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Relevant Analytic Tableaux

"Logic is at root all about trees"

Abstract. Tableau formulations are given for the relevance logics ***E*** (Entailment), ***R*** (Relevant implication) and ***RM*** (Mingle). Proofs of equivalence to modus-ponens-based formulations are via "left-handed" Gentzen sequenzen-kalküle. The tableau formulations depend on a detailed analysis of the structure of tableau rules, leading to certain "global requirements". Relevance is caught by the requirement that each node must be "used"; modality is caught by the requirement that only certain rules can "cross a barrier". Open problems are discussed.

In this paper[1] we present a tableau-style analysis of the implication-negation fragments of the principal relevant logics ***E***, ***R*** and ***RM*** of [1] as well as the classical propositional calculus ***TV***. This analysis, although similar in certain respects to the one given by Smullyan in [32] for ***TV***, differs substantially in that it is purely *proof theoretic* in character, as opposed to that of Smullyan, which is *semantical*. This is then, as far as we know, the first time that a non-semantically based tableau-style analysis has been given for a set of logics. Therefore this paper seems like an appropriate place to distinguish between these two different tableau-style analyses. Henceforth the kinds of structures studied in a *semantically* based tableau-style analysis will be called *analytic semantic tableaux*, while the kinds of structures studied in a *proof theoretically* based tableau-style analysis will be called simply *analytic tableaux*. Further, logical systems in which the theoremhood (validity) of formulas is determined by whether or not analytic tableaux (analytic semantic tableaux) constructed in a prescribed way from these formulas meet certain necessary and sufficient conditions will be called *analytic tableau systems* (*analytic semantic tableau systems*). Hence the *analytic tableau formulations* (*analytic semantic tableau formulations*) of some arbitrary ogic ***S*** is simply

[1] This paper was delivered by McRobbie at the *1976 Vacation School in Logic*, held at the Victoria University of Wellington, New Zealand, from August 15–22, and was announced in [24]. Thanks for encouragement and many helpful discussions are due in particular to Dunn, Martin and Meyer and also to A.S. McRobbie, Mortensen, Rennie, R. Routley and Rubenstein. Thanks also to the National Science Foundation for partial support of Belnap through Grant SOC71 03594 A04.

an analytic tableau system (analytic semantic tableau system) that is provably equivalent to $\boldsymbol{S}$.[2] In what follows, by '*tableau*' we shall be referring to the proof theoretic species and not to the semantic one unless otherwise indicated.

All conventions are adopted from [1]. The systems $\boldsymbol{E}$, $\boldsymbol{R}$, and $\boldsymbol{RM}$ are defined in §27 of [1], and their implication-negation fragments $\boldsymbol{E}_{\overset{\sim}{\rightarrow}}$, $\boldsymbol{R}_{\overset{\sim}{\rightarrow}}$, and $\boldsymbol{RM}_{\overset{\sim}{\rightarrow}}$ are defined in §14 of [1]. The system $\boldsymbol{TV}_{\overset{\sim}{\rightarrow}}$ is the implication-negation fragment of $\boldsymbol{TV}$.

I.

In this section we will present the definitions of the various structures, and the concepts defined on these structures, which we shall be using. These definitions are essentially based on, and are extensions of, those given in Smullyan [32] and Toledo [35].

A *tableau* τ is a finite *tree* to whose *nodes* formulas have been assigned. Some of the nodes may also be *annotated* in a sense to be specified. There may be *barriers* between certain adjacent nodes. We use $<_0$, $<$, and $\leqslant$ respectively for *immediate predecessor*, *predecessor*, and *predecessor-identity*. We let F be the function which assigns formulas to nodes, so that $F(i)$ is the formula assigned to node i. An *end node* is one which is not the predecessor of any node. A *branch* of a tableau is a sequence of nodes commencing with the *origin* such that each node that is not an end node

