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DISPLAY LOGIC

1. INTRODUCTION

I formulate a Gentzen consecution calculus for an indefinite number of logics all mixed together, including boolean (two valued), intuitionistic, relevance, and (various) modal logics. This is accomplished by augmenting and refining the structural ideas of Gentzen (1934). The key feature of the calculus permitting control in the presence of multiple logics is this: every "positive part" of a consecution can be displayed as the consequent, standing alone, of an equivalent consecution; and every "negative part" can be displayed as the antecedent, standing alone, of an equivalent consecution; such a calculus I call a "Display logic". The first spin-off is a generalized Elimination theorem. (Some terminology, such as "consecution calculus", which translates "Sequenzen Kalkül", comes from Anderson and Belnap (1975); other terminology comes from Curry (1963).

Section 2 outlines the grammatical structures needed for Display logic, emphasizing the ways in which it develops themes in Gentzen. Section 3 explains the system itself in a wholesale way, but without attention to the special features of the various calculuses that Display logic can treat. Then the rules are divided into (1) display equivalences, (2) structural rules, and (3) connective postulates. Section 4 states and proves a wholesale (cut) Elimination theorem. Section 5 discusses in more detail how various logics appear in the context of Display logic. Section 6 elaborates a series of possibilities.

2. GRAMMAR

The grammar of any consecution calculus is complex (though the complexity is frequently played down by means of reliance on geometrical intuitions; see Section 6.8 for a related point), and because we are treating so many logics simultaneously, the grammar of Display logic is even more so.

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2.1. Indices and Families

We need a way of distinguishing the connectives of one logic from those of another; to take a familiar example, there is the implication of boolean logic and there is the strict implication of **S4**. To distinguish the similar connectives of these logics, and others, I postulate a set of *indices*, the idea, in first approximation, being that each logic shall be associated with a distinct index. To motivate the second approximation, we should recognize that some well-known logics are "hybrid" in the sense that they involve connectives of more than one kind; the easiest example is a modal logic, which treats both modal connectives and boolean connectives. For this reason, we shall associate indices not with historically given "logics" *per se*, but with *"families"* of connectives. For example, to treat **S4** we shall need indices for two families, the modal **S4** family for its modal connectives and the boolean family for its boolean connectives.

Because many features of Display logic are independent of which families are considered, I shall be vague about precisely which indices are postulated; but as examples I shall use the following indices:

EXAMPLE 2.1. (Indices).

Index	Family
b	boolean (for two valued logic)
$s4$	modal family for **S4** (its modal connectives)
r	relevance family (for relevance connectives of **R**)
h	intuitionist family ("h" for Heyting)

Names for other specific indices, or variables ranging over all indices, can be introduced *ad hoc*.

2.2. Formula-connectives and Structure-connectives

Apart from the multiplicity engendered by the families, the connectives, as (implicitly) in Gentzen, are of two kinds: *formula-connectives*, which take formulas into formulas, and *structure-connectives*, which take structures (analogs of Gentzen's sequences) into structures. The idea of multiple families of *formula*-connectives is old; the idea of characterizing these connectives by using multiple families of *structure*-connectives is due to Dunn (1973, 1975).

There are in each family several formula-connectives (which is usual) and several structure-connectives (which is not usual). Consider Gentzen's "*L*" calculuses. There structure is carried by commas (their meaning is context-sensitive, since they signify conjunction on the left and disjunction on the right; they are of no fixed polyadicity) and by the empty symbol (its meaning is also context-sensitive, since it signifies truth on the left and falsity on the right). Here there are three ways in which these structural ideas are developed.

(1) Positive structuring, carried by a structure-connective $\circ$, is always binary, never, as with Gentzen's commas, of no fixed polyadicity.[1] The meaning of $\circ$ remains, like Gentzen's comma, context-sensitive.

(2) There is a negative structuring $*$; it is 1-place, and is essential in realizing the "display" feature.

(3) There is a zero-place item of structure, I, which replaces the empty symbol of Gentzen (1934). This item carries a heavy burden in accommodating multiple families; and is specially useful in connection with modal logics. Its meaning, too, is context-sensitive.

To repeat: for each family, there are several formula-connectives and several structure-connectives. I catch this idea by postulating a list of *generic connectives*, each of which is a function that maps each family-index into a specific connective of the family associated with that index. With each of these generic connectives is associated a fixed number of *places*. From among the many possibilities for generic connectives, I choose the following (others are added later).

Generic connective	Type of arguments and values	Places	Approximate reading
t	formula	0	truth
f	formula	0	falsity
$\sim$	formula	1	negation
$\Box$	formula	1	necessity
$\Diamond$	formula	1	possibility
$\wedge$	formula	2	conjunction
$\vee$	formula	2	disjunction
$\rightarrow$	formula	2	implication
I	structure	0	truth/falsity
$*$	structure	1	negation
$\circ$	structure	2	conjunction/disjunction

Warning: the given "reading" must be understood in context; for the properties of the connectives differ drastically from family to family.

EXAMPLE 2.2 (Families and logics). Classical two valued logic is formulated using a single family, the boolean family. For example, $\sim_b$ is boolean negation, and $\rightarrow_b$ is material implication. $\Box_b$ in this family is just the identity connective, as is $\Diamond_b$. See 5.1 below.

The logic **S4** is hybrid, including both the boolean and the **S4** families: the extensional connectives belong to the former, the modal connectives to the latter. See 5.6 below. For example, a typical distribution law would relate $\Box_{s4}(A \rightarrow_b B)$ and $\Box_{s4} A \rightarrow_b \Box_{s4} B$.

The logic **R** uses both the boolean and the relevance families. The "standard" vocabulary of the formulation of Anderson and Belnap (1975) involves negation, implication, conjunction, and disjunction; in Display logic, these come through as $\sim_r$ and $\rightarrow_r$ from the relevance family, and $\wedge_b$ and $\vee_b$ from the boolean family; see 5.2 below. For example, we shall want to count $(A \wedge_b B) \rightarrow_r A$ as a logical truth, but not $(A \wedge_r B) \rightarrow_r A$.

Intuitionism is a one-family logic; intuitionist negation, however, is represented in Display logic not by $\sim_h$, but instead by another negation connective, $\tilde{\sim}_h$, introduced below in Section 3.3. See 5.7 below.

In order to avoid indices as much as possible, it is convenient to use the name of each generic connective also as a variable ranging over the specific connectives obtained by applying that generic connective to an index. Thus, "$\circ$" ranges over $\circ_b$, $\circ_{s4}$, etc. One more convention with the same purpose of reducing explicit mention of indices: whenever several names of generic connectives are used together in this way as variables, unless there is special indication to the contrary, they are to be taken as "ranging in tandem"; that is, they are all to be taken as denoting connectives (formula-connectives or structure-connectives) of the same family.

2.3. Formulas, Structures, and Consecutions

There is a set of *variables*, among which some are distinguished as *h-variables*. The *h*-variables play a special role in the intuitionist family.

Formulas are defined as usual: a variable is a formula, and so is the result of applying a formula-connective of any family to an appropriate number of arguments.

Structures are defined inductively: a formula is a structure, and so is the result of applying a structure-connective of any family to the appropriate number of arguments. A substructure (in the obvious sense) of a structure is *positive* or *negative* according as it is inside an even or odd number of $*$'s.

I call '$\vdash$' the *turnstile*. If X and Y are structures, then $X \vdash Y$ is a *consecution*. X is its *antecedent* and Y its *consequent*.[2] An *antecedent* [*consequent*] *part* of a consecution is defined as a positive part of its antecedent or negative part of its consequent [positive part of its consequent or negative part of its antecedent].

A *constituent* of a structure or consecution is an occurrence of a structure therein.

Variables are reserved as follows.

A, B, C, M	formulas
W, X, Y, Z	structures
S	consecutions

2.4. Interpretation

Display logic is essentially a proof theoretical tool; but some relations to semantic concepts are spelled out in Section 5. Furthermore, there are or can be many families, with a distinct "interpretation" for each, so that all that is possible here is an indication of how the various elements of any one family are related. Each structural constituent of a consecution has an interpretation as a formula, depending on whether it is an antecedent part or a consequent part: **I** is interpreted as **t** when an antecedent part and **f** when a consequent part; $*$ is always interpreted as negation $\sim$; and $\circ$ is interpreted as $\wedge$ when an antecedent part and $\vee$ when a consequent part (all with matching markings). In other words, I define two functions $a(X)$ and $c(X)$ from structures to formulas as follows.

$$a(A) = c(A) = A$$
(where A is any formula)

$$a(\mathbf{I}) = \mathbf{t} \qquad c(\mathbf{I}) = \mathbf{f}$$
$$a(X*) = \sim c(X) \qquad c(X*) = \sim a(X)$$
$$a(X \circ Y) = a(X) \wedge a(Y) \qquad c(X \circ Y) = c(X) \vee c(Y)$$

Then the consecution $X \vdash Y$ is interpreted as saying that the formula $a(X)$ implies the formula $c(Y)$, where "implies" is here a family-independent concept. Warning: keep in mind that the interpretation of **I**, as well as that of $\circ$, is context-dependent (like the empty symbol, as well as commas, in Gentzen's calculus); it does *not* behave like a formula.

Furthermore, in each family $\rightarrow$, $\Box$, and $\Diamond$ may be interpreted as follows: $A \rightarrow B$ as $\sim A \vee B$, $\Box A$ as $\mathbf{t} \rightarrow A$, and $\Diamond A$ as $\sim\Box\sim A$. This leaves the following "kernel" connectives of each family as requiring explanation:

Kernel connectives: **t**, **f**, $\sim$, $\wedge$, $\vee$.[3]

I repeat: these do not always have their usual meanings.

3. POSTULATES

3.1. Structural Axioms

There is one schema:

$$A \vdash A$$

where A is any variable.

3.2. Display-equivalence

The essence of Display logic is that for *every* family there are postulated certain bidirectional rules which allow any consequent [antecedent] part of a consecution S to be displayed as the entire consequent [antecedent] itself of a consecution equivalent to S (Theorem 3.2). Consecutions listed below are defined as *display-equivalent$_1$* to the others listed on the same line. (Here and always when I state rules, I am supposing that indices match on those connectives which are explicitly presented.)

$X \circ Y \vdash Z$	$X \vdash Y{*} \circ Z$	
$X \vdash Y \circ Z$	$X \circ Y{*} \vdash Z$	$X \vdash Z \circ Y$
$X \vdash Y$	$Y{*} \vdash X{*}$	$X{**} \vdash Y$

Display-equivalence is defined as the reflexive, transitive and symmetric closure of display-equivalence$_1$; that is, display-equivalent consecutions are those that are mutually inferable by application of the above bidirectional rules. Note that commutativity is postulated as a display-equivalence for $\circ$ only in the consequent.

FACT 3.1. $Y \vdash (X \circ Z{*}){*}$ may be added as a display-equivalence to the first line, $X \circ Z{*} \vdash Y$ to the second, and $X \vdash Y{**}$ to the third. Also items on the same line below are display-equivalent:

$X{*} \vdash Y$	$Y{*} \vdash X$
$X \vdash Y{*}$	$Y \vdash X{*}$

Proof. By diddling.

THEOREM 3.2 (Display theorem). Each antecedent part X of a consecution S can be displayed as the antecedent (itself) of a display-equivalent consecution $X \vdash W$; and the consequent W is determined only by the position of X in S, not by what X looks like. Similarly for consequent parts of S.

That is, let f be an $(n+1)$-ary operation on structures definable by (only) composition from the structure-connectives of the various families. Let the first argument of f always appear as a positive substructure of the result of applying f to $n+1$ arguments. For each such f there is a composition-definable f' such that for all $X, Y_1, \ldots, Y_n$, and Z, the following are display-equivalent:

$$f(X, Y_1, \ldots, Y_n) \vdash Z \qquad X \vdash f'(Z, Y_1, \ldots, Y_n).$$

Analogously for the first argument of f negative, and for $Z \vdash f(X, Y_1, \ldots, Y_n)$, with X positive, or negative.

