quad R(m \otimes n) = \sum n_{<\Delta>} m \otimes n_{<\Delta>}.$$  

For a vector space $V$, $\tau : V \otimes V \to V \otimes V$ will denote the switch map, that is $\tau(v \otimes w) = w \otimes v$ for all $v, w \in V$. If $R : V \otimes V \to V \otimes V$ is a linear map we denote by $R^{12}, R^{13}, R^{23}$ the maps of $\text{End}_k(V \otimes V \otimes V)$ given by

$$R^{12} = R \otimes I, \quad R^{23} = I \otimes R, \quad R^{13} = (I \otimes \tau)(R \otimes I)(I \otimes \tau)$$

where $I$ is the identity map of $V$.

If $M$ is a $k$-vector space and $R \in \text{End}_k(M \otimes M)$ we say that $R$ is a solution of the quantum Yang-Baxter equation if

$$R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12}.$$

The category $\mathcal{H}\mathcal{Y}\mathcal{D}^H$ of Yetter-Drinfel’d modules is deeply involved in the quantum Yang-Baxter equation, but can the category $\mathcal{H}\mathcal{M}^H$ of Hopf modules be also studied in connection
with a non-linear equation?
An affirmative answer has been recently given in [12]. If \((M, \cdot, \rho)\) is a Hopf module then the natural map \(R_{(M, \cdot, \rho)}\) is a solution of the Hopf equation

\[ R^{12} R^{23} = R^{23} R^{13} R^{12}. \]

Conversely, we have the following FRT type theorem.

**THEOREM 1.1.** Let \(M\) be a finite dimensional vector space and let \(R \in \text{End}_k(M \otimes M)\) be a solution for the Hopf equation. Then

(a) There exists an bialgebra \(B(R)\) such that \(M\) has a \(B(R)\)-Hopf module structure \((M, \cdot, \rho)\) and the application \(R\) is the natural map \(R_{(M, \cdot, \rho)}\).

(b) The bialgebra \(B(R)\) is an universal object with this property.

Let us recall now the construction of \(B(R)\).
If \(\{m_1, \ldots, m_n\}\) is a basis of \(M\) let \((x^i_{uv})_{i,j,u,v}\) be a family of scalars from \(k\) such that

\[ R(m_v \otimes m_u) = \sum_{i,j} x^i_{uv} m_i \otimes m_j. \]

Let \((C, \Delta, \varepsilon)\) be the comatrix coalgebra of order \(n\), i.e. \(C\) is the coalgebra with the basis \(\{c_{ij} | i,j=1,\ldots,n\}\) such that

\[ \Delta(c_{kl}) = \sum_u c_{ku} \otimes c_{ul}, \quad \varepsilon(c_{kl}) = \delta_{kl}. \]

Let \(T(C)\) be the bialgebra structure on the tensor algebra \(T(C)\) which extends \(\Delta\) and \(\varepsilon\).

The obstructions \(\chi(i, j, k, l)\) which measure how far away \(M\) is from being a \(T(C)\)-Hopf module are

\[ \chi(i, j, k, l) := \sum_{u,v} x^i_{uv} c_{uk} c_{vl} - \sum_{\alpha} x^i_{kl} c_{\alpha}. \]

If \(I\) is the two-sided ideal of \(T(C)\) generated by all \(\chi(i,j,k,l)\), \(i,j,k,l=1,\ldots,n\), then \(I\) is also a coideal and \(B(R)\) is the quotient bialgebra \(T(C)/I\). In this paper we shall use the notation \((i, j, k, l)\) for the equation given by \(\chi(i,j,k,l)\) which is

\[ \sum_{u,v} x^i_{uv} c_{uk} c_{vl} = \sum_{\alpha} x^i_{kl} c_{\alpha}. \]

Let \(f, g \in \text{End}_k(M)\) and \(R = f \otimes g\). Then by a simple computation we have that \(R\) is a solution of the Hopf equation if and only if \(f^2 = f, g^2 = g\) and \(fg = gf\). This special type of maps are solutions for both Hopf and Yang-Baxter equations \((R = f \otimes g)\) is a solution for the Yang-Baxter equation if and only if \(fg = gf\). We should notice too that if \(f \in \text{End}_k(H)\) is an idempotent then we can associate to \(f\) three natural solutions for the Hopf equations, which are: \(f \otimes I, f \otimes f, f \otimes (I - f)\); in section 2 we will study some of these applications and their associated bialgebras.
2. GEOMETRIC TYPE EXAMPLES

2.1. The two dimensional case. Let $M$ be a two dimensional vector space over a field $k$ and $f \in \text{End}_k(M)$ such that, with respect to the basis $\{m_1, m_2\}$ of $M$, $f$ is represented by the matrix

$$
\begin{pmatrix}
1 - rq & q \\
r(1 - rq) & rq
\end{pmatrix}
$$

where $r$ and $q$ are elements of $k$. Let us observe that $f$ generalises projections in the plane. Then $f^2 = f$ and we can associate to $f$ three bialgebras: the first one corresponds to $R = f \otimes (\text{Id}_M - f)$, which is a solution of the Hopf equation, the second one corresponds to $f \otimes \text{Id}_M$ and the third one to $f \otimes f$. We denote these bialgebras by $B^2_{q,r}(k)$, $D^2_{q,r}(k)$ and respectively $E^2_{q,r}(k)$. The following propositions describe these objects.