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[2] Analytic semantic tableau formulations of the classical propositional calculus and its first-order extension are an adaptation of a method used by Hintikka in [14] for constructing model sets and of the *semantic tableaux* of Beth [3], which themselves derive from the (proof theoretic) tableau-like methods used by Gentzen in [10] for proving theorems of his *Sequenzenkalküle*. Such formulations were first given by Smullyan in a series of papers [28], [29], [30] and [31] and then received comprehensive treatment by him in [32] and [33]. Davidson [5], Davidson, Jackson and Pargetter [6], Fitting [9], Girle [12] and Smullyan [34] have studied similar formulations of a wide range of modal logics (with necessity taken as a systemic primitive), while Smullyan in [34] has done the same for classical and intuitionistic logic. More recently Linden [20] has given analytic semantic tableau formulations of certain infinitary logics (although see his comments pp. 29–33 on the difference between syntactic and semantic transfinite tableaux), while Toledo [35] has done the same for a range of higher-order theories. Under the *nom de plume* of '*truth trees*', analytic semantic tableau formulations of classical logics have been popularized by Gustason and Ulrich [13], Jeffrey [15], Leblanc and Wisdom [19], and Rennie and Girle [26]. As far as we know the only semantically based tableaustyle analysis of any relevant logics or parts thereof is the analytic semantic tableau formulation of the first-degree entailments of $\boldsymbol{T}$, $\boldsymbol{E}$ and $\boldsymbol{R}$ given by Dunn in [7].

of the tableau has as its immediate successor in the branch one of its immediate successors in the tableau.

With each logical system S subject to our investigation we will be associating a tableau system TS. TS will be defined by (a) listing some *rules* which TS admits in the construction of its tableaux, and (b) listing some *global requirements* which the tableaux of TS must satisfy. We pause to characterize the two sorts of rules with which we shall be dealing. It is important to note that the prepositions "*at*" and "*to*" when italicized are being used technically and that special attention needs to be paid when they are used in this way.

A *connective rule* is a rule applied *at* a given node n in some tableau *to* some node $m \leqslant n$, and, in a derivative sense, *to* the formula (of a prescribed form) assigned to m. The application of a connective rule *to* this formula generates (1) less complex formulas of a prescribed kind and (2) successor nodes to n. The generated formulas are assigned to the generated nodes in accordance with some prescribed pattern. The node n is annotated with "$R(m)$", where "R" stands in for the name of the connective rule applied *at* n. (We note that ordinarily (e.g., in [15]) it is the node(s) generated by application of a rule that is (are) annotated, but it will become clear why it is important for us to annotate the node *at* which the rule is applied rather than either the generated node(s) or the node *to* which the rule is applied.) A connective rule may also generate a *barrier* between n and its successor node(s); as is explained below, in certain systems only certain rules are allowed to "cross a barrier."

A *branch closure rule* is a rule applied *at* a given end node n in some tableau *to* each member of a sequence of nodes $m_1, \ldots, m_k \leqslant n$, and derivatively *to* each member of a sequence of formulas (each one of which is of a prescribed form) assigned to the nodes $m_1, \ldots, m_k$. There are no new nodes or formulas generated by an application of a branch closure rule *at* n. The node n is annotated with "$R(m_1, \ldots, m_k)$", where "R" stands in for the name of the particular branch closure rule applied *at* n.

Recall that each TS has associated with it both a family of rules and some global requirements. The former permits us to define, abstractly and inductively, a *TS-tableau*: a finite unannotated chain of nodes to which formulas have been assigned is a TS-tableau, and if n is an end node of a TS-tableau *at* which no rule has been applied, then a new tableau resulting from the application *at* n of one of the rules admitted by TS is also a TS-tableau.

Note that although a formula may have a rule applied *to* it more than once, no more than one rule can be applied *at* a node. Hence, if the end node of some branch of a tableau has a branch closure rule applied *at* it, then no further construction can take place in that branch. Such a branch is said to be *closed*.