Proof. Let X be an antecedent [consequent] part of S. Unless X is already the antecedent [consequent] of S, the "route" to X in S can only be through exactly one of $\circ$ in antecedent or consequent, or $*$ in antecedent or consequent. And in the former cases, X must be either in the left or right. The display-equivalences postulated, or noted in Fact 3.1, suffice for all cases.

To *display* X is to find $X \vdash f'(Z, Y_1, \ldots, Y_n)$, or similarly with X as consequent. The possibility of displaying X is the defining feature of Display logic.

EXAMPLE. The constituent X (second occurrence) in a consecution $(X \circ Y)* \circ (Z* \circ X*) \vdash W \circ Y$ can be displayed as consequent as follows by using the display-equivalences postulated or noted in Fact 3.1:
$Z* \circ X* \vdash ((X \circ Y)* \circ (W \circ Y)*)*$; $Z* \vdash X** \circ ((X \circ Y)* \circ (W \circ Y)*)*$;
$Z* \circ ((X \circ Y)* \circ (W \circ Y)*)** \vdash X**$; $Z* \circ ((X \circ Y)* \circ (W \circ Y)*)** \vdash X$.

3.3. Connective Postulates

There is as usual a pair of rules for each formula-connective. The Display Theorem 3.2, however, allows us to insist as a special feature that the formula with the newly introduced formula-connective stand alone. It is a further striking feature of Display logic that the *same* set of formula-connective postulates is used for *every* family; which is Display logic's own way of making sense out of everyone's sense of family resemblance.

Subscripts on all formula-connectives and structure-connectives in the statement of a given rule are supposed to match.

(t) $\quad \mathbf{I} \vdash \mathbf{t} \qquad \dfrac{\mathbf{I} \vdash X}{\mathbf{t} \vdash X}$

(f) $\quad \dfrac{X \vdash \mathbf{I}}{X \vdash \mathbf{f}} \qquad \mathbf{f} \vdash \mathbf{I}$

(∧) $\dfrac{X \vdash A \quad Y \vdash B}{X \circ Y \vdash A \wedge B}$ $\dfrac{A \circ B \vdash X}{A \wedge B \vdash X}$

(∨) $\dfrac{X \vdash A \circ B}{X \vdash A \vee B}$ $\dfrac{A \vdash X \quad B \vdash Y}{A \vee B \vdash X \circ Y}$

(→) $\dfrac{X \circ A \vdash B}{X \vdash A \to B}$ $\dfrac{X \vdash A \quad B \vdash Y}{A \to B \vdash X* \circ Y}$

(~) $\dfrac{X \vdash A*}{X \vdash {\sim}A}$ $\dfrac{A* \vdash X}{{\sim}A \vdash X}$

($\dot{\sim}$) $\dfrac{X \circ \mathbf{I} \vdash B*}{X \vdash \dot{\sim}B}$ $\dfrac{B* \vdash X}{\dot{\sim}B \vdash \mathbf{I}* \circ X}$

"$\dot{\sim}$" is in practice used only for intuitionist negation.

(□) $\dfrac{X \circ \mathbf{I} \vdash A}{X \vdash \Box A}$ $\dfrac{A \vdash X}{\Box A \vdash \mathbf{I}* \circ X}$

(◇) $\dfrac{X \vdash A}{(X* \circ \mathbf{I}*)* \vdash \Diamond A}$ $\dfrac{X* \circ A \vdash \mathbf{I}*}{\Diamond A \vdash X}$

As an example of how the Display Theorem 3.2 interacts with these postulates, I note the derivability of the form of the rule for → on the left due to Dunn (1975): from (1) $X \vdash A$ and (2) $\ldots B \ldots \vdash Z$ to infer $\ldots((A \to B) \circ X) \ldots \vdash Z$. First write (2) as $f(B) \vdash Z$, and for the f' promised by the Display theorem, get $B \vdash \mathrm{f}'(Z)$. Then $(A \to B) \vdash X* \circ \mathrm{f}'(Z)$ by the rule (→); $((A \to B) \circ X) \vdash \mathrm{f}'(Z)$ by display-equivalence; and $\mathrm{f}((A \to B) \circ X) \vdash Z = \ldots((A \to B) \circ X) \ldots \vdash Z$ by the Display theorem.

THEOREM 3.3 (Identity theorem). Though in Section 3.1 $A \vdash A$ is postulated only for *variables* A, in fact $A \vdash A$ is provable for all formulas A.

I take the Identity theorem to constitute half of what is required to show that the "meaning" of formulas in Display logic is not context-sensitive, but that instead formulas "mean the same" in both antecedent and consequent position. (The Elimination Theorem 4.4 below is the other half of what is required for this purpose.)

3.4. Reduction

As we have seen, the postulates governing formula-connectives are for all families the same; in contrast, families differ in regard to their structural postulates: from among all the possible structural rules one might think of, some are postulated for this family, some for that. At this point to say which structural rules are postulated for which families would only get in our way, since the central theorems of Section 4.3 go through in total independence of this decision, as long as the choice of structural postulates satisfies quite general conditions elaborated below in Section 4.2. Consequently, I shall first list a set of potentially adoptable rules, all of which satisfy these conditions; then I will give a few examples of which rules are postulated for which families, leaving it pretty much up in the air what the full story is. (As usual, all indices of explicitly presented connectives must match.) Some of the names of the rules come via Meyer and Routley (1972) from those for the combinators in Curry and Feys (1958); some do not correspond to any combinator.

(I+) $$\frac{X \vdash Y}{\mathbf{I} \circ X \vdash Y}$$

(I–) $$\frac{\mathbf{I} \circ X \vdash Y}{X \vdash Y}$$

(I–*K*) $$\frac{\mathbf{I} \vdash Y}{X \vdash Y}$$

(I*+) $$\frac{\mathbf{I} \vdash X}{\mathbf{I}* \vdash X}$$

(I*–) $$\frac{\mathbf{I}* \vdash X}{\mathbf{I} \vdash X}$$ [4]

(*B*) $$\frac{W \circ (X \circ Y) \vdash Z}{(W \circ X) \circ Y \vdash Z}$$

(*B*′) $$\frac{W \circ (X \circ Y) \vdash Z}{(X \circ W) \circ Y \vdash Z}$$

(*C*) $$\frac{(W \circ X) \circ Y \vdash Z}{(W \circ Y) \circ X \vdash Z}$$

(CI) $$\frac{X \circ Y \vdash Z}{Y \circ X \vdash Z}$$

(C/h) $$\frac{(W \circ X) \circ Y \vdash Z}{(W \circ Y) \circ X \vdash Z}$$

Restriction on (*C*/*h*): *Y* must be an *h*-variable (Section 2.3).

(CI/I) $$\frac{X \circ \mathbf{I} \vdash Y}{\mathbf{I} \circ X \vdash Y}$$

(C∘) $$\frac{W \circ (X \circ Y) \vdash Z}{X \circ (W \circ Y) \vdash Z}$$

(W) $$\frac{(X \circ Y) \circ Y \vdash Z}{X \circ Y \vdash Z}$$

(WI) $$\frac{X \circ X \vdash Y}{X \vdash Y}$$

(⊢WI) $$\frac{X \vdash Y \circ Y}{X \vdash Y}$$

(W∘) $$\frac{X \circ (X \circ Y) \vdash Z}{X \circ Y \vdash Z}$$

(KI) $$\frac{Y \vdash Z}{X \circ Y \vdash Z}$$

(K) $$\frac{X \vdash Z}{X \circ Y \vdash Z}$$

(⊢K) $$\frac{X \vdash Y}{X \vdash Y \circ Z}$$

(KI⊢WI) $$\frac{Y* \vdash Y}{X \vdash Y}$$

(K⊢WI) $$\frac{X \circ Z* \vdash Z}{X \circ Y \vdash Z}$$

(Brw) $$\frac{X \circ Y \vdash \mathbf{I}*}{Y \circ X \vdash \mathbf{I}*}$$

In Section 5 below there is a full (but hardly complete) discussion of which of these rules are postulated for which families; in the meantime, here is a brief indication for the four families of Example 2.1.

EXAMPLE 3.4 (Reduction examples). For the boolean family, the following rules are postulated, understood as applying when the displayed connectives are indexed with "b": **I**+, **I**−, **I***−, K, W. See 5.1 below. For the **S4** family, indexed with "$s4$", one has K**I**, K**I** $\vdash$ W**I**, $C\circ$, $W\circ$, **I***+, **I***−, **I**−, C**I**/**I**, W, B (or B'). See 5.6 below. For the relevance family, indexed with "r", one has **I**+, **I**−, **I***+, **I***−, B, C, W. See 5.2 below. For the intuitionist family, indexed with "h", one has all the same postulates as for the **S4** family (but of course governing connectives indexed with "h" instead of "$s4$") together with C/h. See 5.7 below.

Suppose it settled which structural rules have been postulated for which families. Then we can say that the premisses of postulated rules *reduce*$_1$ to their respective conclusions. *Reduction* is then the reflexive and transitive closure of display-equivalence together with reduction$_1$. For example, $X \vdash Z$ reduces$_1$ to $X \circ_b Y \vdash Z$, but not to $X \circ_r Y \vdash Z$, since K is postulated for the b-family, but not for the r-family (in the alternate language of note 15, we postulate K_b but not K_r).

A relation of *reduction for structures* can be piggy-backed on that for consecutions. Definitions: $X\ a\!\Rightarrow X'$ just in case $X \vdash Y$ reduces to $X' \vdash Y$, all Y; that is, just in case X reduces to X' in antecedent position. $X\ c\!\Rightarrow X'$ just in case $Y \vdash X$ reduces to $Y \vdash X'$, all Y; that is just in case X reduces to X' in consequent position. $X\ a\!\Leftrightarrow X'$ if both $X\ a\!\Rightarrow X'$ and $X'\ a\!\Rightarrow X$; and similarly for $X\ c\!\Rightarrow X'$; $X \Leftrightarrow X'$ just in case both $X\ a\!\Leftrightarrow X'$ and $X\ c\!\Leftrightarrow X'$.

FACT 3.5 (Reduction anywhere). Let X be an antecedent [consequent] part of S, and let S' result by putting X' for X in S. Then $X\ a\!\Rightarrow X'$ [$X\ c\!\Rightarrow X'$] implies that S reduces to S'. Hence, if $X\ a\!\Leftrightarrow X'$ [$X\ c\!\Leftrightarrow X'$, $X \Leftrightarrow X'$], then X and X' are interreplaceable in any antecedent context [any consequent context, any context].

Proof. The Display Theorem 3.2 suffices. Pictorially: use the display-equivalences to move X to explicit antecedent [consequent] position; change it to X'; then use the reverse sequence of display-equivalences to put X' where X was.

Except for specifying precisely which reduction rules go with which families, this completes the definition of the calculus: the structural axioms, the display-equivalences, and the formula-connective postulates, all of which are common to all families; and the separate reduction rules for each family. Call the system **DL** for "Display logic"; and let **DL**$\{i, j, \ldots\}$ be **DL** with its grammar restricted to the families associated with indices $i, j, \ldots$.

4. SUBFORMULA AND ELIMINATION THEOREMS

The "Subformula theorem" states that derivations are confined to subformulas of their conclusions; it is easy to verify, but the job is nevertheless done explicitly below.

The "Elimination theorem" states that the "Elimination rule" is admissible, where the "Elimination rule" appropriate for **DL** is the following:

$$\text{(ER)} \qquad \frac{(1)\ X \vdash M \qquad (2)\ M \vdash Y}{(3)\ X \vdash Y}$$

(Henceforth a reference to M in (1) or (2) refers to the displayed occurrence.)[5]

Evidently for any family you like, (ER) suffices for the representation in **DL** of the rule of modus ponens in the form, from $\mathbf{I} \vdash A \rightarrow B$ and $\mathbf{I} \vdash A$ to infer $\mathbf{I} \vdash B$, provided the structural rules postulated for the family imply that $\mathbf{I} \circ \mathbf{I}\ a\!\Rightarrow \mathbf{I}$; for $(A \rightarrow B) \circ A \vdash B$ is available in every family.

