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A CONSECUTIVE CALCULUS FOR POSITIVE RELEVANT IMPLICATION WITH NECESSITY*

R (see [1, Section 28]) is one of the principal relevance logics, codifying relations among $\rightarrow$ (relevant but nonmodal implication), &, $\vee$, and $\sim$. $\mathbf{R}^{\Box}$ is its enrichment with an **S4**-ish necessity operator $\Box$ (proposed by Meyer; see [1, Section 27.1.3]) so that entailment can be carried by $\Box(A \rightarrow B)$, and $\mathbf{R}^{\Box}$ may be further extended – conservatively – by the addition of postulates for a constant necessarily true proposition t and a cotenability operator $A \circ B$ [$=_{\text{df}} \sim (A \rightarrow \sim B)$] yielding what we might call $\mathbf{R}^{\Box\circ t}$. No one yet knows a decent consecution formulation (Gentzen 'sequenzen-kalkul' – see Section 7.2 of [1] for our terminology) of **R**, but in [4, 5] the problem is solved for $\mathbf{R}_{+}^{\circ t}$, which is the positive fragment $\mathbf{R}_{+}$ of **R** conservatively enriched by t and $\circ$. In Section 29.10 of [1] it was announced that this result could be extended to $\mathbf{R}_{+}^{\Box\circ t}$; the purpose of this note is to present the proof. That is, we define a consecution calculus $L\mathbf{R}_{+}^{\Box\circ t}$ which by means of an appropriate Elimination theorem we show equivalent to $\mathbf{R}_{+}^{\Box\circ t}$.

A word about history is in order. JMD's result for the system *without* necessity was presented in a colloquium at the University of Pittsburgh in the spring of 1968 and by title at a meeting of the Association for Symbolic Logic in December, 1969 (see [5]). A full treatment appeared in [4] in 1975. The modifications required by the addition of necessity were not quite straightforward; we completed the proof in September, 1972. The final results were written up in the winter of 1973 for circulation, and for presentation to the St. Louis Conference on Relevance Logic in 1975. After it developed that the Proceedings of that conference would not appear, we finally withdrew the paper in order to offer it in this form.

In the meantime, Minc in 1972 (see [7], which gives February 24 as the date of earliest presentation) proved essentially the same theorem we report below, i.e., cut for the system *with* necessity. (It is surely needless to say that our work and his have been totally independent.) And there is the well known 1965 work of Prawitz [9] on normal form theorems for natural deduction forms of relevance logic. We therefore feel called upon to say a few words about what this paper adds.

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First Prawitz. [9] does in fact prove a normal form theory for a relevance logic – but not for the system R_+. [7] even goes so far as to suggest that the totally irrelevant $p \to [(c \,\&\, (q \to c)) \to c]$ is provable in Prawitz' $R_{M,\,S4}$, but we have not been able to reconstruct Minc's reasoning and do not agree. However, we do agree that in fact Prawitz's system is *not* the same as R_+ of [1] – the formula $((A \to (B \vee C)) \,\&\, (C \to D)) \to (A \to (B \vee D))$ (mentioned in [1, p. 341], with a reference to work of Urquhart reported in [10]) distinguishes the two, being provable in Prawitz' system but not in R_+. (Charlwood [2] shows that Prawitz' system is equivalent to that of Urquhart [10], using the work of Fine [6]. These systems are also akin to the *constructive* relevance logics of Pottinger [8].)

Second, Minc. We should first note an obviously needed correction: cut cannot be proved for the system he describes unless, in the statement in the structural rules, variables for formulas of R_+ are replaced by variables for what below we call 'antecedents', i.e., formulas of the consecution calculus GR_+. Otherwise case 1.3 of [7] cannot go through. In the terminology developed below, Section 6, the rules of [7] as stated are not all closed under parametric substitution. For example, it is crucial to be able to contract not only pairs of single formulas, but pairs of lists of such formulas – see [1], p. 390, for an instructive example.

But this correction is easily made, after which the proof of [7] is fine. What we offer below is a new proof, which compares with that of [7] as follows. In the first place, the proof of [7] uses the technique of Curry [3] by which in some cases of the argument one modifies the entire proof-tree in a wholesale manner, substituting, in effect, for what [3] calls 'quasi-parametric ancestors'. Our proof, in contrast, carries out each case by modifying only the immediately preceding steps in a retail way. The trade-off is this: on the wholesale plan, there are many modifications, but each is rather simple. On the retail plan, there are limited modifications, but each must be more complex. What emerges below is precisely the sort of modification which will permit the retail plan to go through.

In the second place, we provide a detailed analysis of the nature of Gentzen rules in the spirit of [3]; and we offer certain easily verifiable properties of rules under which our sort of argument will succeed. This appears to be new.

1. POSTULATES FOR $\mathbf{R}_{+}^{\Box\circ t}$

They are as follows ([1], Section 27.1, 1–4). (We remark that one obtains $\mathbf{R}^{\Box\circ t}$ by adding R12 $A \to {\sim} B \to \cdot B \to {\sim} A$ and R13 ${\sim}{\sim} A \to A$.)

Axioms:

R1	$A \to A$
R2	$(A \to B) \to ((B \to C) \to (A \to C))$
R3	$A \to ((A \to B) \to B)$
R4	$(A \to (A \to B)) \to (A \to B)$
R5	$A \,\&\, B \to A$
R6	$A \,\&\, B \to B$
R7	$((A \to B) \,\&\, (A \to C)) \to (A \to B \,\&\, C)$
R8	$A \to A \vee B$
R9	$B \to A \vee B$
R10	$((A \to C) \,\&\, (B \to C)) \to (A \vee B \to C)$
R11	$A \,\&\, (B \vee C) \to (A \,\&\, B) \vee C$
t1′	t
t2	$t \to (A \to A)$
∘1	$A \to (B \to (A \circ B))$
∘2	$(A \to (B \to C)) \to (A \circ B \to C)$
□1	$\Box A \to A$
□2	$\Box(A \to B) \to (\Box A \to \Box B)$
□3	$(\Box A \,\&\, \Box B) \to \Box(A \,\&\, B)$
□4	$\Box A \to \Box\Box A$

Rule for axioms: if A is an axiom, so is $\Box A$.