For each TS-tableau τ there is a uniquely determined sequence of nodes commencing with the origin of τ and continuing to (and including) the first annotated node of τ – or continuing to the end of τ if no node is annotated. We say that τ *begins with* those nodes, and also that τ *begins with* the sequence α of formulas assigned to those nodes. When thinking of building tableaux, we will also speak of *beginning* a tableau with a sequence of formulas α; i.e., constructing a sequence of nodes to which the members of α are assigned (in order).

Global requirements are brought in to characterize those tableaux which really do some work for us. We say that τ is a *TS-refutation of* a sequence of formulas α if (1) τ is a TS-tableau, (2) τ begins with α, and (3) τ satisfies the global requirements of TS. In these circumstances we say that τ is a *TS-refuting* tableau, and that α is *TS-refutable.* We write

$$\alpha \vdash_{TS}$$

when α is TS-refutable.

Note that the difference between TS-refuting tableaux and (mere) TS-tableax is satisfaction of the global requirements.

Naturally refutability is the dual of provability; so we say that A is a theorem of TS just in case $\overline{A} \vdash_{TS}$, and we write $\vdash_{TS} A$ for theoremhood.

In order to flesh out this abstract characterization, we next give the rules and the global requirements on which we shall be drawing. In our pictures, a solid [dotted] line from i down to j indicates $i <_0 j$ $[i \leqslant j]$ – i.e., our trees grow downwards.

Connective rules

Double Negation ($\approx$).

Let $\overline{\overline{A}}$ and A be assigned to i and j respectively and let n be annotated with $\approx(i)$, where $i \leqslant n <_0 j$. Then we say that $\approx$ has been applied *at n to $\overline{\overline{A}}$ at i*, generating A at j.

$$\begin{array}{lll} i & \overline{\overline{A}} & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ n & & \approx(i) \\ | & & \\ j & A & \end{array}$$

Negated Arrow ($\stackrel{\sim}{\rightarrow}$).

Let $\overline{A\rightarrow B}$, A and $\bar{B}$ be assigned to i, j and k respectively, and let n be annotated with $\underset{\rightarrow}{\sim}(i)$, where $i \leqslant n <_0 j <_0 k$. Then we say that $\underset{\rightarrow}{\sim}$ is applied *at* n *to* $\overline{A\rightarrow B}$ at i, generating A at j and $\bar{B}$ at k.

$$\begin{array}{lll} i & \overline{A\rightarrow B} & \\ \vdots & & \\ n & & \underset{\rightarrow}{\sim}(i) \\ | & & \\ j & A & \\ | & & \\ k & \bar{B} & \end{array}$$

Strict Negated Arrow ($S\underset{\rightarrow}{\sim}$).

This rule is exactly like $\underset{\rightarrow}{\sim}$, except that a barrier is generated between the node n *at* which the rule is applied and its immediate successor node. We may indicate barriers by horizontal lines. Hence an application of the rule $S\underset{\rightarrow}{\sim}$ has the following picture:

$$\begin{array}{lll} i & \overline{A\rightarrow B} & \\ \vdots & & \\ n & & S\underset{\rightarrow}{\sim}(i) \\ | & & \\ \hline | & & \\ j & A & \\ | & & \\ k & \bar{B} & \end{array}$$

Arrow ($\rightarrow$).

Let $A\rightarrow B$, $\bar{A}$ and B be assigned to i, j and k respectively, and let n be annotated with $\rightarrow(i)$, where $i \leqslant n <_0 j$ and $i \leqslant n <_0 k$ and $j \neq k$. Then we say that $\rightarrow$ is applied *at* n *to* $A\rightarrow B$ at i, generating $\bar{A}$ at j and B at k.

$$\begin{array}{ccccc} & i & A\rightarrow B & & \\ & \vdots & & & \\ & n & & & \rightarrow(i) \\ / & & \backslash & & \\ j\ \bar{A} & & kB & & \end{array}$$

Branch closure rules

Closure (*Cl*).