The Subformula and Elimination theorems are proved in three parts. First I give an "analysis" of **DL** that defines the notions of "parameter" and "congruence". Then I state eight conditions on such an analysis, and verify that **DL** in fact satisfies these conditions. Finally, I show that any calculus admitting an analysis satisfying these conditions must support the Subformula and Elimination theorems.

4.1. *Analysis, Parameter, Congruence*

Think of a consecution calculus, for example, **DL**, as being determined by a family of *rules*, under each of which falls a family of *inferences*, each inference with finitely many, possibly zero, premisses.[6] It appears that in any consecution calculus inference there are some constituents that are "held constant" when passing from the premisses to the conclusion – these are called the "parameters" – and, among these so-called parameters, some occurring in the premisses are "identified" with some in the conclusion – a relationship called "congruence". For some rules, e.g. the typical connective rules, there seems only one possible way of defining "parameter" and "congruence", whereas for others, e.g. the structural axiom, or the structural rules K, C, or W, there seem to be alternatives. In the terminology of this paper, it is the job of an "analysis" to provide a definite decision on the matter. Abstractly put, for each inference *Inf* of a calculus, an *analysis* defines constituents of the various consecutions of *Inf* as *parametric* (for *Inf*) or not; and it defines an equivalence relation of *congruence* (for *Inf*) on the parameters of *Inf*.[7] The *congruence class* of a constituent of *Inf* is the set of all constituents congruent to it.

Provision of such an analysis defining "parameter" and "congruence" for each inference sanctioned by its rules is the first step in showing that the consecution calculus **DL** defined in Section 3 admits rule (ER), regardless of which structural rules (drawn from Section 3.4) are postulated for which families. The following account is a bit sketchy, and convoluted by talk of "occurrences", but the idea should be clear.

Let *Inf* be an inference falling under a rule *Ru* of **DL**. *Inf* must then be obtained by assigning structures to the structure-variables and formulas to the formula-variables used in my statement of the rule *Ru*. (This is a remark about the procedure I employed above in describing the structural axioms, the display equivalences, the connective postulates, and the potentially adoptable reduction rules; it is *not* a remark about the nature of rules. See note 6.)

DEFINITION 4.1 (Analysis). According to the *present analysis*,[8] constituents occurring as part of occurrences of structures assigned to structure-variables (as they appear in my statement of the rules) are defined to be *parameters* of *Inf*; all other constituents are defined as nonparametric, including those assigned to formula-variables. Constituents occupying similar positions in occurrences of structures assigned to the same structure-variable are defined as *congruent* in *Inf*.[9]

It is clear that congruence is an equivalence relation.

4.2. Conditions on an Analysis

The analysis 4.1 provided for **DL** satisfies eight crucial conditions. Our strategy will be to state and verify these eight conditions, and then to prove that any consecution calculus possessing an analysis satisfying these conditions must admit the Elimination rule. The eight conditions are supposed to be reminiscent of those of Curry (1963), and should be compared with those of Belnap *et al.* (1980). In order to state the conditions, I let *Ru* be one of DL's rules, and I let *Inf* be an inference falling under *Ru*. Recall that congruence is an equivalence.

C1. *Preservation of formulas.* Each formula which is a constituent of some premiss of *Inf* is a subformula of some formula in the conclusion of *Inf*. That is, structure may disappear, but not formulas. In fact the rules so far given satisfy also the following:

a. Preservation of parameters: each parameter is congruent to a constituent in the conclusion of *Inf*, with which it agrees in shape.

b. Formula-connectives are introduced one at a time: that is, all nonparametric formulas in the premisses of *Inf* are immediate proper subformulas of a nonparametric formula in the conclusion of *Inf*. Definition: nonparametric formulas in the premisses are *components* of *Inf*.

This condition can be verified by eye. It comes to this: in our statement of the rules, no structure- or formula-variable is lost in passing from premiss to conclusion.

C2. *Shape-alikeness of parameters.* Congruent parameters are occurrences of the same structure.

This condition is inescapable from the definition of "congruence" given by our analysis 4.1, without even peeking at the statement of the rules.

C3. *Non-proliferation of parameters.* Each parameter is congruent to at most one constituent in the conclusion of *Inf*; in other words (since congruence is an equivalence relation), no two constituents of the conclusion are congruent to each other. (There can be parameters which occur only in the conclusion; such a parameter is congruent only to itself.) For verification of this condition by eye, observe that each structure-variable occurs exactly once in the conclusion of the statement of each rule. (My argument would not apply to a calculus with the converse of the rule (*WI*); but it would apply to a calculus with the stronger "mingle rule," from $X \vdash Z$ and $Y \vdash Z$ to infer $X \circ Y \vdash Z$, as in Ohnishi and Matsumoto (1962).)

C4. *Position-alikeness of parameters*. Congruent parameters are either all antecedent or all consequent parts of their respective consecutions.
Verification by eye: this is true of each structure-variable used in my statements of the rules.

C5. *Display of principal constituents*. If a formula is nonparametric in the conclusion of *Inf*, it is either the entire antecedent or the entire consequent of that conclusion. Definition: nonparametric formulas in the conclusion are *principal constituents* of *Inf*; evidently there can be at most two such, but this happens only in the structural axiom $A \vdash A$.[10] (This unusual condition is peculiar to Display logic; the Display Theorem 3.2 permits us the luxury of insisting on it without our calculus being too weak.)
Verification: it is easy to see by eye that the only nonparametric formulas in conclusions are (a) the two values of A in the structural axiom, and (b) the formulas introduced by the connective postulates; and these obviously satisfy the condition.

C6. *Closure under substitution for consequent parameters*. Each rule is closed under simultaneous substitution of arbitrary structures for congruent formulas which are *consequent* parts. That is, let *Inf* fall under *Ru*, and let M be a parametric *consequent* part of a consecution of *Inf*. Let *Inf′* result by putting some one structure X for all constituents of *Inf* in the congruence class of the consequent part M. Then *Inf′* also falls under *Ru*. Furthermore, constituents of the sub-

stituted X are all parametric in Inf', and constituents not substituted for are parametric or not in Inf' according as they are parametric or not in Inf.
Verification. As long as any rule is stated with the help of *unrestricted* structure-variables, this condition is bound to be satisfied. The only rule not so stated is (C/h) of Section 3.4, and that rule is all right, too, since no parametric *consequent* parts of the inferences falling under (C/h) are restricted by its "restriction". It is also clear that the "furthermore" clause is in order.

C7. *Closure under substitution for antecedent parameters.* Rules need not be wholly closed under substitution of structures for congruent formulas which are *antecedent* parts, but they must be closed enough. Let Inf fall under a rule Ru and let M be a parametric *antecedent* part of a consecution of Inf. Also let $X \vdash M$ be the conclusion of an inference with M principal (not parametric). Let Inf' result by putting the structure X for all constituents of Inf in the congruence class of the antecedent part M. Then Inf' also falls under Ru. Furthermore (as in C6).[11]
Verification. Again, rules stated without "restrictions" cannot fail to satisfy this condition, so that we must consider only (C/h). But this rule is all right too: Y in (C/h) must be an h-variable M; but structural axiom $M \vdash M$ is the only inference with principal (nonparametric) M as consequent when M is an h- (or indeed any) variable, so that C7 is trivially verified.[12]

C8. *Eliminability of matching principal constituents.* If there are inferences Inf_1 and Inf_2 with respective conclusions (1) $X \vdash M$ and (2) $M \vdash Y$ (the premisses of (ER)) with M principal in both inferences (in the sense of C5), then either (3) $X \vdash Y$ (the conclusion of (ER)) is identical to one of (1) or (2), or it is possible to pass from the premisses of Inf_1 and Inf_2 to (3) by means of inferences falling under the rules, together with the rule (ER)

$$\frac{X' \vdash M' \quad M' \vdash Y'}{X' \vdash Y'}$$

for arbitrary X', Y', but with M' restricted to proper subformulas of M.

C8 needs checking in detail. The only possibilities are when (1) and (2) are both structural axioms (Section 3.1) – but then (3) = (1) = (2) as required; or when (1) and (2) come by matching connective postulates (Section 3.3). Verification of that latter case deserves highlighting.

FACT 4.2. When M is principal in both (1) and (2) for a connective postulate, (3) can be derived from their premisses using only (ER) on components, together with display-equivalence. In particular, none of the reduction rules need be involved.

Proof. I provide an example:

$$\text{(ER)}\ \frac{(\to)\ \dfrac{X \circ A \vdash B}{X \vdash A \to B} \qquad \dfrac{Y \vdash A \quad B \vdash Z}{A \to B \vdash Y * \circ Z}}{X \vdash Y * \circ Z}$$

Use (ER) and display-equivalence as follows (with one premiss indicating display-equivalence, and with two indicating (ER)):

$$\cfrac{\cfrac{Y \vdash A \qquad \cfrac{\cfrac{X \circ A \vdash B \quad B \vdash Z}{X \circ A \vdash Z}}{A \vdash (X \circ Z *) *}}{Y \vdash (X \circ Z *) *}}{X \vdash Y * \circ Z}$$

4.3. Proofs of Subformula and Elimination Theorems

There are two principal consequences of these eight conditions.

THEOREM 4.3 (Subformula theorem). With respect to **DL**, or any calculus which like **DL** admits an analysis satisfying C1, a derivation of $X \vdash Y$ contains no formulas which are not subformulas of constituents of $X \vdash Y$.

Proof needs no more than a citation of C1.[13]

THEOREM 4.4 (Elimination theorem). With respect to **DL**, or any calculus which like **DL** admits an analysis satisfying C2–C8, the "Elimination Rule" (ER) is admissible.

I follow Curry (1963) in dividing the proof into three stages, each of which may be taken as a separate lemma. For all stages, suppose as common hypothesis that C2–C8 of Section 4.2 hold.

Stage 1. Assume as hypotheses of the stage that (H1a) $X \vdash M$ is derivable, and that (H1b) for all X', if there is a derivation of $X' \vdash M$ ending in an inference in which displayed M is *not* parametric, then $X' \vdash Y$ is derivable. Then $X \vdash Y$ is derivable.

Stage 2.[14] Assume as hypotheses of the stage that (H2a) $M \vdash Y$ is derivable, that (H2b) for all Y', if there is a derivation of $M \vdash Y'$ ending in an inference in which displayed M is *not* parametric, then $X \vdash Y'$ is derivable, and that (H2c) $X \vdash M$ is the conclusion of some inference in which M is not parametric. Then $X \vdash Y$ is derivable.

Stage 3. Assume as hypotheses of the stage that for each of (H3a) $X \vdash M$ and (H3b) $M \vdash Y$ there are derivations ending in inferences in which the respective displayed M's are *not* parametric, and that (H3c) for all X', Y', and proper subformulas M' of M, $X' \vdash Y'$ is derivable if $X' \vdash M'$ and $M' \vdash Y'$ are. Then $X \vdash Y$ is derivable.

Proof of the Theorem. It is evident that the theorem follows from the three stages; the grand simplicity of Curry's proof is, however, concealed rather than revealed by any prose I have so far conceived or seen, so that I choose to make it manifest by display in the natural deduction format of Fitch (1952):

1 | | (ER) is admissible for proper subf. of M hyp
  | |—
2 | | | $X \vdash M$ is derivable hyp (= $H1$a)
3 | | | $M \vdash Y$ is derivable hyp (= $H2$a)
  | | |—
4 | | | | $X' \vdash M$ is der. (M nonparam.) hyp
  | | | |—
5 | | | | | $M \vdash Y'$ is der. (M nonparam.) hyp
  | | | | |—
6 | | | | | $X' \vdash Y'$ is der. Stage 3: 4, 5, 1
7 | | | | $H2$b(X' for X) 5–6
8 | | | | $H2$c(X' for X) 4
9 | | | | $X' \vdash Y$ is der. Stage 2: 3, 7, 8
10 | | | $H1$b
11 | | | $X \vdash Y$ is derivable Stage 1: 2, 10
12 | | (ER) is admissible for M 2–11
13 | (ER) is admissible Induction: 1–12

Proof of Stage 1 relies on C2, C3, C4, C5, and C6. Let D_1 be a derivation of $X \vdash M$ as promised by hypothesis $H1$a of the Stage. If M in $X \vdash M$ is not parametric in the inference ending D_1, the conclusion of the Stage is immediate by $H1$b. Suppose then that M is parametric in that inference. Define a set Q inductively by first putting that occurrence of M into Q, and by then working up D_1, adding, for each inference *Inf* in D_1, each constituent of a premiss of *Inf* which is congruent (with respect to *Inf*) to a constituent of the conclusion of *Inf* which is already in Q. (The members of Q are sometimes called "parametric ancestors" of M.) For each consecution S in D_1, let S' result by putting Y for each constituent of S which lies in Q. $(X \vdash M)' = X \vdash Y$, so it suffices to show that S' is derivable for all S in D_1, assuming as inductive hypothesis that the primes of all premisses of S in D_1 are derivable.