Rules:

$\rightarrow E$: from $A \rightarrow B$ and A to infer B

$\& I$: from A and B to infer $A \& B$

2. POSTULATES FOR L ($= L\mathbf{R}^{\square \circ t}$)

Turning now to the consecution calculus $L\mathbf{R}^{\square \circ t}$ we show equivalent to $\mathbf{R}^{\square \circ t}$, let us begin by shortening its name to '**L**'. The formation rules of **L** are a generalization of the usual Gentzen ones inasmuch as (1) there are two kinds of seauences allowed, and (2) we allow sequences of sequences of . . . sequences. We distinguish the two kinds by prefixes: 'I' stands for 'intensional' and corresponds to cotenability, while 'E' stands for 'extensional' and corresponds to conjunction. An *antecedent* then is defined as follows: each formula (in &, ∨, ∘, →, t, and □) is an antecedent; and if $\alpha_1, \ldots, \alpha_n$ are antecedents, so are (where $n \geqslant 1$)

$$I(\alpha_1, \ldots, \alpha_n)$$

and

$$E(\alpha_1, \ldots, \alpha_n).$$

Then a *consecution* in **L** has the form $\alpha \vdash A$, with α an antecedent and A a formula. (*Note*: α cannot be empty in **L**; it is the role of t to allow us to so manage things.)

We use small Greek letters as ranging over antecedents, and capital Greek letters as ranging over (possibly empty) sequences of symbols drawn from the following: the formulas, the symbols I and E, the two parentheses, and the comma. We shall use 'V' as standing indifferently for I or E so as to be able to state rules common to both. And we agree that displayed parentheses are always to be taken as paired.

We now state the axioms and rules for **L**.

The *axioms* have the usual form:

(Id) $A \vdash A$.

The structural rules are manifold. First the familiar ones.

Permutation $(CV\vdash)$

$$\frac{\Gamma_1 V(\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_n)\Gamma_2 \vdash A}{\Gamma_1 V(\alpha_1, \ldots, \alpha_{i+1}, \alpha_i, \ldots, \alpha_n)\Gamma_2 \vdash A}\ (CV\vdash).$$

Contraction $(WV\vdash)$

$$\frac{\Gamma_1 V(\alpha_1,\ldots,\alpha,\alpha,\ldots,\alpha_n)\Gamma_2\vdash A}{\Gamma_1 V(\alpha_1,\ldots,\alpha,\ldots,\alpha_n)\Gamma_2\vdash A}(WV\vdash).$$

Weakening $(KE\vdash)$

$$\frac{\Gamma_1\alpha\Gamma_2\vdash A}{\Gamma_1 E(\alpha,\beta)\Gamma_2\vdash A}(KE\vdash)$$

Note that the α_i, α, and β must be antecedents (*a fortiori* nonempty), whereas Γ_1 and Γ_2 in general will not be antecedents, and may be empty; and recall that 'V' in the above rules varies over E and I.

Now for some structural rules, peculiar to nested sequences, insuring that each antecedent is equivalent to an I-sequence of E-sequences of I-sequences . . . (counting formulas as either); or perhaps an alternation starting with an E-sequence.

$$\frac{\Gamma_1 V(\alpha)\Gamma_2\vdash A}{\Gamma_1\alpha\Gamma_2\vdash A}(V_1\text{ elim})$$

$$\frac{\Gamma_1\alpha\Gamma_2\vdash A}{\Gamma_1 V(\alpha)\Gamma_2\vdash A}(V_1\text{ int})$$

$$\frac{\Gamma_1 V(\alpha_1,\ldots,V(\Sigma),\ldots,\alpha_n)\Gamma_2\vdash A}{\Gamma_1 V(\alpha_1,\ldots,\Sigma,\ldots,\alpha_n)\Gamma_2\vdash A}(V_2\text{ elim})$$

$$\frac{\Gamma_1 V(\alpha_1,\ldots,\Sigma,\ldots,\alpha_n)\Gamma_2\vdash A}{\Gamma_1 V(\alpha_1,\ldots,V(\Sigma),\ldots,\alpha_n)\Gamma_2\vdash A}(V_2\text{ int})$$

Logical rules:

$$\frac{\Gamma_1 I(A,B\Sigma)\Gamma_2\vdash C}{\Gamma_1 I((A\circ B)\Sigma)\Gamma_2\vdash C}(\circ\vdash)\qquad\frac{\alpha\vdash A\quad\beta\vdash B}{I(\alpha,\beta)\vdash(A\circ B)}(\vdash\circ)$$

$$\frac{\Gamma_1 E(A,B\Sigma)\Gamma_2\vdash C}{\Gamma_1 E((A\,\&\,B)\Sigma)\Gamma_2\vdash C}(\&\vdash)\qquad\frac{\alpha\vdash A\quad\beta\vdash B}{E(\alpha,\beta)\vdash(A\,\&\,B)}(\vdash\&)$$

$$\frac{\Gamma_1 A\Gamma_2\vdash C\quad\Gamma_1 B\Gamma_2\vdash C}{\Gamma_1(A\vee B)\Gamma_2\vdash C}(\vee\vdash)\qquad\frac{\alpha\vdash A}{\alpha\vdash(A\vee B)}(\vdash\vee)\qquad\frac{\alpha\vdash B}{\alpha\vdash(A\vee B)}(\vdash\vee)$$

$$\frac{\alpha \vdash A \quad \Gamma_1 B \Gamma_2 \vdash C}{\Gamma_1 I((A \to B), \alpha) \Gamma_2 \vdash C} (\to \vdash) \qquad \frac{I(\Sigma\alpha, A) \vdash B}{I(\Sigma\alpha) \vdash (A \to B)} (\vdash \to)$$

$$\frac{\Gamma_1 \alpha \Gamma_2 \vdash C}{\Gamma_1 I(t, \alpha) \Gamma_2 \vdash C} (t \vdash)$$

$$\frac{\Gamma_1 A \Gamma_2 \vdash C}{\Gamma_1 \Box A \Gamma_2 \vdash C} (\Box \vdash) \qquad \frac{\alpha \vdash A}{\alpha \vdash \Box A} (\vdash \Box)$$

Restriction on $(\vdash \Box)$: every constituent in α must either have the form $\Box B$ or be t.