**Proposition 2.1.** 1. If $q \neq 0$ the bialgebra $B^2_{q,r}(k)$ is the free algebra generated by $A$, $B$ with the relation

$$A^2B = AB.$$

The comultiplication $\Delta$ and the counity $\varepsilon$ are given by

$$\Delta(A) = A \otimes A, \quad \varepsilon(A) = 1$$

$$\Delta(B) = AB \otimes (B - A) + B \otimes A, \quad \varepsilon(B) = 1.$$

2. If $q = 0$ the bialgebra $B^2_{0,r}(k)$ is the free algebra generated by $A$, $B$ and $C$ with the relations

$$AB = 0, \quad AC = C.$$

The comultiplication $\Delta$ and the counity $\varepsilon$ are given by

$$\Delta(A) = A \otimes A, \quad \varepsilon(A) = 1$$

$$\Delta(B) = C \otimes B + B \otimes A, \quad \varepsilon(B) = 0$$

$$\Delta(C) = C \otimes C, \quad \varepsilon(C) = 1.$$

**Proof.** We consider $B = \{m_1 \otimes m_1, m_1 \otimes m_2, m_2 \otimes m_1, m_2 \otimes m_2\}$ a basis of $M \otimes M$. With respect to this basis $R = f \otimes (\text{Id}_M - f)$ is represented by the matrix

$$
\begin{pmatrix}
 rq(1 - rq) & -q(1 - rq) & rq^2 & -q^2 \\
-r(1 - rq)^2 & (1 - rq)^2 & -rq(1 - rq) & q(1 - rq) \\
r^2q(1 - rq) & -rq(1 - rq) & r^2q^2 & -r^2q^2 \\
-r^2(1 - rq)^2 & r(1 - rq)^2 & -r^2q(1 - rq) & rq(1 - rq)
\end{pmatrix}
$$

Now, if we write

$$R(m_v \otimes m_u) = \sum_{i,j=1}^{2} x^{ji}_{uv} m_i \otimes m_j$$

we obtain the scalars $(x^{ji}_{uv})_{i,j,u,v=1,2}$ of $k$. In the relations $\chi(i,j,k,l) = 0$ we denote

$$c_{11} = x, \quad c_{12} = y, \quad c_{21} = z, \quad c_{22} = t.$$

In order to obtain the description of $B^2_{q,r}(k)$ we have to find fewer simplified relations which are equivalent with the initial sixteen relations $\chi(i,j,k,l) = 0$. For the beginning we will show that these sixteen relations can be obtained from only five.
First of all, from the relations $(1, 1, 2, 2)$ and $(2, 1, 2, 2)$ we have a new relation between the four generators $x, y, z, t$. So,

$$(1, 1, 2, 2) : rq(1 - rq)y^2 + rq^2yt - q(1 - rq)ty - q^2t^2 = -q^2x - rq^2y$$

$$(2, 1, 2, 2) : r^2q(1 - rq)y^2 + r^2q^2yt - rq(1 - rq)ty - rq^2t^2 = -q^2z - rq^2t.$$

Because the left part of $(2, 1, 2, 2)$ is the left part of $(1, 1, 2, 2)$ multiplied by $r$, we have the same relation for the right parts, hence

$$-q^2r(x + ry) = -q^2(z + rt).$$

We have two cases.

1. If $q \neq 0$ then the relation between the generators is

$$r(x + ry) = z + rt. \tag{1}$$

With relation (1) the last eight relations (namely from $(2, 1, 1, 1)$ to $(2, 2, 2, 2)$) can be easily obtain from the first eight relations. For example, the relation $(1, 1, 1, 1)$ is

$$(1, 1, 1, 1) : rq(1 - rq)x^2 + rq^2xz - q(1 - rq)zx - q^2z^2 = rq(1 - rq)(x + ry).$$

If it is multiplied by $r$ and in the right term $r(x + ry)$ is replaced by $z + rt$ we get

$$r^2q(1 - rq)x^2 + r^2q^2xz - rq(1 - rq)zx - rq^2z^2 = rq(1 - rq)(z + rt)$$

which is exactly $(2, 1, 1, 1)$.

On the other hand if the first four relations are multiplied by $(rq - 1)/q$ then we find relations $(1, 2, 1, 1), \ldots, (1, 2, 2, 2)$. For example

$$(1, 2, 1, 1) : -q(1 - rq)^2x^2 - rq(1 - rq)xz + (1 - rq)^2zx + q(1 - rq)z^2 = -(1 - rq)^2(x + ry)$$

and this is $(1, 1, 1, 1)$ times $(1 - rq)/q$. So, when $q \neq 0$, the sixteen relations $\chi(i, j, k, l) = 0$ are obtained from the following:

$$(1, 1, 1, 1) : rq(1 - rq)x^2 + rq^2xz - q(1 - rq)zx - q^2z^2 = rq(1 - rq)(x + ry)$$

$$(1, 1, 1, 2) : rq(1 - rq)xy + rq^2xt - q(1 - rq)zy - q^2zt = rq^2(x + ry)$$

$$(1, 1, 2, 1) : rq(1 - rq)yx + rq^2yz - q(1 - rq)tx - q^2tz = -(1 - rq)(x + ry)$$

$$(1, 1, 2, 2) : rq(1 - rq)y^2 + rq^2yt - q(1 - rq)ty - q^2t^2 = -q^2(x + ry)$$

$$(1) : r(x + ry) = z + rt.$$

Note that, from (1), it follows that the algebra $B^2_{qr}(k)$ has at most three generators. Now, if we denote

$$A = t - ry, \quad B = t + \frac{1 - rq}{q}y, \quad C = x + ry$$

then we have

$$x = C - rq(B - A), \quad y = q(B - A), \quad z = q(C - rqB - (1 - rq)A), \quad t = rqB + (1 - rq)A$$

which means that $B^2_{qr}(k)$ is generated by $A, B, C$ with relations obtained by replacing $x, y, z, t$ with $A, B, C$. This relations are

$$rA(C - rqB) = r(1 - rq)C, \quad rqAB = rqC, \quad A(C - rqB) = (1 - rq)C, \quad AB = C.$$
The first two relations are the last two multiplied by $r$ and the fourth one says that $B_{q,r}^2(k)$ is generated by $A$ and $B$. By replacing $C$ with $AB$ in the third relation we get

$$A^2B - rqAB = AB - rqAB$$

and because $q \neq 0$ that means

$$A^2B = AB.$$ 

Thus, we have proved that $B_{q,r}^2(k)$ is the free algebra generated by $A$ and $B$ with the relation

$$A^2B = AB.$$ 

Recall that $(c_{i,j})_{i,j}$ is a comultiplicative matrix, which means, in the two dimensional case that

$$\Delta(x) = x \otimes x + y \otimes z, \quad \varepsilon(x) = 1$$

$$\Delta(y) = x \otimes y + y \otimes t, \quad \varepsilon(y) = 0$$

$$\Delta(z) = z \otimes x + t \otimes z, \quad \varepsilon(z) = 0$$

$$\Delta(t) = z \otimes y + t \otimes t, \quad \varepsilon(t) = 1.$$  

Computing $\Delta$ and $\varepsilon$ for the generators $A$ and $B$ we have

$$\Delta(A) = A \otimes A, \quad \varepsilon(A) = 1$$

$$\Delta(B) = AB \otimes (B - A) + B \otimes A, \quad \varepsilon(B) = 1$$

and this proves the first part of the proposition.