Let S be $W \vdash Z$, and let the inference of D_1 leading to S be

$$Inf \qquad \frac{S_1, \ldots, S_n}{W \vdash Z}$$

Assume *Inf* to fall under rule *Ru*. Let $W' \vdash Z'$ be (not $(W \vdash Z)'$ but) the result of putting Y for all members of Q in S which *in addition* are parametric in *Inf*, and consider

$$Inf' \qquad \frac{S_1', \ldots, S_n'}{W' \vdash Z'}$$

By the bottom-upwards definition of Q together with C3, Q must contain all of a congruence class of *Inf* if it contains any member at all. And by C4, all members of Q are consequent parts; so by C6, *Inf'* falls under the same rule *Ru* as does *Inf*. The inductive hypothesis gives us the derivability of all the premisses (if any) S_i', so that $W' \vdash Z'$ is derivable by *Ru*. If each member of Q in S is parametric for *Inf*, $W' \vdash Z'$ is S', and we are done. Otherwise, by C2, C4, and C5, $Z = Z' = M$, and by the "furthermore" part of C6, that M is not parametric in *Inf'*. Then we have the derivability of $W' \vdash Y$, which is S′, by the hypothesis H1b of the Stage.

Proof of Stage 2 relies on C2, C3, C4, C5, and C7. The proof of this Stage closely follows that of Stage 1. There are two differences: we deal with antecedent parts in Q, and we have the additional hypothesis that M is not parametric in an inference concluding with $X \vdash M$. We may use this information to rely on C7 instead of C6.

Proof of Stage 3. C8 suffices.

This completes the proof that the Elimination Rule (ER) is admissible in **DL**.

5. SOME FAMILIES AND LOGICS

Display logic is in a way a schema; you can include such families as you like, differentiating among them by postulating whatever structural rules you like – so long as they satisfy C1–C8 of Section 4.2. Then you can make up hybrid logics by mixing families as you wish. For definiteness, however, I show how to represent in **DL** a few well-known logics by introducing appropriate families with well-chosen structural postulates. In some cases, what follows are definitions, not claims: though given the Elimination Theorem 4.4, it is easy to see that **DL** is strong enough to represent the desired logic with the help of the family or families indicated, verification of the companion claim of not-too-strong is sometimes on the list of future projects. I will in each case say what claims are plausible, and what is known about those claims.

5.1. *Boolean Family and Two-valued Logic*

Two valued logic is based in **DL** on a single family, the boolean family, indexed with "*b*". For this family, we postulate structural rules.

$$\mathbf{I}+, \mathbf{I}-, \mathbf{I}{*}-, K, W$$ [15]

All others follow. Recall that the index for the boolean family is "*b*", so that the structure-connectives are **I**, $*$, and $\circ$ indexed with "*b*"; but to avoid having to index asterisks in print, we write just "$-$" for the boolean negative structure-connective. Thus, the structure-connectives for the boolean family are $\mathbf{I}_b$, $-$, and $\circ_b$.

The connectives of this family are to be given their standard boolean readings. The following claims are obviously true.

First, consider **DL**$\{b\}$, with its grammar consisting only in the boolean family. $X \vdash Y$ is provable in **DL**$\{b\}$ just in case X tautologically implies Y, and Y is a tautology just in case $\mathbf{I}_b \vdash Y$ holds in **DL** $\{b\}$. The techniques of any standard proof will show this.

Second, **DL**, with all its families, is a conservative extension of **DL** $\{b\}$.

Third, **DL** does not give much information about separation in **DL** $\{b\}$, since the structural elements, which are not subject to the Subformula theorem, comprise all of **t**, **f**, $*$, $\wedge$, and $\vee$ in the boolean family; but of course there are additional separation results known by other techniques.

5.2. *Relevant Implication*

The original "standard" vocabulary for the calculus **R** of relevant implication contains $\sim$, $\rightarrow$, $\wedge$, and $\vee$ – the first two being thought of as "intensional" and the last two as "extensional". In **DL**, **R** will therefore be hybrid, involving first a family indexed by "*r*" for the "intensional" or "relevant" connectives and second the boolean family for the "extensional" connectives. Turning first to the kernel connectives (in the sense of Section 2.4) of the relevance family, we note that $\wedge_r$ and $\vee_r$ are definable by (omitting subscripts) $\sim(A \rightarrow \sim B)$ and $\sim A \rightarrow B$ respectively, while $\mathbf{t}_r$ has to be added –

conservatively, by Anderson and Belnap (1959). This addition has always seemed in the spirit of the relevance enterprise – see Anderson and Belnap (1975), pp. 342–43.

The situation with regard to the boolean family is technically similar but philosophically different. The mathematical fact is that one can add the remaining boolean kernel connectives **t**, **f**, and $\sim$ to **R** conservatively, yielding the "classical relevant logic" **CR**, as in Meyer (1976a), Meyer and Routley (1973), and especially Meyer and Routley (1974). Even philosophically there is no complaint about the addition of boolean **t** and **f**, since these are anyhow definable with the help of propositional quantifiers as $\exists pp$ and $(p)p$; but the addition of boolean negation is controversial – see Belnap and Dunn (1981) for discussion. The problem of finding a consecution calculus for **R** *without* boolean negation remains open. The difficulty is that relevance logic has things to say about boolean conjunction and disjunction, so that we need the boolean family to help us out; but the techniques of this paper do not permit enjoying boolean conjunction and disjunction without carrying along boolean negation as well.

In any event, to repeat, **R** in **DL** is hybrid; its boolean connectives are governed by the postulates of Section 5.1, while for its relevant family we postulate the following rules drawn from Section 3.4:

$$\mathbf{I}+,\ \mathbf{I}-,\ \mathbf{I}{*}+,\ \mathbf{I}{*}-,\ B,\ C,\ W.$$

DL as a whole is obviously a conservative extension of **DL** $\{r, b\}$, since all other connectives can be interpreted as if boolean; hence, we now confine attention to formulas and structures in the vocabulary of **DL** $\{r, b\}$.

The relation of **DL** $\{r, b\}$ to the "classical relevant logic" **CR** is given by: $\mathbf{I}_b \vdash A$ holds in **DL** $\{r, b\}$ just in case A is a theorem of **CR**. One can show **DL** $\{r, b\}$ strong enough by establishing $\mathbf{I}_b \vdash A$ for each axiom A of **CR**, and similarly for its rules; and one can show that **DL** $\{r, b\}$ is not too strong by verifying its axiom and rules in the semantics of Meyer and Routley (1974).

In **CR**, the boolean logic is taken as somehow primary (witness the role of $\mathbf{I}_b$ in the schema relating **CR** to **DL**), while the special relevance theorems A are marked by "$\mathbf{t}_r \rightarrow_b A$" with $\mathbf{t}_r$ relevant and $\rightarrow_b$ boolean. Here A itself would not always be a theorem of the Hilbert calculus **CR** when $\mathbf{t}_r \rightarrow_b A$ was (since $\mathbf{t}_r$ is not a theorem of **CR**), while $\mathbf{t}_r \rightarrow_b A$ would be a theorem for each theorem A. So the marking, in effect, enlarges the set of (call them) quasi-theorems.

As will often be the case for hybrid logics, one obtains an equally satisfying calculus answering to **DL**$\{r, b\}$ by taking relevance logic as primary: calling the Hilbert calculus "**RC**", define A to be a theorem of **RC** just in case $\mathbf{I}_r \vdash A$ holds in **DL**$\{r, b\}$. Then we can mark the special boolean theorems by "$\mathbf{t}_b \rightarrow_r A$" with (this time) $\mathbf{t}_b$ boolean but $\rightarrow_r$ relevant. And everything is reversed: theoremhood in **RC** of $\mathbf{t}_b \rightarrow_r A$ guarantees that of A (since boolean $\mathbf{t}_b$ is itself a theorem of **RC**), but not conversely, so that we are thereby marking off a subset of the theorems as boolean. From the present point of view, the two procedures of finding a Hilbert calculus corresponding to **DL**$\{r, b\}$ are distinct but interchangeable.[16] One more fact. Semantic methods have been used by Meyer to show that there is an axiomatization of **R** – the one of Anderson and Belnap (1975) – which produces all of the pure relevance theorems ($\rightarrow$ and $\sim$ only) from pure relevance axioms and rules. **DL**$\{r\}$ is strong enough to prove these axioms and rules; so it is strong enough to prove all pure relevance theorems without detours. Here is an example (all connectives are in the relevance family).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A \rightarrow B \vdash A{*}\circ B}\ (\rightarrow)}{\mathbf{I}\circ(A \rightarrow B) \vdash A{*}\circ B}\ (\mathbf{I}+)}{(\mathbf{I}\circ(A \rightarrow B))\circ A \vdash B}\ \text{Display}}{(\mathbf{I}\circ A)\circ(A \rightarrow B) \vdash B}\ (C)}{\mathbf{I}\circ A \vdash (A \rightarrow B) \rightarrow B}\ (\rightarrow)}{\mathbf{I} \vdash A \rightarrow ((A \rightarrow B) \rightarrow B)}\ (\rightarrow)$$

Here is a verification of a postulate of **R** involving connectives from both the boolean and the relevance families; in this example, boolean connectives are marked with "b", but relevance connectives have been left unmarked. See note 15.

$(A \rightarrow B)\circ A \vdash B$ and $(A \rightarrow C)\circ A \vdash C$ by 3.3 and $\rightarrow$;
$((A \rightarrow B)\circ A)\circ_b((A \rightarrow C)\circ A) \vdash B \wedge_b C$ by $\wedge_b$;
$(((A \rightarrow B)\circ_b(A \rightarrow C))\circ A)\circ_b(((A \rightarrow B)\circ_b(A \rightarrow C))\circ A) \vdash B \wedge_b C$
by K_b to introduce $(A \rightarrow C)$ and $K\mathbf{I}_b$ to introduce $(A \rightarrow B)$;

$((A \to B) \circ_b (A \to C)) \circ A \vdash B \wedge_b C$ by WI_b;
$((A \to B) \wedge_b (A \to C)) \circ A \vdash B \wedge_b C$ by $\wedge_b$;
$\mathbf{I}_r \vdash ((A \to B) \wedge_b (A \to C)) \to (A \to (B \wedge_b C))$ by $\mathbf{I}+_r$ and $\to_r$.
Observe that **I+** is the only structural postulate required for the relevance family.

Added in proof: The discussion above mis-relates the Meyer/Routley system to **DL**; instead, *their* **CR** is roughly our **RC**.

5.3. *Entailment*

The calculus **E** of entailment of Anderson and Belnap (1975) is hybrid in exactly the same way as **R**. For its intensional connectives (a family with an appropriate index, say "*e*"), the following rules from Section 3.4 are postulated:

$$\mathbf{I}+, \mathbf{I}-, \mathbf{I}{*}+, \mathbf{I}{*}-, C\mathbf{I}/\mathbf{I}, B', W, \vdash W\mathbf{I}.$$

From these B, improbably, follows: $(X \circ (Y \circ Z))$ $a\Rightarrow$ $(\mathbf{I} \circ (X \circ (Y \circ Z)))$ $a\Rightarrow$ $((X \circ \mathbf{I}) \circ (Y \circ Z))$ $a\Rightarrow$ $((Y \circ (X \circ \mathbf{I})) \circ Z)$ $a\Rightarrow$ $(((X \circ Y) \circ \mathbf{I}) \circ Z)$ $a\Rightarrow$ $((\mathbf{I} \circ (X \circ Y)) \circ Z)$ $a\Rightarrow$ $((X \circ Y) \circ Z)$.