3. TRANSLATION AND EQUIVALENCE

I-sequences are to be translated into $\mathbf{R}_{+}^{\Box\circ t}$ via cotenability and E-sequences via conjunction, as in the following definition of a translation function:

$$T(A) = A$$

$$T(V(\alpha)) = T(\alpha)$$

$$T(E(\alpha, \Delta)) = (T(\alpha) \,\&\, T(E(\Delta)))$$

$$T(I(\alpha, \Delta)) = (T(\alpha) \circ T(I(\Delta)))$$

$$T(\alpha \vdash A) = T(\alpha) \to A.$$

EQUIVALENCE THEOREM. Part 1. If $\alpha \vdash A$ is provable in $\mathbf{L}$, then $T(\alpha \vdash A)$ is provable in $\mathbf{R}_{+}^{\Box\circ t}$. Part 2. If A is provable in $\mathbf{R}_{+}^{\Box\circ t}$, then $t \vdash A$ is provable in $\mathbf{L}$. Accordingly, since t is provable in $\mathbf{R}^{\Box\circ t}$, it follows that A is provable in $\mathbf{R}_{+}^{\Box\circ t}$ just in case $t \vdash A$ is provable in $\mathbf{L}$.

Proof of Part 1 is left entirely to the reader.

For Part 2 we must tediously prove $t \vdash A$ in $\mathbf{L}$ for every axiom A of $\mathbf{R}_{+}{}^{\Box\circ t}$, a procedure we omit; and we must show the admissibility in $\mathbf{L}$ of the rules

$$\frac{t \vdash A \quad t \vdash B}{t \vdash A \,\&\, B} \qquad \frac{t \vdash A \quad t \vdash A \to B}{t \vdash B}$$

answering to $\& I$ and $\to E$. The former is trivial; for the latter we must, as usual, prove an Elimination theorem. Its statement involves multiple simultaneous substitution; we prepare by introducing some notation. In the first place, by a *constituent* we shall as in [3] always refer to an occurrence of a formula which does *not* lie in the scope of any logical connective, and by an *M-constituent* we shall mean a constituent which is an occurrence of M. Secondly where $X_1, \ldots, X_{p_0}$ are pairwise disjoint sets of constituents of an antecedent δ, we define

$$\delta(\gamma_1/X_1, \ldots, \gamma_{p_0}/X_{p_0})$$

or often

$$\delta(\gamma_p/X_p)_{p=1}^{p_0}$$

to be the result of simultaneously substituting in δ, for each p $[1 \leqslant p \leqslant p_0]$, the antecedent γ_p for every constituent in X_p. Thirdly, we define Y to be a *n_0-ary sequential partition of* X just in case X is a set and Y is an n_0-tuple of subsets of X (including the possibility that some are empty) which are pairwise disjoint and whose union is X. Where Y is an n_0-tuple, we uniformly take Y_n as its n-th member, so that $Y = \langle Y_1, \ldots, Y_{n_0} \rangle$; hence, Y is an n_0-ary sequential partition of X iff $Y_n \cap Y_{n'} = \phi$ when $n \neq n'$, and $Y_1 \cup \ldots \cup Y_{n_0} = X$.

ELIMINATION THEOREM. Let Y be a n_0-ary sequential partition of a set X of M-constituents in an antecedent δ. Then the following rule is admissible in **L**:

$$\frac{\gamma_p \vdash M\,[1 \leqslant p \leqslant p_0] \quad \delta \vdash D}{\delta(\gamma_p/Y_p)_{p=1}^{p_0} \vdash D}.$$

What makes this Elimination Theorem different from its cousins is that it countenances not just one but a sequence of left premisses. This generalization of the usual theorem seems required by the interaction of the rules $(\vee\vdash)$ and $(\vdash\Box)$. The proof will be by a triple induction (double would do, but the extra level is convenient). The Outer induction is on the length of M, the Middle induction on the 'combined rank' of M in the left and right premisses, and the Inner induction on the number of left premisses at maximum 'rank'. The argument requires an analysis of the concept of 'rank', to which we next proceed by way of some auxiliary notions.

4. SOME DEFINITIONS AND THE NORMALITY PROPERTY

An *inference* is an ordered pair consisting of a finite (non-null) sequence of consecutions – the *premisses* – as left entry and a consecution – the *conclusion* – as right entry. A *rule* is a set of inferences; its members are called *instances*. A *calculus* is a set of consecutions – the *axioms* – together with a set of rules; and an inference *of* a calculus is an instance of one of its rules. A *derivation in* a calculus **S** is, as usual, a tree of consecutions, each branch of which terminates in an axiom of **S**, and each nonterminal node being such that the pair consisting of a sequence of nodes immediately above it as left entry and the given node as right entry constitutes an inference of **S**.

Rank in a derivation has to be defined relative to an 'analysis' of the inferences of **S**: hence the following. In the first place, a pair $\langle P, C \rangle$ is an *analysis of* an inference *Inf* if P is a subset of the constituents of *Inf* and C is an equivalence relation on P. Secondly, $\langle P, C \rangle$ is a *normal analysis of* an inference *Inf* just in case the following hold.