Let A and $\bar{A}$ be assigned to i and j respectively, and let n be annotated with $Cl(i, j)$ where $i, j \leqslant n$. Then we say that Cl is applied *at* n *to* A at i and $\bar{A}$ at j.

$$\begin{array}{lll} i & A & \\ \vdots & & \\ j & \bar{A} & \\ \vdots & & \\ n & & Cl(i,j) \end{array} \qquad\qquad \begin{array}{lll} j & \bar{A} & \\ \vdots & & \\ i & A & \\ \vdots & & \\ n & & Cl(i,j) \end{array}$$

Mingle Closure (*MCl*).

Let $A_1, \ldots, A_m, \bar{A}_1, \ldots, \bar{A}_m$ be assigned to (not necessarily distinct) nodes $i_1, \ldots, i_m, i_{m+1}, \ldots, i_{2m}$ $(m \geqslant 1)$ respectively, and let n be annotated with $MCl(i_1, \ldots, i_{2m})$, where for any j between 1 and $2m$, $i_j \leqslant n$. Then we say that MCl is applied *at* n *to* $A_1, \ldots, \bar{A}_m$ at $i_1, \ldots, i_{2m}$ respectively.
The following illustrates one possibility:

$$\begin{array}{lll} i_1 & A_1 & \\ \vdots & & \\ i_m & A_m & \\ \vdots & & \\ i_{m+1} & \bar{A}_1 & \\ \vdots & & \\ i_{2m} & \bar{A}_m & \\ \vdots & & \\ n & & MCl(i_1, \ldots, i_{2m}) \end{array}$$

Next we list the global requirements on which we shall draw. (Not all will apply to all systems.)

Global requirements

Closure requirement. Every branch of τ is closed, i.e., has a branch closure rule applied *at* its end node.

Use requirement. If a formula at some node in τ has a rule applied *to* it, then both the formula and the node will be said to be *used*. The requirement is that *each* node in τ (and hence the formula assigned to it) must be used at least once.

This is the requirement with which we catch the concept of relevance in a tableau. If τ satisfies the Use requirement, then it has no inessential ingredients, no loose pieces, no irrelevant or extraneous bits.

Barrier requirement. A TS-tableau τ satisfies the barrier requirement iff the only rule which *crosses a barrier* is $\rightarrow$. That is, if there is a barrier between j and k, and if any rule is applied *at k to j*, then the rule must be $\rightarrow$.[3]

Finally we may define the four tableau systems by stating for each (a) which rules it admits and (b) which global requirements it imposes on its tableaux. This may be summed up in the following table:

	Rules	Global requirements
$\boldsymbol{TTV}_{\underset{\rightarrow}{\sim}}$	$\approx, \rightarrow, \underset{\rightarrow}{\sim}, Cl$	Closure
$\boldsymbol{TR}_{\underset{\rightarrow}{\sim}}$	$\approx, \rightarrow, \underset{\rightarrow}{\sim}, Cl$	Closure, Use
$\boldsymbol{TRM}_{\underset{\rightarrow}{\sim}}$	$\approx, \rightarrow, \underset{\rightarrow}{\sim}, MCl$	Closure, Use
$\boldsymbol{TE}_{\underset{\rightarrow}{\sim}}$	$\approx, \rightarrow, S\underset{\rightarrow}{\sim}, Cl$	Closure, Use, Barrier

THEOREM 1. *Let $\boldsymbol{S}$ be $\boldsymbol{E}_{\underset{\rightarrow}{\sim}}, \boldsymbol{R}_{\underset{\rightarrow}{\sim}}, \boldsymbol{RM}_{\underset{\rightarrow}{\sim}}$ or $\boldsymbol{TV}_{\underset{\rightarrow}{\sim}}$. For all formulas A, $\vdash_{TS} A$ iff $\vdash_{\boldsymbol{S}} A$, i.e., iff there is a proof of A in the corresponding Hilbert system $\boldsymbol{S}$.*

The proof will be expedited by a detour.