Certainly these postulates for **E**'s intensional connectives, together with the boolean postulates for its extensional connectives, are strong enough to prove a standard set of axioms and rules. In contrast to the situation with **R**, however, it is not known that **DL**$\{e, b\}$ is not too strong; in particular, it is not known whether the addition of boolean negation to **E** in the way suggested by **DL** is or is not conservative.[17]

See Section 6.6 below for another possible formulation of **E**.

5.4. *Ticket Entailment*

The calculus **T** of ticket entailment of Anderson and Belnap (1975) is hybrid in precisely the same sense as **R**; and since it is known[18] that boolean negation can be added to **T** as it can be added to **R**, our state of information is precisely analogous. The following are the postulates for the intensional family of connectives of **T**:

$$\mathbf{I}+, \mathbf{I}-, \mathbf{I}{*}+, \mathbf{I}{*}-, B, B', W, \vdash W\mathbf{I}.$$

5.5. Semantics of Relevance Logics

One paradigmatic form of semantic investigation of relevance logics such as **R**, **E**, and **T** has been based on a three termed relation; this form, due to Routley and Meyer, appears to fit well with Display logic; but the matter has been little investigated, so that I here present only some suggestive definitions and no facts.

A *model set* is a quadruple $(K, D, R, *)$ satisfying the "display postulates" $(R*)$ $Rxyz$ only if $Rxz*y*$ and $(**)$ $x** = x$.[19]

The kernel connectives are evaluated as follows, where "A_x" means that A is true at x: $\mathbf{t}_x$ if and only if x is in D; $\mathbf{f}_x$ if and only if $x*$ is not in D; $(\sim A)_x$ if and only if not A_{x*}; $(A \wedge B)_z$ if and only if for some x, y in K, $Rxyz$ and A_x and B_y; $(A \vee B)_x$ if and only if for all y and z in K, if $Rxy*z$ then either A_y or B_z.

It is easy to see that the display equivalences are sound on these semantics, when one interprets all connectives via the kernel connectives as in Section 2.4, and interprets $X \vdash Y$ as: for all x in K, X_x only if Y_x. Presumably completeness is also at hand, but time is finite; still, the matter ought to be pursued because, although *so far* the semantics seem to have little intuitive appeal, they are certainly known to have great technical power.

5.6. Modal Logics

I discuss only modal logics based on a binary relational structure. In **DL** these logics are hybrid: their extensional connectives are part of the boolean family of Section 5.1, while for their modal connectives the idea is to add a family interpreted in a relational structure (K, D, R), with K the set of all points, D a set of "normal" points, and R a binary relation on K (as in Kripke, 1965). The "kernel" connectives of Section 2.4 are explained as follows. $\mathbf{t}$ holds at just the points in D, and $\mathbf{f}$ just in $K-D$. $\sim A$ holds at x just in case A doesn't. $A \wedge B$ holds at a point y just in case for some x in K, Rxy, A holds at x, and B holds at y. $A \vee B$ holds at x just in case for every y in K such that Rxy, either A holds at y or B holds at y.

This induces the following explanation of the structure-connectives. **I** in antecedent [consequent] position holds [doesn't hold] at all points in D. In antecedent position, $(X \circ Y)$ holds at a point y just in case for some x in K, Rxy, X holds at x, and Y holds at y. In consequent position, $(X \circ Y)$ holds at x just in case for every y in K such that Rxy, either X holds at y

or Y holds at y. $X*$ holds at x just in case X doesn't hold at x.

The induced account of $\rightarrow$, $\Box$, and $\Diamond$ agrees with that of Kripke (1965) only for "normal" logics where $D = K$. Nonnormal logics are discussed below. The connective $\wedge$ is not always definable in the "standard" vocabulary; see note 13 for a discussion of this point.

For every modal family discussed in this section, we postulate

$$K\mathbf{I}, K\mathbf{I} \vdash W\mathbf{I}, C\circ, W\circ, \mathbf{I}{*}{+}, \mathbf{I}{*}{-}.$$

So much for what is common to the modal families of all the modal logics of the sort we are treating. In addition we are supposing that each such logic is fitted out with the boolean family, and that the connectives of this family are given their usual extensional interpretation in a relational structure (K, D, R). These postulates (both modal and boolean) are valid, and the display equivalences preserve validity, where to say that $X \vdash Y$ is valid is to say that X at x implies Y at x, all x in K, for each relational structure (K, D, R). Presumably completeness is available, probably easily, but this claim is on a long list of future projects.

Before dealing with individual modal systems, I offer a few facts applying to any family satisfying the modal postulates listed above. Let $(\mathbf{I}, *, \circ)$ be the modal structure-connectives, and recall that $(\mathbf{I}_b, -, \circ_b)$ are boolean.

$\mathbf{I}+$ is a special case of $K\mathbf{I}$.

From $K\mathbf{I}\vdash W\mathbf{I}$ and $K\mathbf{I}$ we obtain $K \vdash W\mathbf{I}$ as follows: $X \vdash Z \circ Z$; $Z* \vdash (X \circ Z*)*$; $X \circ Z* \vdash (X \circ Z*)*$ by $K\mathbf{I}$; $(X \circ Z*)** \vdash (X \circ Z*)*$; $Y \vdash (X \circ Z*)*$ by $K\mathbf{I} \vdash W\mathbf{I}$; $X \circ Y \vdash Z$.

Also, given only $K\mathbf{I}\vdash W\mathbf{I}$, we can calculate that modal $*$ is just boolean negation: $X* \Leftrightarrow X-$. Start with $X* \vdash Y$; $(X \circ_b Y)* \vdash (X \circ_b Y)$ by boolean moves; $X- \vdash (X \circ_b Y)$ by $K\mathbf{I} \vdash W\mathbf{I}$; $X- \vdash Y$ by boolean moves. Now start with $X \vdash Y*$; $(X \circ_b Y) \vdash (X \circ_b Y)*$ by boolean moves; $Y-* \vdash (X \circ_b Y)*$ by $K\mathbf{I} \vdash W\mathbf{I}$; $(X \circ_b Y) \vdash Y-$; $X \vdash Y-$ by boolean moves. So $X* \; a\Rightarrow X-$ and $X* \; c\Rightarrow X-$. The first of these implies that $X- \; c\Rightarrow X*$ and the second that $X- \; a\Rightarrow X*$; so $X* \Leftrightarrow X-$ as required.

Given $K\mathbf{I} \vdash W\mathbf{I}$ and $K\mathbf{I}$, $(X \circ Y) \; a\Leftrightarrow ((X \circ \mathbf{I}_b) \circ_b Y)$. Start with $X \circ Y \vdash Z$; $X \circ Y** \vdash Z$; $X \circ (Y* \circ_b Z)* \vdash (Y* \circ_b Z)$ by boolean moves; $X \circ \mathbf{I}_b \vdash (Y* \circ_b Z)$ by $K \vdash W\mathbf{I}$; $((X \circ \mathbf{I}_b) \circ_b Y) \vdash Z$. Now start with $((X \circ \mathbf{I}_b) \circ_b Y) \vdash Z$; $((X \circ \mathbf{I}_b) \circ_b (X \circ Y)) \vdash Z$ by $K\mathbf{I}$; $X \circ Y \vdash Z$ by boolean moves.

Consequently, given $K\mathbf{I} \vdash W\mathbf{I}$ and $K\mathbf{I}$, in the presence of the boolean family, $C\circ$ and $W\circ$ are redundant.

For the normal logics, where all points are normal ($D = K$), add ($\mathbf{I}$–K) as a postulate. This clearly suffices to identify $\mathbf{I}$ and $\mathbf{I}_b$.
For von Wright's **M** (Kripke 1965), add the "reflexivity" postulates $\mathbf{I}$–, $C\mathbf{I}/\mathbf{I}$, and W. ($\mathbf{I}$–K) follows, using $K\mathbf{I}$, $C\mathbf{I}/\mathbf{I}$, $\mathbf{I}$–, as do $W\mathbf{I}$ and $\vdash W\mathbf{I}$.[20] For **S4**, add the "reflexivity" postulates $\mathbf{I}$–, $C\mathbf{I}/\mathbf{I}$, and W, and a transitivity postulate, either B or B'. ($\mathbf{I}$–K), $W\mathbf{I}$, $\vdash W\mathbf{I}$, and the other one of B' or B follow.[21] For the Brouwersche logic ($D = K, R$ reflexive and symmetric) add $\mathbf{I}$–, $C\mathbf{I}/\mathbf{I}$, W, and Brw. ($\mathbf{I}$–K), $W\mathbf{I}$, and $\vdash W\mathbf{I}$ follow.[22] Here is a proof of the Brouwersche postulate: $A \vdash A$; $(A*\circ \mathbf{I}*)* \vdash \Diamond A$ by ($\Diamond$); $(\Diamond A)* \vdash (A*\circ \mathbf{I}*)$; $(\Diamond A)*\circ A \vdash \mathbf{I}*$; $A \circ \Diamond A* \vdash \mathbf{I}*$ by (Brw); $A \circ \mathbf{I} \vdash \Diamond A$; $A \vdash \Box\Diamond A$ by ($\Box$).

For **S5** ($D = K, R$ an equivalence relation) add $\mathbf{I}$–, $C\mathbf{I}/\mathbf{I}$, W, Brw, B or B'. $W\mathbf{I}$, $\vdash W\mathbf{I}$, and the other of B or B' follow.[23]

For **S2** and **S3**, where the relation is reflexive on the points in D (the normal points), but not all points are normal, complications are required. In particular, the truth conditions of some of the modal connectives are altered (see Kripke, 1965). To avoid confusion, I introduce $\Box' A$, $\Diamond' A$, and $A \to' B$ as variants of $\Box$, $\Diamond$, and $\to$. $\Box' A$ holds at x if x is in D, and if Rxy implies that A holds at y, all y in K; $\Diamond' A$ holds at x if x is not in D, or if Rxy and A holds at y, some y in K; and $A \to' B$ holds at x if x is in D, and if Rxy and A holds at y only if B holds at y, all y in K. These variant connectives are the usual ones of **S2** and **S3** – their cousins, with truth conditions as given at the beginning of this section, are definable; see Anderson and Belnap (1975), pp. 115–117.