1. $\langle P, C \rangle$ is an analysis of *Inf*; i.e., P is a subset of the constituents of *Inf*, and C is an equivalence relation on P.

In the following we call members of P 'parameters' and say that constituents related by C are 'congruent'.

2. Congruent parameters are occurrences of the same formula.
3. Congruent parameters are on the same side of $\vdash$.
4. Each parameter is congruent to exactly one parameter in the conclusion.

These conditions are adapted from [3, p. 197]; we comment several paragraphs below.

The above gives us the notion of an analysis of a single inference; moving up two levels, we say that f is an *analysis-function* for a calculus **S** if f is a function defined on all inferences of **S**, and such that $f(Inf)$ is always an analysis of *Inf*. (This disallows the possibilities of analyses relative to rules, and of multiple analyses, perhaps wanted in the more general case but avoidable in application to **L** by one *ad hoc* step taken below.) Further, f is a *normal analysis-function* for **S** if it maps each inference of **S** into a normal analysis of that inference.

We introduce the following terminology in the spirit of [3]. Let

$f(Inf) = \langle P, C \rangle$. Then the members of P are *f-parameters* and more particularly *f-premiss-parameters* or *f-conclusion-parameters* according to where they lie. Further, constituents of *Inf* which are *not* in P are called *f-principal* if they are in the conclusion of *Inf* and *f-subalterns* if in one of the premisses.

We choose a normal analysis-function $f_{\mathbf{L}}$ for **L** by specifying $f_{\mathbf{L}}(Inf) = \langle P, C \rangle$ as follows. Turning first to the conclusion of *Inf*, we put in P and thereby make f-conclusion-parameters all conclusion-constituents of *Inf except* any constituent newly introduced by one of the logical rules; hence such a constituent (if there is one) is $f_{\mathbf{L}}$-principal. Further, with respect to any premiss of *Inf*, we put in P and thereby make $f_{\mathbf{L}}$-premiss-parameters just those constituents (if any) matching in a way obvious from the statement of the rules a $f_{\mathbf{L}}$-conclusion-parameter of *Inf*; hence the others (if any) are $f_{\mathbf{L}}$-subalterns. Lastly, we define C by taking the reflexive, symmetrical, and transitive closure of the following relation C_0: each $f_{\mathbf{L}}$-premiss-parameter bears C_0 to the $f_{\mathbf{L}}$-conclusion-parameter which it matches in way obvious (again) from the statement of the rules.

In the above we have relied on 'the statement of the rules'. In order to be sure this procedure makes sense, we should first verify that no inference falls under more than one rule of **L**, and that 'the statement of a rule' provides a unique congruence relation; and this is not so for those instances of one of the rules ($CV \vdash$) in which (1) the conclusion is identical to the premiss in virtue of the permutation of adjacent instances of the *same* antecedent (i.e., with reference to the statement of the rule, $\alpha_i = \alpha_{i+1}$), and (2) there is in the premiss more than one case of like adjacent constituents; for these instances of the rules ($CV \vdash$) would allow more than one analysis – and indeed could fall under both ($CE \vdash$) and ($CI \vdash$) – in accordance with 'the statement of the rules'. To avoid this difficulty we must do something or other *ad hoc*; our choice is to modify the above by declaring that in such cases each premiss-constituent shall bear C_0 to the similarity positioned conclusion-constituent.

NORMALITY PROPERTY. $f_{\mathbf{L}}$ is a normal analysis-function for **L**.

Proof by inspection.

We note the following in regard to our definition of normality and our particular analysis of **L**. (a) Condition 2, while sensible and faithful to [3], is not used below. (b) In virtue of 1 and 4, a conclusion-parameter can be congruent only to itself; more generally, congruence is determined uniquely by specifying for each premiss-parameter the conclusion-parameter to which it is congruent; one then takes the reflexive, symmetric, and transitive closure. (c) Condition 4 is important to our argument, and 3 is used though it could be avoided (but it is sensible; indeed, one might think of strengthening it in our context of two kinds of sequences). (d) [3] requires that every conclusion-parameter shall be congruent to at least one parameter in at least one premiss; our analysis of ($KE \vdash$) does not satisfy this condition. (e) [3] requires that a parameter shall be congruent to at most one parameter in any one premiss; our analysis of ($WV \vdash$) does not satisfy this condition. (f) [3] requires that there be at most one principal (nonparametric) constituent in the conclusion; our analysis satisfies this further condition, but we do not add it since alternative analyses, especially of ($KE \vdash$), not satisfying this condition would still be sensible and permit our argument to go through with hardly any modification.

We now define the concept of f-rank, where f is an analysis-function for a calculus **S**. Let *Der* be a derivation in **S**; let S be the final consecution in *Der*; unless S is an axiom, let *Inf* be the inference in *Der* of which S is the conclusion; and let X be a set of constituents in S. Define the *f-rank of X in Der* as follows. If X is empty its f-rank in *Der* is 0. If X is nonempty but contains no f-conclusion-parameters – i.e., if all the constituents in X are f-principal or if S is an axiom of **S** – then the f-rank of X in *Der* is 1. Otherwise let the inference *Inf* be

$$(1) \qquad \frac{S_n \quad [1 \leqslant n \leqslant n_0]}{S}$$

and for each n $[1 \leqslant n \leqslant n_0]$ let Der_n be the subderivation of *Der* terminating in S_n, and let X_n be the (possibly empty) set of f-premiss-parameters in S_n which are f-congruent to some member of X. Let r be the maximum among the f-ranks of the various X_n in their respective Der_n $[1 \leqslant n \leqslant n_0]$. Then the f-rank in *Der* is defined as $r + 1$. (The feature of the definition giving inductive control is that if X is nonempty, its rank in *Der* is always greater than that of any of the X_n in their respective Der_n.)