II.

With Theorem 1 in mind, in this section we prseent "lefthanded" Gentzen formulations of the four systems of interest as intermediaries between the Hilbert calculi and the tableau systems. Notation is from

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[3] This requirement turns out to be modal in character, answering to the necessitive character of entailments. The concept of a barrier was suggested by Meyer (to whom the appropriate thanks go) as a simplification of a more complex earlier device. By varying the conditions on what rules can cross barriers and certain other conditions, it is possible to provide analytic tableau formulations of many strict implicational calculi and of the same calculi alternatively formulated using $\Box$ and $\Diamond$. See McRobbie [23].

§13.1 of [1]. Greek letters stand for sequences of formulas; all members of $\vec{\alpha}$ have the form $A \to B$; $\tilde{\alpha}$ is the sequence of negations of members of α.

We give a set of axioms and rules from which the various left handed Gentzen formulations of $\mathbf{E}_{\overset{\sim}{\to}}$, $\mathbf{R}_{\overset{\sim}{\to}}$, $\mathbf{RM}_{\overset{\sim}{\to}}$ and $\mathbf{TV}_{\overset{\sim}{\to}}$ are defined.

AXIOMS

$A, \bar{A} \vdash$ (Ax $\vdash$) $\qquad \alpha, \tilde{\alpha} \vdash$ (MAx $\vdash$)

STRUCTURAL RULES

Permutation ($C \vdash$) $\qquad$ *Contraction* ($W \vdash$) $\qquad$ *Weakening* ($K \vdash$)

$$\frac{\alpha, A, B, \beta \vdash}{\alpha, B, A, \beta \vdash} \qquad \frac{\alpha, A, A \vdash}{\alpha, A \vdash} \qquad \frac{\alpha \vdash}{\alpha, A \vdash}$$

CONNECTIVE RULES

Double Negation ($\approx \vdash$)

$$\frac{\alpha, A \vdash}{\alpha, \bar{\bar{A}} \vdash}$$

Arrow ($\to \vdash$)

$$\frac{\alpha, \bar{A} \vdash \beta, B \vdash}{\alpha, \beta, A \to B \vdash}$$

Negated Arrow ($\overset{\sim}{\to} \vdash$)

$$\frac{\alpha, A, \bar{B} \vdash}{\alpha, \overline{A \to B} \vdash}$$

Strict Negated Arrow ($S\overset{\sim}{\to} \vdash$)

$$\frac{\vec{\alpha}, A, \bar{B} \vdash}{\vec{\alpha}, \overline{A \to B} \vdash}$$

The left handed Gentzen systems for $\mathbf{E}_{\overset{\sim}{\to}}$, $\mathbf{R}_{\overset{\sim}{\to}}$, $\mathbf{RM}_{\overset{\sim}{\to}}$ and $\mathbf{TV}_{\overset{\sim}{\to}}$ are defined from these as follows.

$$\begin{aligned}
L_l\mathbf{E}_{\overset{\sim}{\to}} &= \text{Ax}, C \vdash, W \vdash, \approx \vdash, \to \vdash, S\overset{\sim}{\to} \vdash \\
L_l\mathbf{R}_{\overset{\sim}{\to}} &= \text{Ax}, C \vdash, W \vdash, \approx \vdash, \to \vdash, \overset{\sim}{\to} \vdash \\
L_l\mathbf{RM}_{\overset{\sim}{\to}} &= \text{MAx}, C \vdash, W \vdash, \approx \vdash, \to \vdash, \overset{\sim}{\to} \vdash \\
L_l\mathbf{TV}_{\overset{\sim}{\to}} &= \text{Ax}, C \vdash, W \vdash, K \vdash, \approx \vdash, \to \vdash, \overset{\sim}{\to} \vdash
\end{aligned}$$

We write $\alpha \vdash_{L_l\mathbf{S}}$ when $\alpha \vdash$ is a theorem of $L_l\mathbf{S}$.