We are speaking of "nonnormal" systems, and postulates for these connectives must refer to normality. I give two ways of doing so. One, which adds a premiss $X \vdash \mathbf{t}$ or $\mathbf{f} \vdash X$ to certain rules, requires us to think of $\mathbf{t}$ and $\mathbf{f}$ as "automatic subformulas" in order to satisfy condition C1 for the Subformula theorem 4.3 (or one could just weaken the theorem explicitly instead of indirectly). The other adds a conjoined or disjoined (by the boolean $\circ_b$) modal $\mathbf{I}$ in the conclusion, and hence involves two families.
I present both ways at once, with the understanding that only one of the square bracketed pieces is to be included in each rule. (These rules are to be added to the basic modal postulates given at the beginning of this section.)

$$(\Box'/\mathbf{I}) \qquad \frac{\mathbf{I} \vdash X}{\Box' A \vdash X}$$

$$(\rightarrow'/\mathbf{I}) \qquad \frac{\mathbf{I} \vdash Y}{A \rightarrow' B \vdash X}$$

$$(\Diamond'/\mathbf{I}) \qquad \frac{X \vdash \mathbf{I}}{X \vdash \Diamond' A}$$

$$(\Box') \qquad \frac{X \vdash A \circ A \quad [X \vdash \mathbf{t}]}{X[\circ_b \mathbf{I}] \vdash \Box' A} \qquad \frac{A \vdash Y}{\Box' A \vdash X \circ Y}$$

$$(\Diamond') \qquad \frac{X \vdash A}{(X * \circ Y) * \vdash \Diamond' A} \qquad \frac{X * \circ A \vdash A * \quad [\mathbf{f} \vdash X]}{\Diamond' A \vdash X[\circ_b \mathbf{I}]}$$

$$(\rightarrow') \qquad \frac{X \circ A \vdash B \quad [X \vdash \mathbf{t}]}{X[\circ_b \mathbf{I}] \vdash A \rightarrow' B} \qquad \frac{X \vdash A \quad B \vdash Y}{A \rightarrow' B \vdash X * \circ Y}$$

The **S2** structural postulates are modified versions of the "reflexivity" postulates **I**–, C**I**/**I**, and W, the modification corresponding to requiring that reflexivity hold only for points in D (normal points) instead of for all points in K. As above, the square brackets indicate a choice of ways to deal with normality.

$$(\mathbf{I}-/\mathbf{t}) \qquad \frac{\mathbf{I} \circ X \vdash Y \quad [X \vdash \mathbf{t}]}{X[\circ_b \mathbf{I}] \vdash Y}$$

$$(C\mathbf{I}/\mathbf{I}/\mathbf{t}) \qquad \frac{X \circ \mathbf{I} \vdash Y \quad [\mathbf{I} \circ X \vdash \mathbf{t}]}{(\mathbf{I} \circ X)[\circ_b \mathbf{I}] \vdash Y}$$

$$(W/\mathbf{t}) \qquad \frac{(X \circ Y) \circ Y \vdash Z \quad [X \circ Y \vdash \mathbf{t}]}{(X \circ Y)[\circ_b \mathbf{I}] \vdash Z}$$

It needs to be verified that these postulates satisfy the conditions C1–C8. C1 is either not quite satisfied, or requires the "automatic subformulahood" of **t** and **f**, as noted, for the extra-premiss version. The rest except C8 are straightforward. But verifying C8 for ($\vdash\Box'$) and ($\Diamond'\vdash$) requires appeal to ($K \vdash W$**I**), so that Fact 4.2 no longer holds. Furthermore, verification of C8 for the extra-premiss version requires adding the following pair of rules, which use the "automatic subformulahood" of **t** and **f** (in order to satisfy

C1 – they satisfy C2–C8 as well) in a particularly repugnant (but still technically harmless) fashion:

$$(\text{ER/t})\ \frac{X \vdash \mathbf{t} \quad \mathbf{t} \vdash Y}{X \vdash Y} \qquad (\text{ER/f})\ \frac{X \vdash \mathbf{f} \quad \mathbf{f} \vdash Y}{X \vdash Y}$$

For **S3**, add a transitivity postulate B or B' (Section 3.4) to the **S2** postulates.[24]

Theoremhood of A in **S2** or **S3** is defined by $\mathbf{I} \vdash A$, with the **I** from the modal family (truth at all *normal* points). **E2** and **E3** of Lemmon (1957), cited by Kripke (1965), are obtained by taking the boolean $\mathbf{I}_b$ (instead of the modal **I**) as the sign of theoremhood: $\mathbf{I}_b \vdash A$ (truth at *all* points).

Deontic logics. "Obligatory implies permitted" is obtained by postulating the rule, from $(X \circ (Y \circ \mathbf{I})) \vdash \mathbf{I}*$ to infer $X \vdash Y*$.

5.7. *Intuitionist Logic*

An *h-formula* is constructed from h-variables (Section 2.3) by the standard intuitionist connectives $\wedge_h, \vee_h, \rightarrow_h$, and $\sim_h$. Plain negation $\sim_h$ is explicitly excluded (see 3.3 for the two negations; as indicated above, given $K\mathbf{I} \vdash W\mathbf{I}$, $\sim_h$ agrees with boolean $\sim_b$). Intuitionist logic is given semantically (Kripke) by the conditions $D = K$, R reflexive and transitive, all intuitionistic formulas "persistent": Rxy and A at x imply A at y for A an h-formula. This is a single-family logic; consequently, for the remainder of this section we save subscripts by assuming that all connectives are indexed with "h", and that all formulas are h-formulas. To obtain the structural postulates for the h-family, add (restricted) C/h to the postulates for **S4**. I observe that Display logic permits the happy coexistence of intuitionist and boolean logic.

As an example of how things go, we will verify all of the Łukasiewicz postulates for intuitionist logic as presented in Prior 1955. But first an essential

LEMMA 5.1 (Persistence).

$$A \circ \mathbf{I} \vdash A.$$

Proof. Argue inductively on the definition of "*h*-formula".

Suppose A is an h-variable. $(\mathbf{I}\circ\mathbf{I})\circ A \vdash A$ by 3.3 and $K\mathbf{I}$; $(\mathbf{I}\circ A)\circ\mathbf{I} \vdash A$ by C/h; $A\circ\mathbf{I} \vdash$ *by* $\mathbf{I}-$.

$A \vdash A$ by 3.3 and $B\circ\mathbf{I} \vdash B$ by hyp. of ind.; $A\circ(B\circ\mathbf{I}) \vdash A\wedge B$ by $\wedge$; $(A\circ B)\circ\mathbf{I} \vdash A\wedge B$ by B; $(A\wedge B)\circ\mathbf{I} \vdash A\wedge B$ by $\wedge$.

$A\circ\mathbf{I} \vdash A$ by hyp. of ind.; $A\circ(\mathbf{I}\circ B*) \vdash A$ by $\mathbf{I}$–K; $(A\circ\mathbf{I})\circ B* \vdash A$ by B; $A \vdash \mathbf{I}*\circ(A\circ B)$; $B \vdash \mathbf{I}*\circ(A\circ B)$ similarly; $A\vee B \vdash (\mathbf{I}*\circ(A\circ B))\circ(\mathbf{I}*\circ(A\circ B))$ by $\vee$; $(A\vee B)\circ\mathbf{I} \vdash A\circ B$ by $\vdash W\mathbf{I}$ and display; $(A\vee B)\circ\mathbf{I} \vdash A\vee B$ by $\vee$.

$(A\to B)\circ A \vdash B$ by 3.3 and $\to$; $(A\to B)\circ(\mathbf{I}\circ A) \vdash B$ by $K\mathbf{I}$; $((A\to B)\circ\mathbf{I})\circ A \vdash B$ by B; $(A\to B)\circ\mathbf{I} \vdash A\to B$ by $\to$.

$A* \vdash A*$ by 3.3 and display. $\sim A \vdash \mathbf{I}*\circ A*$ by $\sim$; $\sim A\circ\mathbf{I} \vdash A*$; $\sim A\circ(\mathbf{I}\circ\mathbf{I}) \vdash A*$ by $\mathbf{I}+$; $(\sim A\circ\mathbf{I})\circ\mathbf{I} \vdash A*$ by B; $\sim A\circ\mathbf{I} \vdash \sim A$ by $\sim$.

The postulates as numbered in Prior (1955), p. 308, may now be verified as follows.

1. $A\circ\mathbf{I} \vdash A$ by Persistence 5.1; $A\circ B \vdash A$ by $\mathbf{I}$–K; $\mathbf{I} \vdash A\to(B\to A)$ by $\mathbf{I}+$ and $\to$.
2. $((A\to(B\to C))\circ A)\circ B \vdash C$ by 3.3 and $\to$;
   $((A\to(B\to C))\circ A)\circ((A\to B)\circ A) \vdash C$ by 3.3 and $\to$;
   $((A\to B)\circ((A\to(B\to C))\circ A))\circ A \vdash C$ by B';
   $(((A\to B)\circ(A\to(B\to C)))\circ A)\circ A \vdash C$ by B;
   $((A\to B)\circ(A\to(B\to C)))\circ A \vdash C$ by W;
   $\mathbf{I} \vdash (A\to B)\to((A\to(B\to C))\to(A\to C))$ by $\mathbf{I}+$ and $\to$.
3. $A\circ\mathbf{I} \vdash A$ by Persistence 5.1; $A\circ B \vdash A$ by $\mathbf{I}$–K;
   $\mathbf{I} \vdash (A\wedge B)\to A$ by $\wedge$, $\mathbf{I}+$, and $\to$.
4. $A\circ B \vdash B$ by 3.3 and $K\mathbf{I}$; $\mathbf{I} \vdash (A\wedge B)\to B$ by $\wedge$, $\mathbf{I}+$, and $\to$.
5. $A\circ B \vdash A\wedge B$ by 3.3 and $\wedge$; $\mathbf{I} \vdash A\to(B\to(A\wedge B))$ by $\mathbf{I}+$ and $\to$.
6. $A\circ\mathbf{I} \vdash A$ by Persistence 5.1; $A\circ B* \vdash A$ by $\mathbf{I}$–K; $A \vdash A\circ B$ by display; $\mathbf{I} \vdash A\to(A\vee B)$ by $\vee$, $\mathbf{I}+$, and $\to$.
7. $A \vdash A\circ B$ as for 5a; $A \vdash B\circ A$ by display; $\mathbf{I} \vdash A\to(B\vee A)$ by $\vee$, $\mathbf{I}+$, and $\to$.

8. $(A\to C)\circ A \vdash C$ by 3.3 and $\to$; $A\vdash ((A\to C)\circ C*)*$ by display; $B\vdash ((B\to C)\circ C*)*$ similarly; $(A\vee B)\vdash ((A\to C)\circ C*)*\circ((B\to C)\circ C*)*$ by $\vee$; $(A\vee B)\vdash (((A\to C)\circ(B\to C))\circ C*)*\circ(((A\to C)\circ(B\to C))\circ C*)*$ by Persistence 5.1 and **I**–K to introduce $(B\to C)$, and K**I** to introduce $(A\to C)$; $(A\vee B)\vdash (((A\to C)\circ(B\to C))\circ C*)*$ by $\vdash W$**I**; $((A\to C)\circ(B\to C))\circ(A\vee B)\vdash C$ by display; $\mathbf{I}\vdash (A\to C)\to((B\to C)\to((A\vee B)\to C))$ by **I**+ and $\to$.
9. $B*\vdash B*$ by 3.3 and display; $\sim B\circ\mathbf{I}\vdash B*$ by $\sim$ and display; $\mathbf{I}\circ\sim B\vdash B*$ by C**I**/**I**; $B\vdash(\sim B)*$ by **I**– and display; $(A\to B)\circ A\vdash(\sim B)*$ by 3.3 and $\to$; $(A\to B)\circ\sim B\vdash A*$ by display; $(A\to B)\circ((A\to\sim B)\circ A)\vdash A*$ by 3.3 and $\to$; $((A\to B)\circ(A\to\sim B))\circ A\vdash A*$ by B; $((A\to B)\circ(A\to\sim B))\circ A**\vdash A*$ by display; $((A\to B)\circ(A\to\sim B))\circ\mathbf{I}\vdash A*$ by $K\vdash W$**I**; $(A\to B)\circ(A\to\sim B)\vdash\sim A$ by $\sim$; $\mathbf{I}\vdash(A\to B)\to((A\to\sim B)\to\sim A)$ by **I**+ and $\to$.
10. $A\circ\mathbf{I}\vdash A$ by Persistence 5.1; $A*\vdash(A\circ\mathbf{I})*$; $\sim A\vdash\mathbf{I}*\circ(A\circ\mathbf{I})*$ by $\sim$; $\sim A\circ(A\circ\mathbf{I})\vdash\mathbf{I}*$; $(A\circ\sim A)\circ\mathbf{I}\vdash\mathbf{I}*$ by B; $(A\circ\sim A)\circ B*\vdash\mathbf{I}*$ by **I**–K; $(A\circ\sim A)\circ\mathbf{I}\vdash B$; $A\circ\sim A\vdash B$ by C**I**/**I** and **I**–; $\mathbf{I}\vdash A\to(\sim A\to B)$ by **I**+ and $\to$.

Modus ponens. Suppose $\mathbf{I}\vdash A$ and $\mathbf{I}\vdash A\to B$; but $(A\to B)\circ A\vdash B$ by 3.3 and $\to$; so $\mathbf{I}\circ\mathbf{I}\vdash B$ by the Elimination Theorem 4.4 and display; so $\mathbf{I}\vdash B$ by **I**–.