It is convenient also to define *the consequent f-rank of Der* as the f-rank of X in *Der* for the particular case where X is the unit set of the consequent of the final consecution S of *Der*.

When f is in particular $f_{\mathbf{L}}$, we drop it as a prefix on rank, congruence, parameter, etc.

5. ELIMINATION THEOREM; OUTLINE OF PROOF

We may use these concepts to rephrase the Elimination Theorem in a way convenient for the upcoming proof. Make the following abbreviations.

$\Phi(M, k, j, i)$: $\delta, \gamma_1, \ldots, \gamma_{p_0}$ are antecedents and M and D are formulas; X is a set of M-constituents of δ, and Y is a p_0-ary sequential partition of X; for $1 \leqslant p \leqslant p_0$, Der_p is a derivation in **L** of $\gamma_p \vdash M$, with k the maximum of the consequent ranks of all the Der_p, and i the number of the Der_p having this maximum consequent rank; and finally, Der is a derivation in **L** of $\delta \vdash D$, with j the rank of X in Der.

$\psi(M, k, j, i)$: for all $p_0, \delta, \gamma_1, \ldots, \gamma_{p_0}, D, X$, and Y; if $\Phi(M, k, j, i)$, then $\delta(\gamma_p/Y_p)_{p=1}^{p_0} \vdash D$ is provable in **L**.

With these abbreviations we rephrase the Elimination theorem in a way convenient for the upcoming inductive proof.

ELIMINATION THEOREM. For every M, k $[k \geqslant 0]$, j $[j \geqslant 0]$, and i $[i \geqslant 1]$: $\psi(M, k, j, i)$.

Proof. We proceed by a nested induction. First choose arbitrary M, and suppose

Outer hypothesis: for all M' shorter than M, for all k, j, i, $\psi(M', k, j, i)$. Next choose arbitrary k and j and suppose

Middle hypothesis: for all k' and j' such that $(k' + j') < (k + j)$, for all i, $\psi(M, k', j', i)$.

Inner hypothesis: for all i' such that $i' < i$, $\psi(M, k, j, i')$. Lastly, choose arbitrary $p_0, \delta, \gamma_1, \ldots, \gamma_{p_0}, D, X$, and Y, and suppose

Step hypothesis: $\Phi(M, k, j, i)$.

In order to establish the theorem, it suffices to show that under these hypotheses, $\delta(\gamma_p/Y_p)_{p=1}^{p_0} \vdash D$ is provable in **L**. The argument is by cases; but first for convenience we define

$$\textit{L-premisses}: \ \gamma_p \vdash M \ [1 \leqslant p \leqslant p_0] \qquad \textit{R-premiss}: \ \delta \vdash D$$

$$\textit{Conclusion}: \quad \delta(\gamma_p/Y_p)_{p=1}^{p_0} \vdash D$$

In the first place, if X is empty then R-premiss = Conclusion, so the Step hypothesis may be used. Suppose now that X is nonempty.

Case 1. Each of the derivations of the L-premisses has a consequent rank of 1, hence is either an axiom or comes by a logical rule with the consequent occurrence of M principal. Two subcases.

Case 1.1. X contains no conclusion-parameters; hence R-premiss is either an axiom or comes by a logical inference the principal constituent of which is the unique member of X. Since X has exactly one member, all but one of the Y_p $[1 \leqslant p \leqslant p_0]$ is empty, so that if $p_0 \geqslant 2$, the Inner hypothesis can be used with the one remaining needed L-premiss. Suppose then that $p_0 = 1$; i.e., there is a unique L-premiss which by 1 is either an axiom or comes by a logical rule. Three subcases.

Case 1.11. The L-premiss is an axiom. Hence R-premiss = Conclusion; use the Step hypothesis.

Case 1.12. R-premiss is an axiom. Hence the L-premiss = Conclusion; use the Step hypothesis.

Case 1.13. Each of the L-premiss and R-premiss comes by a logical inference, with the consequent occurrence of M principal in one and the unique M-constituent in X principal in the other. Evidently the inferences must be instances of matching logical rules. Five subcases (for $\circ$, &, $\vee$, $\rightarrow$, and $\Box$). In each case the argument, using the Outer hypothesis, is straightforward, with an occasional use of one of the rules (V_2 int).

Case 1.2. X contains at least one conclusion-parameter. See below.

Case 2. At least one of the derivations of the L-premisses has a consequent rank of at least 2; hence $k \geqslant 2$. See below.

In the sequel we treat only the cases 1.2 and 2; to deal with these we elaborate on the concepts of left and right regularity of [3], Section 28.5.3. The plan is to state easily verifiable properties of the rules of **L**, and then to see how more complex properties of the rules needed in the treatment of the two cases mentioned are corollaries of the easily verified properties.

6. CLOSURE UNDER SUBSTITUTION AND CASE 1.2

Let f be an analysis-function for **S**. We shall say that a rule Ru of S is *closed under f-parametric substitution* under the following conditions. Let Inf be an instance of Ru, and let X be a f-congruence class of constituents of Inf; i.e., the set of all constituents of Inf which are f-congruent to some constituent of Inf. Then for arbitrary antecedent β, the inference $Inf(\beta/X)$ which results from substituting β for all members of X is itself an instance of Ru. (Of course on the *right* only *formulas* β may be substituted.) Furthermore, f-parameter and f-congruence for $Inf(\beta/X)$ are as follows: in the first place, constituents of $Inf(\beta/X)$ lying within any substituted occurrence of β are f-parameters, and are congruent to just those constituents of $Inf(\beta/X)$ occupying like positions in substituted occurrences of β. Secondly, note that substitution induces a natural 1–1 correspondence between (a) constituents of $Inf(\beta/X)$ *not* lying within any substituted occurrence of β and (b) the unsubstituted-for occurrences of *Inf*; this correspondence is an isomorphism with respect to both f-parameterhood and f-congruence. I.e., such a constituent of $Inf(\beta/X)$ is an f-parameter iff its correspondent in *Inf* is, and such constituents of $Inf(\beta/X)$ are f-congruent iff their correspondents are. (Compare [3, p. 198], (r6).)