2. If $q = 0$ we have the relation (1), too. This time the relation follows from

$$(1, 2, 2, 1) : -ryx + tx = x + ry \quad \text{and} \quad (2, 2, 2, 1) : -r^2yx + rty = z + rt.$$ 

But if there is the same relation between $x, y, z, t$ that means that it is enough to look at the first eight relations. On the other hand, because $q = 0$ we have for all $i, j = 1, 2$

$$x_{11}^{11} = x_{12}^{12} = 0$$

showing that the first four relations are in fact

$$0 = 0.$$ 

Thus, if $q = 0$ the relations $\chi(i, j, k, l) = 0$ can be obtained from the following five:

$$(1, 2, 1, 1) : -rx^2 + zx = -r(x + ry)$$

$$(1, 2, 1, 2) : -rxy + zy = 0$$

$$(1, 2, 2, 1) : -ryx + tx = x + ry$$

$$(1, 2, 2, 2) : -ry^2 + ty = 0$$

$$(1) : r(x + ty) = z + rt.$$ 

If, we denote

$$A = t - ry, \quad B = y, \quad C = x + ry$$

it follows that

$$x = C - rB, \quad y = B$$

$$z = rC - r(A + rB), \quad t = A + rB$$

$$0 = 0.$$
which says that the algebra $B^2_{0,r}(k)$ is generated by $A, B$ and $C$.

In order to find the relations that define $B^2_{0,r}(k)$ we have to replace in our five relations the original generators $x, y, z, t$ with their expression in terms of $A, B, C$. In this way we get

\[-rA(C - rB) = -rC, \quad -rAB = 0\]

\[A(C - rB) = C, \quad AB = 0.\]

But, if we use the last relation in the other ones we get only two independent relations

\[AC = C, \quad AB = 0.\]

Computing the coalgebra structure on $B^2_{0,r}(k)$ we find our statement. □

**Proposition 2.2.** 1. If $q \neq 0$ the bialgebra $D^2_{q,r}(k)$ is the free algebra generated by $A, B$ with relations

\[A^2B = AB, \quad B^3 = B^2.\]

The comultiplication $\Delta$ and the counity $\varepsilon$ are given by

\[\Delta(A) = B^2 \otimes A + A \otimes B - A \otimes A, \quad \varepsilon(A) = 0\]

\[\Delta(B) = B^2 \otimes A + B \otimes B - B \otimes A, \quad \varepsilon(B) = 1.\]

2. If $q = 0$ the bialgebra $D^2_{0,r}(k)$ is the free algebra generated by $A, B$ and $C$ with relations

\[B^2 = BC = CB = AB = 0, \quad C^2 = AC = C.\]

The comultiplication $\Delta$ and the counity $\varepsilon$ are given by

\[\Delta(A) = A \otimes A, \quad \varepsilon(A) = 1\]

\[\Delta(B) = C \otimes B + B \otimes A, \quad \varepsilon(B) = 0\]

\[\Delta(C) = C \otimes C, \quad \varepsilon(C) = 1.\]

**Proof.** We consider $B = \{ m_1 \otimes m_1, m_1 \otimes m_2, m_2 \otimes m_1, m_2 \otimes m_2 \}$ a basis of $M \otimes M$. With respect to this basis $R = f \otimes \text{Id}_M$ is represented by the matrix

\[
\begin{pmatrix}
1 - rq & 0 & q & 0 \\
0 & 1 - rq & 0 & q \\
r(1 - rq) & 0 & rq & 0 \\
0 & r(1 - rq) & 0 & rq
\end{pmatrix}
\]

Now, if we write

\[R(m_v \otimes m_u) = \sum_{i,j=1}^{2} x_{uv}^{ji} m_i \otimes m_j\]

we obtain the scalars $(x_{uv}^{ji})_{i,j,u,v=1,2}$ of $k$. In the relations $\chi(i, j, k, l) = 0$ we denote

\[c_{11} = x, \quad c_{12} = y, \quad c_{21} = z, \quad c_{22} = t.\]

As in the previous proposition we find a relation between the generators. From the relations

\[(2, 2, 2, 2) : r(1 - rq)ty + rqt^2 = qz - rqt \quad \text{and} \quad (1, 2, 2, 2) : (1 - rq)ty - qt^2 = qx + rqt\]