THEOREM 2. *Let* $\mathbf{S}$ *be* $\mathbf{E}_{\overset{\sim}{\to}}$, $\mathbf{R}_{\overset{\sim}{\to}}$, $\mathbf{RM}_{\overset{\sim}{\to}}$ *or* $\mathbf{TV}_{\overset{\sim}{\to}}$. *For all formulas* A, $\vdash_{\mathbf{S}} A$ *iff* $A \vdash_{L_l\mathbf{S}}$. *That is, the left-handed systems exactly correspond to the respective Hilbert systems.*

PROOF. The cases for $\mathbf{E}_{\overset{\sim}{\to}}$ and $\mathbf{R}_{\overset{\sim}{\to}}$ can be recovered from §13 of [1], since the systems $L_l\mathbf{S}$ are the duals of the right handed systems $L_r\mathbf{S}$ treated there. The result for $\mathbf{RM}_{\overset{\sim}{\to}}$ is new, but straightforward. The case for $\mathbf{TV}$ can be extracted from Gentzen [10].

The next theorem gives us half of the equivalence between the tableau systems and the left-handed systems.

THEOREM 3. *For each considered* $\mathbf{S}$, *if* $\alpha \vdash_{L_l\mathbf{S}}$ *then* $\alpha \vdash_{T\mathbf{S}}$.

PROOF is by straightforward induction on the length of the proof of $\alpha \vdash$ in $L_l S$.

If $\alpha \vdash$ is an axiom of $L_l S$, then beginning a tableau with α and applying the appropriate branch closure rule *at* its last node *to* all its nodes constitutes a refutation, in the appropriate system, of α.

For the remaining rules of the $L_l S$-calculi, we assume we have TS-refutations of their premiss or premisses and then show how to construct a TS-tableau which will refute its conclusion because it satisfies the various global requirements of TS.

Permutation. Begin a tableau with its conclusion, and continue it in exactly the same way as for the tableau refuting its premiss, except for switching annotations referring to the permuted items. Because only the last node of the beginning of the tableau is annotated, all *to*-nodes will remain above their *at*-nodes, and there will be no problem about barriers. Nor is there any problem about any of the global requirements.

Contraction. Begin a tableau with its conclusion, and continue it in exactly the same way as the tableau refuting its premiss, except change references to one of the A's to become references to the other. Note in particular that MCl permits repetition of a node in its annotation. There is no problem about any of the global requirements.

Weakening. Begin a tableau with its conclusion, and continue it in exactly the same way as the tableau refuting its premiss. The inserted step will not be used, but this is not a problem since if $L_l S$ has Weakening as a rule, then TS does not have the Use requirement.

Of the connective rules, we go through only Arrow and Strict Negated Arrow.

Arrow. By hypothesis we have refutations τ_1 of α, $\bar{A}$ and τ_2 of β, B. Begin a tableau with the conclusion of Arrow, and apply $\rightarrow$ at its last step. Now continue down the left as in τ_1, and down the right as in τ_2, changing annotations to suit. Clearly the Closure and Barrier requirements will be satisfied by the constructed tableau if satisfied by the given ones, and so will the Use requirement, since members of α and $\bar{A}$ are used (just) down the left side, members of β and B are used (just) down the right side, and $A \rightarrow B$ is used *at* itself.

Strict Negated Arrow. By hypothesis we have a refutation τ of the premiss. Begin a tableau with the conclusion of Strict Negated Arrow, and apply $S\bar{\rightarrow}$ at its last step. Now continue with τ, changing annotations to suit. Because of the restriction on Strict Negated Arrow, the Barrier requirement will be satisfied by the constructed tableau if satisfied by τ; and so will the Use and Closure requirements.

We leave verification of the other rules to the reader.[4]

THEOREM 4. *For each considered* $\boldsymbol{S}$, *if* $a \vdash_{TS}$ *then* $a \vdash_{L_lS}$.