See Section 6.6 for an alternate formulation of intuitionism within **DL**.

5.8. Interfamilial Relations

Here I record a staccato of interfamilial facts. I will let $(\mathbf{I}_b, -, \circ_b)$ be of the boolean family of Section 5.1, for which **I**+, **I**–, **I***–, K, and W, and hence all structural rules hold, and I will let $(\mathbf{I}, *, \circ)$ be of some other family. See Section 3.4 for the notation "$\Leftrightarrow$" and its cousins.

FACT 5.2 (Uniqueness of boolean family). There is only one boolean family: if **I**+, **I**–, **I***–, K, and W are postulated for $(\mathbf{I}, *, \circ)$, then $\mathbf{I}\Leftrightarrow\mathbf{I}_b$, $(X\circ Y)\Leftrightarrow(X\circ_b Y)$, and $X*\Leftrightarrow X-$.

It seems likely (but I do not have a proof) that this is the only uniqueness result of its kind: if a set of structural rules on the connectives of some one family is enough to confer uniqueness for all three elements of the family, then that set of rules implies I+, I–, K, and W. Otherwise put: it is my belief that for every family other than the boolean, it is possible to have multiple families satisfying the same one-family structural postulates without interreplaceability.

FACT 5.3 (Binary reduction). Suppose WI [or $\vdash W$I] for $\circ$. Then $(X \circ Y)\ a\Rightarrow (X \circ_b Y)$ [or $(X \circ Y)\ c\Rightarrow (X \circ_b Y)$]. (The only features required of the boolean family are K (hence $\vdash K$) and KI.)

FACT 5.4 (Star distribution over boolean family). The $*$ of any family distributes over the boolean connectives: $\mathbf{I}_b * \Leftrightarrow \mathbf{I}_b$, $(X \circ_b Y)* \Leftrightarrow (X* \circ_b Y*)$, and $X{-}* \Leftrightarrow X*{-}$.

The following give us that $\mathbf{I}_b * \Leftrightarrow \mathbf{I}_b$: $\mathbf{I}_b *\ a\Rightarrow (\mathbf{I}_b \circ_b \mathbf{I}_b *)*\ a\Rightarrow \mathbf{I}_b **\ a\Rightarrow \mathbf{I}_b$; $\mathbf{I}_b\ a\Rightarrow (\mathbf{I}_b \circ_b \mathbf{I}_b *)\ a\Rightarrow \mathbf{I}_b *$; and exactly similar moves yield $\mathbf{I}_b *\ c\Leftrightarrow \mathbf{I}_b$ as well.

Next consider $(X \circ_b Y)*$ and $(X* \circ_b Y*)$. For right-to-left, each of X and Y reduces to $(X \circ_b Y)$ in both antecedent and consequent positions; so WI and $\vdash W$I now suffice. For left-to-right, each of X and Y reduces to $(X* \circ_b Y*)*$ in both antecedent and consequent positions; so again WI and $\vdash W$I now suffice.

Lastly, consider $X*{-}$ and $X{-}*$. Start with $X*{-} \vdash Y$. Then: $Y{-} \vdash X*$; $X \vdash Y{-}*$; $X \vdash (Y*{-} \circ_b Y{-}*)$; $Y* \vdash (X{-} \circ_b Y{-}*)$; $(X{-} \circ_b Y{-}*)* \vdash Y$; $(X{-}* \circ_b Y{-}**) \vdash Y$ (by the distribution of star over $\circ_b$, just proved); $(X{-}* \circ_b Y{-}) \vdash Y$; $X{-}* \vdash (Y \circ_b Y)$; $X{-}* \vdash Y$; so $X*{-}\ a\Rightarrow X{-}*$. And $X*{-}\ c\Rightarrow X{-}*$, by an analogous argument. The first of these implies that $X{-}*\ c\Rightarrow X*{-}$, and the second that $X{-}*\ a\Rightarrow X*{-}$; so we are done with proving that $X*{-} \Leftrightarrow X{-}*$.

These arguments were uncovered by reflecting on the proof of Meyer (1976a) that the relevance and boolean negations permute. See also Meyer and Routley (1973, 1974).

FACT 5.5 (Star distribution with CI). If CI holds, then $(X \circ Y)* \Leftrightarrow (X* \circ Y*)$.

Hence, under the same assumption, if the boolean family is the only other family present, and assuming the rules I$*$+ and I$*$–, all $*$'s may be pushed inside to formulas. (But note: it does not follow that the boolean negative structuring, –, can be pushed inside structure-connectives from other families.)

FACT 5.6 (Equivalence of I's). Let I+, I–, I*+, I*–, CI/I, and KI hold for each of two (e.g. modal) families. Then their I's are equivalent.

6. FURTHER DEVELOPMENTS

This section raises some possibilities and questions.

6.1. Demarcation

It would be a matter of great interest to characterize those logics which can and those which cannot be codified by means of the techniques of Display logic. On the other hand, I do not think that Display logic should be viewed as itself setting the boundary of the province of logic (Kneale) in the style of Hacking (1979). Logic is that discipline which tries to shed light on the problem of separating the good inferences from the bad; I do not therefore propose to use some technical property not closely connected with that aim to mark off Logic from Nonlogic, or to use such a property to defend an historically given logic as somehow privileged.[25] For example, of those logics offered as philosophically interesting, quantum logic is one that I see no way of catching by the techniques of Display logic (it also eludes Hacking, 1979). This is equally true of the logic answering to the theory of modular lattices, which Dunn pointed out (in conversation) presents a somewhat simpler version of the same problem. But we should *not* conclude that quantum logic is not a logic. Whether it is or is not of significance in sorting good from bad arguments must be argued on quite other grounds.

6.2. Quantifiers

Quantifiers may be added with the obvious rules:

$$(UQ)\ \frac{Aa \vdash X}{(x)Ax \vdash X} \qquad \frac{X \vdash Aa}{X \vdash (x)Ax}$$

provided, for the right rule, that a does not occur free in the conclusion.

(The rule for the intuitionist universal quantifier, however, would involve **I**.) The rule for the existential quantifier would be dual. The abstract details of C6, C7, and C8 would need complicating, but not the ideas. One might talk about *variants* of inferences being isomorphic with respect to the analysis into parameters and congruence classes.

On the other hand, as yet this addition provides no extra illumination. I think that is because these rules for quantifiers are "structure free" (no structure-connectives are involved; see also Section 6.5). One upshot is that adding these quantifier rules to modal logic brings along Barcan and its converse (see Hughes and Cresswell, 1968) willy-nilly, which is an indication of an unrefined account; alternatives therefore need investigating. Introducing a family for each constant helps.

6.3. *Interpolation*

I believe or hope that Display logic can be used as a basis for establishing an interpolation theorem; but that remains to be seen. One source of the belief is that one needs "enough connectives" (via structure) for the Elimination Theorem 4.4, and also "enough connectives" for interpolation. Another source is the use by McRobbie (1979) of a consecution calculus for interpolation in a nonclassical context. There is on the other hand the negative result of Fine (1979) for quantified **S5**.

6.4. *Algebra*

Evidently algebra is in the air, especially residuation. See Dunn's section 28.2 of Anderson and Belnap (1975). The most immediate inspiration for the algebraic flavor is Meyer and Routley (1972).

If one did not have $*$, one would have some residuals in each family, using the Display Theorem 3.2 as a guide. For example, suppose that we replace $*$ in each family by a pair of binary structural connectives $X - Y$ and $X -- Y$, thinking of X as positive and Y as negative substructures. Then the following equivalences would (for example) suffice: $X \vdash Y \circ Z$ and $X \vdash Z \circ Y$, as before; $X \vdash Y \circ Z$ and $X - Y \vdash Z$ and $X -- Y \vdash Z$ (the two new connectives are not different on the left); $X \circ Y \vdash Z$ and $X \vdash Z - Y$ and $Y \vdash Z -- X$.

In the same spirit, one might look at the case when one refuses to postulate commutativity for $\circ$ on the right of the turnstile.

6.5. *Other Connectives*

One sees that the basic three place relation is $X \circ Y \vdash Z$, or, with equal fundamentality, $X \vdash Y \circ Z$. So for the premiss for the rule for a binary connective in which the components are together, there are two possibilities: in the place of X and Y (or Z), and in the place of Y and Z. When one adds $*$ to get the effect of positive and negative, one gets many possibilities. Only some are directly realized in our formula-connective rules; for example, we miss an arrow $A \rightarrow B$ with rule $A \circ X \vdash B$ yielding $X \vdash A \rightarrow B$. Of course in the presence of (*C*I) such an arrow would not be much of an addition. There are also other possibilities involving **I**.

There is also the possibility of "structure-free" formula-connectives, the rules for which involve no structure-connectives; for example, the rules of Gentzen (1934) for conjunction were such:

$$\frac{A \vdash X}{A \wedge B \vdash X} \qquad \frac{B \vdash X}{A \wedge B \vdash X} \qquad \frac{X \vdash A \quad X \vdash B}{X \vdash A \wedge B}$$

Such formula-connectives should doubtless be specially marked (or unmarked) to indicate their independence of any family. I think that conjunction and disjunction (with dual rules) are the only two possibilities; in particular, that there is no structure-free negation connective; nor structure-free implication.

Note that distribution cannot be obtained for these formula-connectives without appeal to structural elements; and that in the presence of the boolean family, not only is distribution forthcoming, but these structure-free formula-connectives agree with the corresponding boolean connectives. (This is a paradigm case of failure of conservative extension in **DL**.)

In any event, the spirit of **DL** suggests that only those formula-connective rules be postulated which allow Fact 4.2 to go through, thus strengthening C8 by forbidding use of any but display-equivalences in reducing the complexity of the formula eliminated in (ER). But see the treatment of **S2** and **S3** in Section 5.6 for some rules not following this suggestion.

6.6. *Restricted Rules*

Curry (1963) and others obtain modal logic by restricting the rule for □ on the right – requiring every formula on the left (thinking only of commas) to have the form $\Box B$. The Elimination Theorem 4.4 survives in the presence of such rules. That is, instead of adding a structural family for modality, one can keep the nonmodal family only, say the boolean family, or the relevance family, and instead place restrictions on the rules. Exactly how these restrictions have to go is controlled by conditions C6 and C7 (Section 4.2).

For example, to get **R** with an **S4** necessity (see Anderson and Belnap, 1975) based on a modal family of its own seems to require interfamilial postulates. But one can get it instead by using just the relevance and boolean families, and restricting the rule for $\Box A$ on the right as follows: the parameter X on the left can contain no formula as negative part; and each formula it does contain must have the form $\Box B$. It can then be seen that this rule satisfies C6 (trivially : there are no parametric formulas which are consequent parts) and C7 (not quite so trivially, but still easily) of Section 4.2. It does not seem possible to add an **S5** necessity (Bacon, 1966) in the same way; positing a separate family appears to be the only way.

For intuitionism, instead of omitting structural rules from the full boolean set, one can restrict the rules for introducing the intuitionistic connectives on the right. The restriction would be this: the antecedent of the conclusion can contain no formulas as negative parts; and each formula it does contain must be an h-formula (Section 5.7). Again verification of C6 and C7 is straightforward.

We can still show that **DL** is a conservative extension of **DL**$\{h\}$ as follows: by the Subformula theorem, we need pay attention only to consecutions involving h-variables and intuitionistic formula-connectives (but with the possibility of structure-connectives from other families). Re-interpret all such consecutions in this way: S means that the conjunction of all its formula antecedent parts implies the disjunction of all its formula consequent parts. Then the restrictions guarantee that all rules are verified intuitionistically. (That is, we do not need to give separate interpretation to $*$ at all.)

For the formulation of **E** of Section 5.3 above, one would not have "the Ackermann property" discussed in Anderson and Belnap (1975) according to which one does not have a theorem $A \to (B \to C)$ unless (in the "standard" vocabulary) A contains some implicative formula; for of course there is $\mathbf{I} \vdash A \to (B \to (A \wedge B))$. To restore this (I would say) happy property, one might restrict the rule for implication on the right in the manner suggested by the above discussion.