it follows that

\[r(q(x + ry)) = q(z + rt).\]
We have two cases.
1. If \( q \neq 0 \) then the relation between the generators is the same as in the previous proposition
\[
(2) \quad r(x + ry) = z + rt.
\]
With relation (2) the relations \( \chi(i, j, k, l) = 0 \) are obtained from \((1, 1, 1, 1), \ldots, (1, 2, 2, 2)\) and (2). Now, if we denote
\[
A = y/q, \quad B = t + (1 - rq)/qy, \quad C = x + ry
\]
then we have
\[
x = C - rqA, \quad y = qA, \quad z = rC - rB + r(1 - rq)A, \quad t = B - (1 - rq)A
\]
which means that \( D^2_{q,r}(k) \) is generated by \( A, B, C \) with relations obtained by replacing \( x, y, z, t \) with \( A, B, C \). Looking closely at the relations obtained and keeping in mind that \( q \neq 0 \) we obtain that \( D^2_{q,r}(k) \) is generated by \( A \) and \( B \) with relations
\[
AB = 0, \quad B^3 = B^2.
\]
Computing \( \Delta \) and \( \varepsilon \) for \( A \) and \( B \) we have
\[
\Delta(A) = B^2 \otimes A + A \otimes B - A \otimes A, \quad \varepsilon(A) = 0
\]
\[
\Delta(B) = B^2 \otimes A + B \otimes B - B \otimes A, \quad \varepsilon(B) = 1.
\]
2. If \( q = 0 \) we have the relation (1), too. This time the relation follows from
\[
(1,1,1,1) : x^2 = x + ry \quad \text{and} \quad (2,1,1,1) : rx^2 = z + rt.
\]
Because \( q = 0 \) then among the elements \((x^k_{u,v})\) the nonzero elements are
\[
x^{11}_{11} = x^{21}_{21} = 1, \quad x^{12}_{11} = x^{22}_{21} = r
\]
It follows like in the first case that the relations \( \chi(i, j, k, l) = 0 \) are equivalent with the next nine relations:
\[
x^2 = x + ry, \quad xy = 0, \quad yx = 0, \quad y^2 = 0
\]
\[
zx = 0, \quad zy = 0, \quad tx = x + ry, \quad ty = 0
\]
\[
(2) : r(x + ty) = z + rt
\]
If, we denote
\[
A = t - ry, \quad B = y, \quad C = x + ry
\]
it follows that
\[
x = C - rB, \quad y = B, \quad z = r(C - A - rB), \quad t = A + rB
\]
which shows that the algebra \( D^2_{0,r}(k) \) is generated by \( A, B \) and \( C \).
After rewriting the relations and keeping the independent ones we obtain that the relations that define the algebra \( D^2_{0,r}(k) \) are
\[
B^2 = CB = BC = AB = 0, \quad C^2 = AC = C.
\]
Computing the coalgebra structure on \( D^2_{0,r}(k) \) we find our statement. □
PROPOSITION 2.3. 1. If \( q \neq 0 \) the bialgebra \( E^2_{q,t}(k) \) is the free algebra generated by \( A, B \) with the relation
\[
B^3 = B^2.
\]
The comultiplication \( \Delta \) and the counity \( \varepsilon \) are given by
\[
\begin{align*}
\Delta(A) &= B^2 \otimes A + A \otimes B - A \otimes A, \quad \varepsilon(A) = 0 \\
\Delta(B) &= B^2 \otimes A + B \otimes B - B \otimes A, \quad \varepsilon(B) = 1.
\end{align*}
\]
2. If \( q = 0 \) the bialgebra \( E^2_{0,t}(k) \) is the free algebra generated by \( A, B \) and \( C \) with the relations
\[
B^2 = BC = CB = 0, \quad C^2 = C.
\]
The comultiplication \( \Delta \) and the counity \( \varepsilon \) are given by
\[
\begin{align*}
\Delta(A) &= A \otimes A, \quad \varepsilon(A) = 1 \\
\Delta(B) &= C \otimes B + B \otimes A, \quad \varepsilon(B) = 0 \\
\Delta(C) &= C \otimes C, \quad \varepsilon(C) = 1.
\end{align*}
\]
Proof. We consider \( B = \{ m_1 \otimes m_1, m_1 \otimes m_2, m_2 \otimes m_1, m_2 \otimes m_2 \} \) a basis of \( M \otimes M \). With respect to this basis \( R = f \otimes f \) is represented by the matrix
\[
\begin{pmatrix}
(1 - rq)^2 & q(1 - rq) & q(1 - rq) & q^2 \\
r(1 - rq)^2 & rq(1 - rq) & rq(1 - rq) & rq^2 \\
r(1 - rq)^2 & rq(1 - rq) & rq(1 - rq) & rq^2 \\
r^2(1 - rq)^2 & r^2q(1 - rq) & r^2q(1 - rq) & r^2q^2
\end{pmatrix}
\]
In the relations \( \chi(i,j,k,l) = 0 \) we denote
\[
c_{11} = x, \quad c_{12} = y, \quad c_{21} = z, \quad c_{22} = t.
\]
From the relations \((1,1,2,2)\) and \((2,1,2,2)\) we obtain
\[
rq^2(x + ry) = q^2(z + rt).
\]
1. If \( q \neq 0 \) the above relation becomes
\[(3) \quad r(x + ry) = z + rt.
\]
With relation \((1)\) the sixteen relations \( \chi(i,j,k,l) = 0 \) are obtained from the following: \((1,1,1,1), (1,1,1,2), (1,1,2,1), (1,1,2,2)\). Now, denote
\[
A = \frac{y}{q}, \quad B = t + \frac{1 - rq}{q}y, \quad C = x + ry.
\]
After computations it follows that \( E^2_{q,r}(k) \) is the free algebra generated by \( A \) and \( B \) with the relation
\[
B^3 = B^2
\]
Computing \( \Delta \) and \( \varepsilon \) for the generators \( A \) and \( B \) we have
\[
\begin{align*}
\Delta(A) &= B^2 \otimes A + A \otimes B - A \otimes A, \quad \varepsilon(A) = 0 \\
\Delta(B) &= B^2 \otimes A + B \otimes B - B \otimes A, \quad \varepsilon(B) = 1.
\end{align*}
\]
2. If \( q = 0 \) we have the relation \((3)\), too. This time the relation follows from \((1,1,1,1)\) and \((2,1,1,1)\). Because \( q = 0 \) then among the elements \( (x^i_{j,kr}) \) the nonzero elements are
\[
x_{11}^{11} = 1, \quad x_{11}^{12} = x_{11}^{21} = r, \quad x_{11}^{22} = r^2
\]
\[9\]
It follows like in the first case that $E^2_{q,r}(k)$ is given by the relations $(1,1,1,1),(1,1,1,2), (1,2,1,1),(1,1,2,2)$ and (3). If, we denote
\[ A = t - ry, \quad B = y, \quad C = x + ry \]
it follows that $E^2_{0,r}(k)$ is generated by $A, B$ and $C$ with the relations
\[ B^2 = CB = BC = 0, \quad C^2 = C. \]
Computing the coalgebra structure on $E^2_{0,r}(k)$ we find our statement. \( \square \)

**Remark 2.4** Let us notice that $r$ does not interfere in the description of $E^2_{q,r}(k)$, so, we can denote $E^2_{q,r}(k)$ by $E^2_q(k)$. Also, for any $q$ and $q't$ nonzero elements of $k$, the bialgebras $E^2_q(k)$ and $E^2_{q'}(k)$ are isomorphic. Thus, we can resume to study $E^2_1(k)$ and $E^2_0(k)$.