CLOSURE UNDER PARAMETRIC SUBSTITUTION PROPERTY. Each rule of **L** except ($\vdash \Box$) is closed under $f_{\mathbf{L}}$-parametric substitution.

Proof by inspection of the rules of **L**. Verification is perhaps easier if we use normality, parts 3 and 4, to restate the property. Let

$$(1)\qquad \frac{\alpha_n \vdash C_n \quad [1 \leqslant n \leqslant n_0]}{\delta \vdash D}$$

be an instance of *Ru* and let y be a f-conclusion-parameter. Let X_n be the set of f-parameters in the premiss $\alpha_n \vdash C_n$ which are congruent to y; then

$$(2)\qquad \frac{\alpha_n(\beta/X_n) \vdash C_n \quad [1 \leqslant n \leqslant n_0]}{\delta(\beta/\{y\}) \vdash D} \quad \text{or}$$

$$\frac{\alpha_n \vdash C_n(\beta/X_n) \quad [1 \leqslant n \leqslant n_0]}{\delta \vdash \beta}$$

(according as y is on the left or right; if on the right, β must be a formula) is an instance of *Ru*, and with f-parameterhood and f-congruence as stated.

One form in which we shall need this property is stated in the following

COROLLARY. Let

$$(3)\qquad \frac{\alpha_n \vdash C_n \quad [1 \leqslant n \leqslant n_0]}{\delta \vdash D}$$

be an instance of a rule *Ru* of **L** other than ($\vdash \Box$), let X be a set of conclusion parameters in δ, and let Y be a p_0-ary sequential partition of X. Let Y_{pn} $[1 \leqslant n \leqslant n_0, 1 \leqslant p \leqslant p_0]$ be the set of parameters in α_n which are congruent to one of those in Y_p, and let $\gamma_1, \ldots, \gamma_{p_0}$ be antecedents. Then the following will also be an instance of *Ru*:

$$(4) \qquad \frac{\alpha_n(\gamma_p/Y_{pn})_{p=1}^{p_0} \vdash C_n \quad [1 \leqslant n \leqslant n_0]}{\delta(\gamma_p/Y_p)_{p=1}^{p_0} \vdash D}$$

Furthermore, parameterhood and congruence are undisturbed for unsubstituted-for constituents.

Proof. One only needs to verify that the simultaneous substitution of the Corollary can be reduced to successive single substitutions as authorized by the Closure under parametric substitution property; and this is guaranteed by normality, especially part 4, which implies that for each n $[1 \leqslant n \leqslant n_0]$, all the Y_{pn} $[1 \leqslant p \leqslant p_0]$ are pairwise disjoint.

We can now treat Case 1.2. By the hypothesis of the case, we know that the derivation *Der* of *R*-premiss terminates in an inference *Inf* with respect to which at least one *M*-constituent in X is a conclusion-parameter. Suppose first that *Inf* is an instance of a rule *Ru* other than ($\vdash \square$), and let *Inf* be (3). Define $\bar{X}$ as the set of conclusion-parameters in X, and let $\bar{Y}$ be that p_0-ary sequential partition of $\bar{X}$ such that $\bar{Y}_p = Y_p \cap \bar{X}$ $[1 \leqslant p \leqslant p_0]$. For $1 \leqslant n \leqslant n_0$, let $\bar{Y}_{pn}$ be the set of premiss-parameters in $\alpha_n \vdash C_n$ which are congruent to a member of $\bar{Y}_p$. By using the **L**-premisses and $\alpha_n \vdash C_n$ with the Middle hypothesis, obtain the **L**-provability of

$$(5) \qquad \alpha_n(\gamma_p/\bar{Y}_{pn})_{p=1}^{p_0} \vdash C_n \quad [1 \leqslant n \leqslant n_0]$$

The inference from the premisses (5) to

$$(6) \qquad \delta(\gamma_p/\bar{Y}_p)_{p=1}^{p_0} \vdash D$$

is, by the Corollary to the Closure under parametric substitution property, also an instance of *Ru*, so that (6) is provable in **L**. Now if $X = \bar{X}$, then (6) = Conclusion, and we are done.

Otherwise, let $\bar{\bar{X}}$ be the set of *M*-constituents in (6) corresponding to those in $X - \bar{X}$, and let $\bar{\bar{Y}}$ be the p_0-ary sequential partition of $\bar{\bar{X}}$ defined by letting $\bar{\bar{Y}}_p$ be the set of *M*-constituents in (6) corresponding to those in $Y_p - \bar{X}$. By the 'furthermore' part of the cited Corollary, all members of $\bar{\bar{X}}$ (actually there will be exactly one, but we do not use this information) must be nonparametric (principal) in the inference from (5) to (6), so that the rank of $\bar{\bar{X}}$ in the derivation of (6) terminating in the inference from (5) to (6) is 1. We may therefore use the **L**-premisses with (6) and the Middle hypothesis (1 being less than $2 \leqslant j$) to obtain

$$(\delta(\gamma_p/\bar{Y}_p)_{p=1}^{p_0})(\gamma_p/\bar{\bar{Y}}_p)_{p=1}^{p_0} \vdash D$$

which is just Conclusion.