6.7. *Incompatibility*

There is some value in working through the "incompatibility version" of the above proceedings. This corresponds to (but does not imitate) the "left handed systems" explained in McRobbie and Belnap (1979).

The idea is straightforward: define an incompatibility relation $X \vdash Y$ as $X \vdash Y*$. Evidently the relation is family-relative, unlike the turnstile; which makes the whole thing less interesting. In the single-family case, however, or in the case when the boolean logic is taken as "primary", it is worthwhile working through what things look like in this new guise. For one thing, $*$ tends to disappear except on formulas, and a new positive binary structure-connective $(X : Y) = (X* \circ Y*)*$ turns out to do a lot of work.

Since there is such a close relation between "analytic tableaux" and one-sided consecution calculuses, perhaps this suggests that the proper way to arrive at an analytic tableaux formulation for **DL** on the model of McRobbie and Belnap (1979) would be to use an essentially relational idea as in Dunn (1976).

6.8. *Binary Structuring and Infinite Premiss-sets*

Why didn't Gentzen 1934 use a binary structure-connective instead of polyvalent commas? (The idea is due to Meyer (1976b).) Of course, for the fellow who leaps platonistically to think of the stuff on the left of the turnstile as intending a *set*, there would be no point to binary structuring. And even if one thought of what is on the left of the turnstile as a *sequence*, in the abstract sense, binary structuring would not be likely to emerge. Perhaps this was Gentzen's picture, for he was careful in his formalistic way to *postulate* the rules *W***I** (contraction) and *C***I** (permutation), while evading the necessity of worrying about an associativity rule such as *B* only by the gimmick of using commas as polyvalent. (Not to be misleading, let me note that *B* in fact follows from *W***I**, *K*, and *K***I**, in contrast to the definability situation in combinatory logic.)

Perhaps Gentzen did not much worry about the theory of the grammar of his *L* calculuses. For example, although Gentzen (1934) once speaks of his comma as an auxiliary symbol (2.3, p. 71), he does not list it with the two parentheses and the arrow when he is officially listing the "auxiliary symbols" of his language (1.1, p. 70). (References are to the Szabo translation.)

There is tension here, and several ways to resolve it. One is by construing the left as a set-name from the beginning, as some have done. That misses possibilities, but is coherent. Another is to invent the notion of a "fireset", as in Mcrobbie (1979). That is also coherent; but misses possibilities. The only device which misses nothing is to take structuring as binary instead of polyvalent. And I think this course, on reflection, to be more in the spirit of Gentzen's cautious postulation of *W*I and *C*I than are the later leaps to sets or firesets.

I am not arguing that binary structuring is more intuitive, but instead that it is more satisfactory from a mathematical point of view. I am recommending binary structuring on quite the same grounds that lead nearly everyone to prefer binary conjunction in formal systems to a polyvalent ("run on") conjunction.

It might be objected that the limitation to binary structuring prevents generalization to infinite sets of premisses; but this is not so. To guide imagination, picture a structure X as a tree; now (while keeping at most binary forking at each node) let branches be infinite. Why not?[26]

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NOTES

[1] See Section 6.8 for a discussion of the matter.

[2] The locution "$\vdash Y$" is not admitted; its work is done by "$\mathbf{I} \vdash Y$".

[3] Equivalently, and in some respects more perspicuously, one must explain the structural connectives: **I** in antecedent and consequent, $*$, and $\circ$ in antecedent and consequent.

[4] I have not taken the rules **I**$*$+ and **I**$*$– as display-equivalences, but I have as a matter of fact postulated them for every family I have considered. On the other hand, I have avoided relying on these rules in any interesting way by using **I**$*$ instead of **I** in rules like (Brw) below; so as far as the families I have defined are concerned, the rules **I**$*$+ and **I**$*$– are apparently useful only in showing that $\mathbf{f} \Leftrightarrow \sim\mathbf{t}$. I do not know if they might have other interesting applications, or if it might be interesting to withhold them.

[5] It is worthwhile noting that (ER) can be strengthened in various ways in the direction of Gentzen's "mix" – but not in every way.

First, there could be many M's in (2), and they could be located anywhere, as long as they were antecedent parts:

$$\frac{X \vdash M \quad (\ldots M \ldots M \ldots) \vdash Y}{(\ldots X \ldots X \ldots) \vdash Y}$$

Second, keeping the right premiss as in the above example, the M in the left premiss could be buried, as long as it was a *single* consequent part:

$$\frac{X \vdash (\text{---} M \text{---}) \quad (\ldots M \ldots M \ldots) \vdash Y}{(\ldots X' \ldots X' \ldots) \vdash Y}$$

where $X' \vdash M$ is display-equivalent to $X \vdash (\text{---}M\text{---})$, as promised by the Display Theorem 3.2. Also duals of these, and certain other small generalizations, are easily available. In contrast, however, the following premisses

$$\frac{X \vdash (\text{---}M\text{---}M\text{---}) \quad (\ldots M \ldots M \ldots) \vdash Y}{?}$$

"have no conclusion" in the general case – as far as I know. Observe that it is the Display Theorem 3.2 which allows us to define a calculus for which the admissibility of (ER) is directly provable, without recourse to "mix" or the like.

[6] An *inference* is an ordered pair consisting of a set of consecutions (its premisses) and a consecution (its conclusion). A *rule* is a set of inferences.

[7] In applications, looking alike will be a necessary but not sufficient condition of congruence. Keep in mind that the congruence supplied by an analysis must by definition be an equivalence on the parameters.

[8] The intent of the preceding discussion was to leave room for alternate analyses.

[9] This definition strictly speaking presupposes that each inference falls under only one rule; but it doesn't matter.

[10] A formulation alternative to that of S2 and S3 in Section 5.6 below could involve replacing the rule ($\Box'$/I) there with an axiom $\Box' A \vdash \mathrm{t}$ that would also contain two principal constituents.

[11] It suffices that C6 and C7 hold good for the union of all the rules of the calculus; but it is easier to verify the stronger rule-by-rule conditions stated.

[12] Conditions C6 and C7 are more severe and thereby more interesting than a first glance might indicate. For example, they tend to block structural rules that can operate only on formulas (e.g., Gentzen's own). Minc (1972), and, independently, Dunn (1973), were the first to see the necessity of structural rules operating on entire structures; and theirs were also the first accounts of calculuses with more than one family of structures. Note 14 comments on the asymmetry of C6 and C7. (Let me take this opportunity to correct an error in Belnap *et al.* (1980): that paper *wrongly* stated that the rules of Minc (1972) were inaccurate. "Our" mistake was due to *my* inattentive conflation of Minc's clearly expressed distinction between "formula" and "formula of $R+$".)

[13] The interest of a Subformula theorem such as 4.3 evidently depends inversely on what might disappear, namely, the structural elements. In **DL** there are three in each family, which is a lot. Some may not be expressible in standard vocabularies, e.g., boolean negation in intuitionism. For another example, consider the modal family for **S4**, Section 5.6, and note that one must have $X \circ Y$ at y just in case both Y at y, and also Rxy and X at x, some x (so that $\circ$, in effect, moves us backwards). This $X \circ Y$ is

not expressible in standard S4 vocabularies; but of course is expressible in S5, where R is symmetric.

The presence of the extra structure, whether expressible in standard vocacularies or not, counts as a technical demerit, inasmuch as it weakens the force of the Subformula Theorem 4.3; e.g., the Subformula theorem does not by itself take us very far toward decision procedures. Furthermore, there is not even a *general* guarantee that derivations of $X \vdash Y$ will contain only structure-connectives (as opposed to formula-connectives) from families already represented, through either formula or structure connectives, in $X \vdash Y$. Separate, piecemeal argument appears to be needed in each case (and is often available).

The presence of structural elements not expressible in standard vocabularies seems to me not, in contrast, any sort of demerit; for the standard vocabularies are defined historically, not by "logic itself". It may well be that the connectives suggested by Display logic will turn out to have their uses; see Saarinen (1978) for a suggestion about "backwards looking operators" such as the one for modal logics noted in Section 5.6.

[14] Others (Curry, 1963, p. 173, cites Lorenzen) have been able to find a special priority for the rules introducing connectives on the right. It might appear that I share this vision, given the asymmetry in the conditions C6 and C7 (Section 4.2) and the related asymmetry between Stages 1 and 2. But the appearance is illusory: although one of the logics I treat, namely intuitionism, is asymmetrical in this way (it is the only one of the logics treated in Section 5 which requires the asymmetry, but see Section 6.6 for others), the method is not in itself asymmetrical. That is, there could well be another logic requiring giving priority to Stage 2 over Stage 1, a kind of dual of intuitionism. These methods could treat that logic equally well (but could not treat both that logic and intuitionism at the same time.)

Perhaps it is worth noting here that my primary treatment of modal logics S4 and S5, in Section 5.6, does *not* involve an asymmetry – none of the rules are restricted in any way.

A related view is that the left rule for a connective can somehow be "deduced" from the right rule. Some weak version of this is likely correct, but the rule ($\Box'$) for S2–S3 in Section 5.6 comes close to providing counterevidence. Nor does the possibility of this "deduction" suggest an asymmetry, unless one were prepared to argue that the reverse "deduction" was not equally possible.

[15] In the context of multiple-family proofs, it is convenient to index rules as well as connectives; thus, the rule K_b sanctions the inference from $X \vdash Z$ to $X \circ_b Y \vdash Z$, while (say) K_r would sanction the inference from $X \vdash Z$ to $X \circ_r Y \vdash Z$. (The former is a rule of DL; as we see below, the latter is not postulated for DL.) But we avoid this indexing as much as possible.

[16] In this special case, the $\mathbf{I}_r$ of the relevance family and the $\mathbf{I}_b$ of the boolean family are comparable: $\mathbf{I}_b\ a\!\Rightarrow \mathbf{I}_r$. But in the general case, when there are various families each with its own **I**, and all incomparable, we can only say that each choice of **I** defines a Hilbert calculus via the schema "A is a theorem just in case $\mathbf{I} \vdash$ A holds in DL", and that each of the others is marked therein by appropriate "$\mathrm{t} \rightarrow A$" – the t corresponding to the other **I**, the arrow corresponding to our chosen **I**. This discussion assumes that the rules **I**+ and **I**– of Section 3.4 are postulated for both **I**'s; for otherwise, it would seem that **I** does not sufficiently resemble Gentzen's empty symbol to warrant a role in defining a notion of theoremhood.

[17] Both R. K. Meyer and S. Giambrone have written in confirmation of our ignorance on this matter. Meyer has also written that there are or might be two distinct ways to add a boolean negation to E, but I have no further information.
[18] Letter from S. Giambrone, April 27, 1981. Presumably this result will appear in Giambrone's dissertation for the Australian National University, in preparation.
[19] Warning: this star is used for historical reasons, and has nothing whatsoever to do with *; indeed, it is more of an identity operator (but not quite) than a negation operator.
[20] Query: can the "reflexivity" postulates be *usefully* simplified?
[21] I do not know of another consecution formulation of **M** or of **S4** (for which an Elimination theorem is provable) (a) with unrestricted rules or (b) with rules for possibility as a primitive.
[22] I do not know of another consecution formulation of the Brouwersche logic.
[23] I do not know of another consecution formulation of **S5** that uses only techniques already introduced for other purposes – and for which an Elimination theorem is provable. See Sato (1980) for a survey.
[24] One can hope that there is a set of postulates for these "nonnormal systems" which neither involves two-family postulates nor cheats on the subformula theorem even a little; nevertheless, those who have seen other consecution formulations of **S2** or **S3** will agree that the present formulation represents an improvement with respect to simplicity. For example, Zeman (1973), p. 114, has the following proviso for a key rule: "The left premiss is optional; however, if the rule is once applied with the left premiss not holding, it may not be used again at all in the same proof string."
[25] Church's thesis is fine; but it would be absurd to use it to argue that a particular method was not (intuitively) effective.
[26] Thanks to J. Horty for numerous suggestions.

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