### 2.2. The $n$ dimensional case.

Let $n \in \mathbb{N}$ and $M$ be a $n$-dimensional vector space over a field $k$ and $\{m_1, m_2, \ldots, m_n\}$ a basis of $M$. Let
\[ \phi : \{1, \ldots, n\} \to \{1, \ldots, n\} \]
\[ \beta : \{1, \ldots, n\} \times \{1, \ldots, n\} \to k \]
be functions such that $\phi^2(i) = \phi(i)$, $\beta(i,i) = 1$, for every $i \in \{1, \ldots, n\}$. We can define $f_{\phi, \beta} \in \text{End}_k(M)$ in the following way
\[ f_{\phi, \beta}(m_i) = \delta_{i \in \text{Im} \phi} \sum_{a \in \phi^{-1}(i)} \beta(i,a)m_a. \]

It can be easily shown that $f_{\phi, \beta}$ is an idempotent. These endomorphisms have a strongly geometric semification. We will evince now just one example, other geometrical meanings of $f_{\phi, \beta}$ are left to the reader.

Let $h \in \{1, \ldots, n\}$, and let $\phi$ be the map defined by $\phi(i) = i$, if $i \leq h \phi(i) = h$, if $i > h$.

Let $\beta(i,j) = \delta_{i,j}$, for $i, j \in \{1, \ldots, n\}$. Then $f_{\phi, \beta}$ is the projection on the $h$-plane
\[ m_{h+1} = \ldots m_n = 0. \]

We denote
\[ E^n_{\phi, \beta}(k) = B(f_{\phi, \beta} \otimes f_{\phi, \beta}). \]

To give the description of this bialgebra is now a standard computation. We will only report the results.

**PROPOSITION 2.5.** The bialgebra $E^n_{\phi, \beta}(k)$ is the free algebra generated by $c_{ij} \in \otimes_{i,j=1,n} \text{ and relations } (i,j,k,l)$ : 
\[ \sum_{u,v} \delta_{v \in \text{Im}(\phi)} \delta_{u \in \text{Im}(\phi)} \sum_{\substack{i \in \phi^{-1}(u) \atop j \in \phi^{-1}(v)}} \beta(v,i) \beta(u,j) c_{uk} c_{ul} = \delta_{k \in \text{Im}(\phi)} \sum_{a \in \phi^{-1}(l)} \beta(l,\alpha) \beta(k,i) c_{ia}. \]
3. FINITE DIMENSIONAL QUOTIENTS

In this section we will construct examples of finite dimensional noncommutative nonco-commutative bialgebras arising from quotients of \( E_q^2(k) \). It can be observed that our description of \( E_q^2(k) \) is slightly different from the one given in [6]. The reason of using this description is that it makes easier the study of biideals. In [10] the author proved that, over an algebraically closed field of characteristic zero, any \( p \) dimensional Hopf algebra is isomorphic to the groupal algebra \( k[Z_p] \). We will construct examples of noncommutative nonco-commutative bialgebras in prime dimension (we remind that in [6] the author constructed an example of such a bialgebra, but under the assumption \( char(k) = 2 \).

**Lemma 3.1.** 1. For any \( n \geq 2 \), \( I_{1,2n+1} \) the two-sided ideal of \( E_q^2(k) \) generated by \( A^n, AB^2 \) and \( BA \) is a biideal.

2. For any \( n \geq 3 \), \( I_{4,n+3} \) the two-sided ideal of \( E_q^2(k) \) generated by \( A^n, AB^2, BA \) and \( A^2B \) is a biideal.

**Proof.** 1. First of all notice that \( B^2 \) is a grouplike element

\[
\Delta(B^2) = B^2 \otimes B^2.
\]

It follows that

\[
\Delta(AB^2) = B^2 \otimes AB^2 + AB^2 \otimes B^2 - AB^2 \otimes AB^2
\]

\[
\Delta(BA) = B^2A \otimes AB - B^2A \otimes A^2 - BA \otimes AB + BA \otimes A^2 + B^2 \otimes BA + BA \otimes B^2 - BA \otimes BA.
\]

Hence \( \Delta(AB^2) \) and \( \Delta(BA) \) belong to \( I_{1,2n+1} \otimes E_q^2(k) + E_q^2(k) \otimes I_{1,2n+1} \).

Let us look now at \( \Delta(A^n) \)

\[
\Delta(A^n) = (B^2 \otimes A + A \otimes B - A \otimes A)^n
\]

So, \( \Delta(A^n) \) belongs to \( \text{Span}(X_1^{n_1}X_2^{n_2}\cdots X_k^{n_k} \mid \text{where } n, k \in \mathbb{N}, \sum_{i=1}^{k} n_i = n \text{ and } X_i \in \{B^2 \otimes A, A \otimes B, A \otimes A\}) \). It is enough to prove that every such an element belongs to \( I_{1,2n+1} \otimes E_q^2(k) + E_q^2(k) \otimes I_{1,2n+1} \).

If \( k = 1 \) then \( X_1^n \in I_{1,2n+1} \otimes E_q^2(k) + E_q^2(k) \otimes I_{1,2n+1} \), for any \( X_1 \in \{B^2 \otimes A, A \otimes B, A \otimes A\} \).