Suppose second and last that *Inf* is an instance of ($\vdash\Box$), and in particular is

$$\frac{\delta \vdash C}{\delta \vdash \Box C} \quad (R\text{-premiss})$$

Apply the Middle hypothesis to the L-premisses and $\delta \vdash C$, obtaining the **L**-provability of

$$(7) \qquad \delta(\gamma_p/Y_p)_{p=1}^{p_0} \vdash C$$

We wish to show that (7) is a suitable premiss for ($\vdash\Box$), since if it is we may thereby obtain Conclusion. In the first place, we observe that by the conditions on ($\vdash\Box$), every constituent in δ must have either the form $\Box A$ or be $\boldsymbol{t}$; in particular such is true for all of the M-constituents in X. We now invoke the case hypothesis 1: each **L**-premiss is either an axiom or has its consequent M-constituent as principal constituent for a logical rule. Since there is no right rule for $\boldsymbol{t}$, all of the nonaxiomatic L-premisses must come by ($\vdash\Box$). Accordingly, by the restriction on this rule, every constituent in each γ_p must either have the form $\Box A$ or be $\boldsymbol{t}$, so that the same is true for every constituent in (7); hence (7) is indeed an appropriate premiss for an inference by ($\vdash\Box$) to Conclusion.

This completes our treatment of Case 1.2.

CLOSURE UNDER EMBEDDING AND CASE 2. Let f be an analysis-function for S. We shall say that a rule *Ru* of S is *closed under embedding in a larger f-parametric context* if the following holds. Suppose *Ru* has as an instance the inference

$$(8) \qquad \frac{\delta_m \vdash A_m \quad [1 \leqslant m \leqslant m_0] \qquad \alpha_n \vdash C \quad [1 \leqslant n \leqslant n_0]}{\gamma \vdash C}$$

where (a) the displayed occurrence of C in $\gamma \vdash C$ is a f-conclusion-parameter, where (b) the $\alpha_n \vdash C$ are all the premisses – we suppose there is at least one – containing a f-premiss-parameter on the right of $\vdash$ congruent to the aforementioned occurrence of C, and where (c) the $\delta_m \vdash A_m$ (if any; there may be none) are the premisses in which the right side of $\vdash$ is *not* a f-premiss-parameter (hence a subaltern). (Subsequent references to (8) are all supposed to include these provisos; we call the $\alpha_n \vdash C$ the *f-parametric* *premisses* and the $\delta_m \vdash A_m$ the *f-nonparametric premisses*.) Let β be an antecedent and let y be a constituent of β. Then closure under embedding in a larger f-parametric context requires that

$$(9) \qquad \frac{\delta_m \vdash A_m \quad [1 \leqslant m \leqslant m_0] \qquad \beta(\alpha_n/\{y\}) \vdash C \quad [1 \leqslant n \leqslant n_0]}{\beta(\gamma/\{y\}) \vdash C}$$

also be an instance of Ru. (We note that this property, though related to those of [3, pp. 197–198], has no quite clear analogue there. Its closest cousin is the part of (r6) which speaks of 'inserting' new parameters.)

CLOSURE UNDER EMBEDDING PROPERTY. All the rules of **L** are closed under embedding in a larger $f_{\mathbf{L}}$-parametric context.

Proof. The right logical rules of **L** satisfy the condition vacuously. For the other rules, write (9) as

$$(9') \qquad \frac{\delta_m \vdash A_m \quad [1 \leqslant m \leqslant m_0] \qquad \Gamma\alpha_n\Delta \vdash C \quad [1 \leqslant n \leqslant n_0]}{\Gamma\gamma\Delta \vdash C}.$$

Now verification of closure under embedding can be obtained by inspection of these rules, noting that whenever (8) is an instance of a rule Ru of L, so is (9′).

We remark that there is only one rule of **L**, namely ($\rightarrow\vdash$), having instances (8) with any nonparametric premisses at all, and in that case there is only one. And there is only one rule of **L**, namely ($\vee\vdash$), having instances (8) with more than a single parametric premiss; and even in that case, there are but two.

For application we are going to need a corollary of the Closure under embedding property, which will be an immediate consequence of a certain fact to the effect that if a rule is closed under embedding in a larger f-parametric context in the sense defined above, then it is also closed, in a sense, under a more complex sort of embedding. For statement of the fact, we define $\mathrm{Part}_{n_0}(X)$ as the set of all n_0-ary sequenced partitions of X.

FACT. Let f be an analysis-function for a system **S** of which Ru is a rule, and let Ru be closed under embedding in a larger f-parametric context. For each antecedent β and nonempty set of occurrences X in β, if (8) is an instance of Ru, then (10) below is also in a wider sense; that is, the conclusion of (10) may be obtained from the premisses of (10) by a *series* of one or more applications of Ru.

$$(10)\quad \frac{\delta_m \vdash A_m \quad [1 \leqslant m \leqslant m_0] \quad \beta(\alpha_n/Y_n)_{n=1}^{n_0} \vdash C \quad [Y \in \mathrm{Part}_{n_0}(X)]}{\beta(\gamma/X) \vdash C}.$$

The notation is intended to suggest that in addition to the m_0 f-nonparametric premisses which come over unchanged from (8), there is a premiss $\beta(\alpha_n/Y_n)_{n=1}^{n_0} \vdash C$ for each member Y of the set Part_{n_0} of n_0-ary sequential partitions of X.

We note that in the special case $n_0 = 1$ – i.e., when there is only one f-parametric premiss – (8) and (10) respectively assume the simpler forms

$$(8')\quad \frac{\delta_m \vdash A_m \quad [1 \leqslant m \leqslant m_0] \qquad \alpha \vdash C}{\gamma \vdash C}$$

$$(10')\quad \frac{\delta_m \vdash A_m \quad [1 \leqslant m \leqslant m_0] \qquad \Gamma_1\alpha\Gamma_2\alpha\Gamma_3 \ldots \Gamma_{n-1}\alpha\Gamma_n \vdash C}{\Gamma_1\gamma\Gamma_2\gamma\Gamma_3 \ldots \Gamma_{n-1}\gamma\Gamma_n \vdash C}.$$

Among rules of L having instances of the form (8) there is only one, namely, ($\vee\vdash$) not falling under the special case (8′)–(10′) of closure under embedding; and even then n_0 is but 2.