PROOF. We shall see that a $\boldsymbol{TS}$-refuting tableau can be looked at as a sort of Gentzen proof turned upside down; the global restrictions will come heavily into play.

From this point onward we follow Curry's lead [4] by "identifying" sequences which are permutes of each other (all our L_l-systems have permutation) — but we shall have to keep track of which formulas occur in our sequences, and how many times each occurs (some of our L_l-systems do not have Weakening).

Given any $\boldsymbol{TS}$-refuting tableau, we define a function *Seq* from its nodes into sequences of formulas as follows: for each node n, A is to have n_A occurrences in $Seq(n)$ just in case there are n_A distinct applications of rules *at* nodes $\geqslant n$ *to* nodes $\leqslant n$ to which A is assigned (counting separately for multiple mentions of the same node in *MCl*); that is, just in case n is caught n_A times between the *at* and the *to* of an application of a rule *to* A; that is, just in case there are n_A triples $\langle i, j, m\rangle$ where i and j are nodes and m is an integer, such that $i \leqslant n \leqslant j$, $F(i) = A$, and the mth member of the annotation *at* j is i.

LEMMA. *If* τ *is a* $\boldsymbol{TS}$*-refuting tableau, then for each annotated node* n *of* τ, *i.e., for each node* n *at which a rule is applied,* $Seq(n) \vdash_{L_lS}$.

PROOF is by induction on the number of annotated nodes succeeding a given annotated node.

Suppose first that n is an end node. By the Closure requirement, n must be annotated by an appropriate branch closure rule *Cl* or *MCl*; since there will be in $Seq(n)$ an occurrence of a formula corresponding to each reference *to* it in the annotation of n, $Seq(n) \vdash$ will be an axio m of $L_l\boldsymbol{S}$.

Second, suppose n is annotated with $\rightarrow(i)$, where $F(i) = A \rightarrow B$. Let j and k be the immediate successors of n, so that $F(j) = \overline{A}$ and $F(k) = B$. Let $Seq(j) = a, \overline{A}, \ldots, \overline{A}$, with $j_{\overline{A}}$ $\overline{A}s$ corresponding to references *to* j, and let $Seq(k) = \beta, B, \ldots, B$, with k_B Bs corresponding to references *to* k, where $j_{\overline{A}}, k_B \geqslant 0$, and where a and β contain formulas corresponding to references *to* nodes $\leqslant n$. Clearly $Seq(n) = a, \beta, A \rightarrow B$. Now by inductive hypothesis, $Seq(j) \vdash_{L_lS}$ and $Seq(k) \vdash_{L_lS}$. If $\boldsymbol{TS}$ requires Use, then $j_{\overline{A}}$, $k_B > 1$. If not, $L_l\boldsymbol{S}$ has Weakening, so we may anyhow suppose $j_{\overline{A}}, k_B > 1$. By Contraction and $\rightarrow\vdash$, $Seq(n) \vdash_{L_lS}$.

[4] Theorem 3 can be established *directly* in the fashion of McRobbie [22]; i.e., a detour *via* $L_l\boldsymbol{S}$ is not necessary. What is primarily involved in this proof is proving a tableau-theoretic equivalent of Gentzen's *Hauptsatz* (see Gentzen [10]) for $\boldsymbol{TS}'$, where $\boldsymbol{TS}'$ is just $\boldsymbol{TS}$ plus a tableau-theoretic equivalent of Gentzen's rule *cut*.