If \( k \geq 2 \) and \( X_k \) it is not \( A \otimes B \) then it follows that \( X_1^{n_1}X_2^{n_2}\cdots X_k^{n_k} \in E_q^2(k) \otimes I_{1,2n+1} \), since \( X_1^{n_1}X_2^{n_2}\cdots X_k^{n_k} \) has the form \( t \otimes u \) and \( u \) is either \( A^n \) or an element from the ideal generated by \( BA \).

If \( k \geq 2 \) and \( X_k = A \otimes B \) then \( X_1^{n_1}X_2^{n_2}\cdots X_k^{n_k} = t \otimes u \) and \( t \) is either \( A^n \) or an element from the ideal generated by \( BA \). So, we have proved that

\[
\Delta(I_{1,2n+1}) \subseteq I_{1,2n+1} \otimes E_q^2(k) + E_q^2(k) \otimes I_{1,2n+1}.
\]

Of course, we also have \( \varepsilon(I_{1,2n+1}) = 0 \).

2. First notice that \( I_{1,2n+1} \subset I_{4,n+3} \). So, after an easy computation for \( \Delta(A^2B) \) the conclusion follows. \( \square \)

**Lemma 3.2.** 1. For any \( n \geq 2 \), \( J_{1,2n+1} \) the two-sided ideal of \( E_0^2(k) \) generated by \( A^n - A, AC - C, CA - C, BA \) is a biideal.

2. For any \( n \geq 3 \), \( J_{4,n+3} \) the two-sided ideal of \( E_0^2(k) \) generated by \( A^n - A^2, AC - C, CA - C, BA, A^2B \) is a biideal.
Proof. Because $A$ and $C$ are grouplikes the proof has no difficulty. □

With the notations from the previous lemmas we have

PROPOSITION 3.3. 1. The bialgebra $E_1^2(k)/I_{1,2n+1}$ which is the free algebra generated by $A$ and $B$ with relations

$$B^3 = B^2, \quad A^n = AB^2 = BA = 0$$

has dimension $2n + 1$.

2. The bialgebra $E_0^2(k)/J_{1,2n+1}$ which is the free algebra generated by $A, B$ and $C$ with relations

$$B^2 = BC = CB = BA = 0, \quad CA = AC = C^2 = C, \quad A^n = A$$

has dimension $2n + 1$.

Proof. In both cases the proof is based on the ”Diamond Lemma” which gives us a way of finding bases in associatives $k$-algebras. The technique is standard and it is fully illustrated in [2]. That is why we will only report the results of our computations.

1. $\{1, A, A^2, \ldots, A^{n-1}, B, AB, A^2B, \ldots A^{n-1}B, B^2\}$ is a basis for $E_1^2(k)/I_{1,2n+1}$.

2. $\{1, A, A^2, \ldots, A^{n-1}, B, AB, A^2B, \ldots A^{n-1}B, C\}$ is a basis for $E_0^2(k)/J_{1,2n+1}$. □

PROPOSITION 3.4. 1. The bialgebra $E_1^2(k)/I_{4,n+3}$ which is the free algebra generated by $A$ and $B$ with relations

$$B^3 = B^2, \quad A^n = BA = AB^2 = A^2B = 0$$

has dimension $n + 3$.

2. The bialgebra $E_0^2(k)/J_{4,n+3}$ which is the free algebra generated by $A, B$ and $C$ with relations

$$B^2 = BC = CB = BA = A^2B = 0, \quad C^2 = AC = CA = C, \quad A^n = A^2$$

has dimension $n + 3$.

Proof. 1. $\{1, A, A^2, \ldots, A^{n-1}, B, AB, B^2\}$ is a basis for $E_1^2(k)/I_{4,n+3}$.

2. $\{1, A, A^2, \ldots, A^{n-1}, B, AB, C\}$ is a basis for $E_0^2(k)/J_{4,n+3}$. □

Remark 3.5 All the bialgebras described in this section are noncommutative and nonco-commutative.

4. $M_2(k)$ AND $GL_2(k)$

In the previous section we presented some quotients of $E_q^2(k)$. Since $E_q^2(k) = E_{\phi,0}^2(k)$ for $\phi(1) = \phi(2) = 2$ and $\beta(2, 1) = \beta(2, 2) = 1, \beta(1, 1) = \beta(1, 2) = 0$, we may ask what is happening to the same bialgebra in the three dimensional case. So, in this section we will study $E_{\phi,0}^3(k)$ for $\phi(i) = 3$ and $\beta(3, i) = 1, \beta(2, i) = \beta(1, i) = 0$, for $i \in \{1, 2, 3\}$ (let us denote this bialgebra by $E^3(k)$). We will obtain the bialgebra $M_2(k)$ as a quotient of $E^3(k)$.

PROPOSITION 4.1. The bialgebra $E^3(k)$ is the free algebra generated by $A, B, C; D, E, F$ with relations

$$E^3 = E^2, \quad EF = FE = F^2 = 0.$$
The comultiplication $\Delta$ and the counit $\varepsilon$ are given by

$$
\begin{align*}
\Delta(A) &= B \otimes A + A \otimes C + (E^2 - 1) \otimes A + (1 - B) \otimes F, \quad \varepsilon(A) = 0 \\
\Delta(B) &= B \otimes B + A \otimes D + (1 - E^2) \otimes (1 - B) + (1 - B) \otimes (1 - E), \quad \varepsilon(B) = 1 \\
\Delta(C) &= D \otimes A + C \otimes C - D \otimes F, \quad \varepsilon(C) = 1 \\
\Delta(D) &= C \otimes D + D \otimes B + D \otimes (E - 1), \quad \varepsilon(D) = 0 \\
\Delta(E) &= E \otimes E + (E^2 - E) \otimes (1 - B) - F \otimes D, \quad \varepsilon(E) = 1 \\
\Delta(F) &= (E^2 - E) \otimes A + E \otimes F + F \otimes C, \quad \varepsilon(F) = 0.
\end{align*}
$$