Proof. By simple induction on the cardinality of X. If there is but one member in X, then (10) = (9) and the hypothesis of the Fact suffices. Suppose the Fact true for X with q members; we show it continues to hold for X with $q + 1$ members. Choose $y \in X$, and rewrite the f-parametric premisses of (10) in n_0 batches according to which of the sets $Y_1, \ldots, Y_{n_0}$ contains y:

(11)

$$\beta(\alpha_1/\{y\}, \alpha_1/(Y_1 - \{y\}), (\alpha_i/Y_i)_{i \neq 1}) \vdash C \quad [y \in Y_1 \text{ and } Y \in \mathrm{Part}_{n_0}(X)]$$

$$\vdots$$

$$\beta(\alpha_{n_0}/\{y\}, \alpha_{n_0}/(Y_{n_0} - \{y\}), (\alpha_i/Y_i)_{i \neq n_0}) \vdash C \quad [y \in Y_{n_0} \text{ and } Y \in \mathrm{Part}_{n_0}(X)].$$

For each n $[1 \leqslant n \leqslant n_0]$, consider the n-th batch of premisses, drawn from (11),

$(11)_n$

$$\beta(\alpha_n/\{y\}, \alpha_n/Y_n - \{y\}, (\alpha_i/Y_i)_{i \neq n} \vdash C \quad [y \in Y_n \text{ and } Y \in \mathrm{Part}_{n_0}(X)].$$

These may be rewritten

$(11')_n \qquad \beta(\alpha_n/\{y\}, \alpha_n/Z_n, (\alpha_i/Z_i)_{i\neq n} \vdash C \quad [Z \in \mathrm{Part}_{n_0}(X-\{y\})]$

that is,

$(11'')_n \qquad \beta(\alpha_n/\{y\}, (\alpha_i/Z_i)_{i=1}^{n_0}) \vdash C \quad [Z \in \mathrm{Part}_{n_0}(X-\{y\})]$.

The cardinality of $X-\{y\}$ is q, so that by the hypothesis of the induction we may obtain from the premisses $(11'')_n$, together with the m_0 nonparametric premisses of (10), by means of a series of applications of *Ru*, the following:

$(12)_n \qquad \beta(\alpha_n/\{y\}, \gamma/(X-\{y\})) \vdash C.$

We do this for each n $[1 \leqslant n \leqslant n_0]$. Now we know by hypothesis that *Ru* is closed under embedding in a larger *f*-parametric context; consequently, from all of the consecutions $(12)_n$ $[1 \leqslant n \leqslant n_0]$ together with the non-parametric premisses of (10), we may obtain by one further application of *Ru*

$$\beta(\gamma/\{y\}, \gamma/(X-\{y\})) \vdash C$$

which is just the conclusion of (10), as desired, and which finishes the proof of the Fact.

COROLLARY. Every rule *Ru* of **L** is closed under embedding in the wider sense that if (8) is an instance of *Ru*, then one may obtain the conclusion of (10) from its premisses by a series of zero or more applications of *Ru*.

Proof. Immediate from the Closure under embedding property and the Fact.

We are now in a position to deal with Case 2, the hypothesis of which is that at least one of the derivations of the L-premisses has a consequent rank of at least 2. Choose one of these whose consequent rank is the maximum, k, and for notational convenience (only) let us pretend we have chosen the derivation Der_1 of the *first* L-premiss, $\gamma_1 \vdash M$. Let Der_1 terminate in an inference

$$(13) \qquad \frac{\beta_m \vdash A_m \quad [1 \leqslant m \leqslant m_0] \qquad \alpha_n \vdash M \quad [1 \leqslant n \leqslant n_0]}{\gamma_1 \vdash M}$$

with the nonparametric premisses (if any) collected on the left and the parametric premisses – there must by the case hypothesis be at least one – collected on the right. Consider the $n_0 + p_0 - 1$ premisses

$$\alpha_n \vdash M \quad [1 \leqslant n \leqslant n_0] \qquad \gamma_p \vdash M \quad [2 \leqslant p \leqslant p_0]$$

and choose an arbitrary n_0-ary sequential partition Z of the set Y_1. When put together with the R-premiss, either the Middle hypothesis (if the number i of left derivations with maximum consequent rank is 1) or the Inner hypothesis (otherwise) will justify our claim that the following are all provable:

$$(14) \qquad \delta((\alpha_n/Z_n)_{n=1}^{n_0}, (\gamma_p/Y_p)_{p=2}^{p_0}) \vdash D \quad [Z \in \mathrm{Part}_{n_0}(Y_1)].$$

Now we argue as follows. In the first place, since (13) is an instance of some rule Ru of **L**, so also is the result of substituting D for the exhibited parametric M – by the Parametric substitution closure property; i.e., we have

$$\frac{\beta_m \vdash A_m \quad [1 \leqslant m \leqslant m_0] \qquad \alpha_n \vdash D \quad [1 \leqslant n \leqslant n_0]}{\gamma_1 \vdash D}$$

as an instance of Ru, and with the sorting into parametric and non-parametric premisses unchanged (by the 'furthermore' clause). Consequently, by the Corollary to the Closure under embedding property, the following consecution can be obtained from the premisses (14) by a series of one or more applications of Ru:

$$\delta(\gamma_1/Y_1, (\gamma_p/Y_p)_{p=2}^{p_0}) \vdash D.$$

But this is just Conclusion; which completes the proof of the Elimination theorem and of the equivalence of **L** to $\mathbf{R}_+^{\Box\circ t}$.

We observe that in common with other consecution treatments of the more complicated relevance calculuses, no decision procedure appears to be immediately forthcoming.

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