Proof. We know from proposition 2.5 that the non trivial relations are

$$
\begin{align*}
c_{31}c_{31} &= 0, \quad c_{31}c_{32} = 0, \quad c_{31}c_{33} = 0, \\
c_{32}c_{31} &= 0, \quad c_{32}c_{32} = 0, \quad c_{32}c_{33} = 0, \\
c_{33}c_{31} &= 0, \quad c_{33}c_{32} = 0, \quad c_{33}c_{33} = c_{11} + c_{12} + c_{13}, \\
c_{33}c_{33} &= c_{21} + c_{22} + c_{23}, \quad c_{33}c_{33} = c_{31}c_{32}c_{33}.
\end{align*}
$$

We denote

$$
\begin{align*}
A &= c_{12}, \quad B = 1 - c_{13}, \quad C = c_{22} - c_{12}, \\
D &= c_{13} - c_{23}, \quad E = c_{33}, \quad F = c_{32}.
\end{align*}
$$

Writing again the relations, the comultiplication and the counit with these notations we obtain our statement. $\Box$

Remark 4.2 The bialgebra $E^3(k)$ it is not a Hopf algebra. One can try to localize it in order to obtain a Hopf algebra. Since $E^2$ is grouplike it should become invertible and so the following two relations should be added

$$
E = 1, \quad F = 0.
$$

Let $I$ be the two-sided ideal generated by $E - 1$ and $F$. Then $I$ is also a biideal and $E^3k/I$ is isomorphic as bialgebras to $M_2(k)$. Thus, our attempt to obtain a Hopf algebra conductes to $GL_2(k)$.

5. A GENERAL OPERATOR FOR THE HOPF EQUATION

All the solutions of the Hopf equation that have been presented until here were of the type $f \otimes g$. There exists an example of a solution which is not of this form, given in [6] but the field there has characteristic two. In this final section our goal is to define a general operator which is not of the type $f \otimes g$, without any assumption on the characteristic of the field.

THEOREM 5.1. Let $n \in \mathbb{N}$, $n \geq 2$. Let $M$ be an $n$-dimensional vector space over $k$, with basis $\{m_1, m_2, \ldots, m_n\}$. Let $\phi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be a map such that $\phi^2 = \phi$. Then the map

$$
R_\phi : M \otimes M \to M \otimes M, \quad R_\phi(m_i \otimes m_j) = \delta_{\phi^{-1}(i)} \delta_{ij} \sum_{a \in \phi^{-1}(i)} \sum_{b=1}^{n} m_a \otimes m_b
$$

is a solution of the Hopf equation.
Proof. We denote \( R = R_\phi \). We will verify that \( R^{12} R^{23} = R^{23} R^{13} R^{12} \). All the computations are based on the identity \( \phi^2 = \phi \).

\[
R^{12} R^{23}(m_i \otimes m_j \otimes m_k) = R^{12}(\delta_{j \in \text{Im} \phi} \delta_{jk} \sum_{a \in \phi^{-1}(j)} \sum_{y=1,n} m_i \otimes m_a \otimes m_y)
\]

\[
= \delta_{i \in \text{Im} \phi} \delta_{j \in \text{Im} \phi} \delta_{jk} \sum_{a \in \phi^{-1}(j)} \sum_{c \in \phi^{-1}(i)} \sum_{x=1,n} m_c \otimes m_x \otimes m_y
\]

\[
= \delta_{i \in \text{Im} \phi} \delta_{j \in \text{Im} \phi} \delta_{jk} \delta_{\phi(i)j} \sum_{c \in \phi^{-1}(i)} m_c \otimes m_x \otimes m_y.
\]

Using \( \phi^2 = \phi \) we get

\[
\delta_{i \in \text{Im} \phi} \delta_{j \in \text{Im} \phi} \delta_{jk} \delta_{\phi(i)j} = \delta_{ij} \delta_{jk} \delta_{\text{Im} \phi}.
\]

Thus,

\[
(4) \quad R^{12} R^{23}(m_i \otimes m_j \otimes m_k) = \delta_{ij} \delta_{jk} \delta_{\text{Im} \phi} \sum_{c \in \phi^{-1}(i)} m_c \otimes m_x \otimes m_y.
\]

\[
R^{23} R^{12}(m_i \otimes m_j \otimes m_k) = R^{23} \sum_{a \in \phi^{-1}(i)} \sum_{b \in \phi^{-1}(j)} \sum_{c \in \phi^{-1}(a)} \sum_{d \in \phi^{-1}(b)} \sum_{x,y=1,n} m_c \otimes m_b \otimes m_d.
\]

Thus, we have

\[
R^{23} R^{12}(m_i \otimes m_j \otimes m_k) = \delta_{i \in \text{Im} \phi} \delta_{ij} \sum_{a \in \phi^{-1}(i)} \delta_{a \in \text{Im} \phi} \delta_{ak} \sum_{c \in \phi^{-1}(a)} \delta_{b \in \text{Im} \phi} \delta_{bd} \sum_{x \in \phi^{-1}(b)} m_c \otimes m_x \otimes m_y.
\]

That is

\[
R^{23} R^{12}(m_i \otimes m_j \otimes m_k) = \delta_{i \in \text{Im} \phi} \delta_{ij} \delta_{\phi(k)j} \delta_{\phi(k)i} \sum_{c \in \phi^{-1}(k)} \delta_{b \in \text{Im} \phi} \delta_{bd} \sum_{x \in \phi^{-1}(b)} m_c \otimes m_x \otimes m_y.
\]

Using \( \phi^2 = \phi \) we have the following

\[
\delta_{i \in \text{Im} \phi} \delta_{k \in \text{Im} \phi} \delta_{ij} \delta_{\phi(k)i} = \delta_{ij} \delta_{jk} \delta_{\text{Im} \phi} = \delta_{ij} \delta_{jk} \delta_{\text{Im} \phi}.
\]

We can also observe that

\[
\sum_{c \in \phi^{-1}(k)} m_c \otimes m_x \otimes m_y = \sum_{b \in \phi^{-1}(k)} \sum_{c \in \phi^{-1}(k)} m_c \otimes m_x \otimes m_y
\]

\[
= \sum_{c \in \phi^{-1}(k)} m_c \otimes m_x \otimes m_y
\]
because

$$\bigcup_{b \in \text{Im} \phi} \phi^{-1}(b) = \{1, 2, \ldots, n\}. $$

Thus, we finally have

(5) \[ R^{12} R^{13} R^{12}(m_i \otimes m_j \otimes m_k) = \delta_{ij} \delta_{jk} \sum_{c \in \phi^{-1}(k)} \sum_{x, y = 1, n} m_c \otimes m_x \otimes m_y. \]

Using (4) and (5) we can conclude that $R^{12} R^{23} = R^{23} R^{13} R^{12}$. \( \square \)

**Remark 5.2.** If $\text{Im} \phi$ has at least two elements then $R_\phi$ is not of the type $f \otimes g$. Indeed, assume that $R = f \otimes g$. We denote

$$f(m_i) = \sum_{a=1}^{n} \alpha_{ia} m_a, \quad g(m_j) = \sum_{b=1}^{n} \beta_{jb} m_b,$$

where $\alpha_{ia}, \beta_{jb}$ are scalars from $k$. With these notations we get

$$f(m_i) \otimes g(m_j) = \sum_{a, b=1}^{n} \alpha_{ia} \beta_{jb} m_a \otimes m_b$$

and using the definition of $R$ it follows that

$$\alpha_{ia} \beta_{jb} = \delta_{i \in \text{Im} \phi} \delta_{ij} \delta_{a \in \phi^{-1}(i)},$$

for any $i, j, a, b \in \{1, \ldots, n\}$. Let $i, j$ be two different elements from $\text{Im} \phi$ and let $a \in \phi^{-1}(i), b \in \phi^{-1}(j)$. Then $\alpha_{ia} \beta_{ib} = 1, \alpha_{ia} \beta_{jb} = 0, \alpha_{ja} \beta_{jb} = 1$. From the third relation follows that $\beta_{jb}$ is not zero and from the first one follows that $\alpha_{ia}$ is not zero, which is in contradiction with the second relation.

**COROLLARY 5.3.** Let $R_\phi$ be the solution of the Hopf equation described above. If $B(R_\phi)$ is the bialgebra associated with $R_\phi$, then $B(R_\phi)$ is the free algebra generated by $\{c_{ij}\}_{i,j=1,n}$ with relations

$$(i, j, k, l) : \ c_{\phi(i)k} c_{\phi(l)i} = \delta_{kl} \sum_{a=1}^{n} \delta_{\phi(a)l} c_{ia}$$

**Proof.** Writing

$$R(m_u \otimes m_u) = \sum_{i,j=1}^{2} x_{uv} m_i \otimes m_j$$

we get $X_{uv}^{ji} = \delta_{\phi(i)u} \delta_{u, v}$, for every $i, j, u, v = 1, n$. Replacing these in the original relations $\chi(i, j, k, l)$ we find that they have the stated form. \( \square \)

**COROLLARY 5.4.** Let $\phi = id$ the identity on $\{1, \ldots, n\}$. Then $B(R_{id})$ is the free algebra generated by $\{c_{ij}\}_{i,j=1,n}$ with relations

$$(i, j, k, l) : \ c_{ik} c_{il} = \delta_{kl} c_{ik}.$$
COROLLARY 5.5. Let \( \phi(i) = n - 1 \) for \( i \in \{1, \ldots, n - 1\} \), \( \phi(n) = n \). Then \( B(R_\phi) \) is the free algebra generated by \( \{c_{ij}\}_{i,j=1,n} \) with relations
\[
\begin{align*}
(i,j,k,l) & : \quad c_{n-1,k}c_{n-1,l} = \delta_{kl}, \quad i \neq n, l \neq n, n-1 \\
(i,j,k,n-1) & : \quad c_{n-1,k}c_{n,n-1} = \delta_{kn}c_{n-1} - \sum_{\alpha=1}^{n-1} c_{\alpha}, \quad i \neq n \\
(i,j,k,n) & : \quad c_{n,k}c_{n-1,n} = \delta_{kn}c_{in}, \quad i \neq n \\
(n,j,k,l) & : \quad c_{n,k}c_{n,l} = \delta_{kl}, \quad l \neq n, n-1 \\
(n,j,k,n-1) & : \quad c_{n,k}c_{n,n-1} = \delta_{kn}c_{n} - \sum_{\alpha=1}^{n-1} c_{\alpha} \\
(n,j,k,n) & : \quad c_{n,k}c_{n,n} = \delta_{kn}c_{nn}.
\end{align*}
\]

COROLLARY 5.6. Let \( n = 4 \), \( \phi(i) = 1 \) for \( i = 1, 2 \), \( \phi(j) = 3 \), for \( j = 3, 4 \). Then \( B(R_\phi) \) is the free algebra generated by \( \{c_{ij}\}_{i,j=1,n} \) with relations
\[
\begin{align*}
(i,j,k,l) & : \quad c_{1,k}c_{1,l} = \delta_{kl}, \quad i = 1, 2, l \neq 1, 3 \\
(i,j,k,1) & : \quad c_{1,k}c_{1,1} = \delta_{k1}(c_{i1} + c_{i2}), \quad i = 1, 2 \\
(i,j,k,3) & : \quad c_{1,k}c_{1,3} = \delta_{k3}(c_{i3} + c_{i4}), \quad i = 1, 2 \\
(i,j,k,l) & : \quad c_{2,k}c_{2,l} = \delta_{kl}, \quad i = 3, 4, l \neq 1, 3 \\
(i,j,k,1) & : \quad c_{2,k}c_{2,1} = \delta_{k1}(c_{i1} + c_{i2}), \quad i = 3, 4 \\
(i,j,k,3) & : \quad c_{1,k}c_{2,3} = \delta_{k3}(c_{i3} + c_{i4}), \quad i = 3, 4.
\end{align*}
\]

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