Third, suppose n is annotated with $\tilde{\to}(i)$ or $S\tilde{\to}(i)$, where $F(i) = A \to B$. Let j be the successor of n, and k the successor of j, so that $F(j) = A$ and $F(k) = \bar{B}$. Let $Seq(k) = a, A, \ldots, A, \bar{B}, \ldots, \bar{B}$, with k_A As corresponding to references to j, and $k_{\bar{B}}$ $\bar{B}$s, corresponding to references to k, each > 0. Since j is not annotated, $Seq(n) = a, \overline{A \to B}$. By the hypothesis of the induction, k being annotated, $Seq(k) \vdash_{L_lS}$. As before, if TS requires Use, $k_A, k_{\bar{B}} \geqslant 1$, while if not L_lS has Weakening, so we may anyhow suppose $k_A, k_{\bar{B}} \geqslant 1$. By contraction, $a, A, \bar{B} \vdash_{L_lS}$. If TS requires Barrier, every member of a is an implication, since all the references it represents will cross the barrier generated by the application of $S\tilde{\to}$ *at* n; so that $Seq(n) \vdash_{L_lS}$ by $S\tilde{\to}\vdash$. Otherwise $\tilde{\to}\vdash$ produces the same result.

The case when n is annotated with $\approx$ is left to the reader.

Returning to the proof of Theorem 4, we suppose $a \vdash_{TS}$, and note by the Lemma that $Seq(n) \vdash_{L_lS}$, where n is the first annotated node in a tableau TS-refuting a; by Closure there will be such a node. $Seq(n)$ can contain no formula not in a. For each formula A in a, let m_A be its number of occurrences in a and let n_A be its number of occurrences in $Seq(n)$. If TS imposes the Use requirement, $n_A \geqslant m_A \geqslant 1$; and if not, Weakening is available in L_lS, so that it is anyhow harmless to suppose $n_A \geqslant m_A \geqslant 1$. So $a \vdash_{L_lS}$ by Contraction.

Finally, we note that Theorem 1 is an immediate consequence of Theorems 2–4: the tableau and Hilbert systems are in the appropriate sense equivalent.

III.

We conclude with a short list of some of the more interesting problems raised by the results we have presented in this paper.

1. The analytic tableau formulation of $\boldsymbol{R}_{\tilde{\to}}$ given in this paper has been extended to the system $\boldsymbol{R}_{+}$ by McRobbie in [21]. Can it be extended further to all of $\boldsymbol{R}$?

2. There is a precise translation between analytic semantic tableau formulations and analytic tableau formulations of $\boldsymbol{TV}$ and a large number of modal logics, (e.g., see McRobbie [23]). Analytic semantic tableau formulations of $\boldsymbol{E}_{\tilde{\to}}$, $\boldsymbol{R}_{\tilde{\to}}$ and $\boldsymbol{RM}_{\tilde{\to}}$ can be quite straightforwardly extracted from the semantics given for their parent systems in Routley and Meyer [27]. Are the analytic semantic tableau formulations and the analytic tableau formulations of these logics intertranslatable? Put more generally, what do the systems $\boldsymbol{TE}_{\tilde{\to}}$, $\boldsymbol{TR}_{\tilde{\to}}$ and $\boldsymbol{TRM}_{\tilde{\to}}$ mean from the point of view of the Routley/Meyer semantics?

3. By dropping the rule $\approx$ from $\boldsymbol{TE}_{\overset{\sim}{\to}}$, $\boldsymbol{TR}_{\overset{\sim}{\to}}$ and $\boldsymbol{TRM}_{\overset{\sim}{\to}}$, and adjusting *Cl* and *MCl* so that closure can only take place on propositional variables and their negates, it can easily be shown that we have the analytic tableau formulations of $\boldsymbol{E}_{\to}$, $\boldsymbol{R}_{\to}$ and $\boldsymbol{RM}_{\to}$. What do these formulations mean from the point of view of the semilattice semantics given for these logics by Urquhart in [36]? What does the system $\boldsymbol{TR}_{+}$ mean from the point of view of the theory of Dunn monoids given by Meyer in [25]? The relation between ***TTV*** and Boolean algebras is discussed by Eytan in [8].

4. Decidability for various analytic tableau formulations of various strict implicational calculi can be straightforwardly extracted from Davidson, Jackson and Pargetter [6], and it is a trivial exercise to show that ***TTV*** can be used to show ***TV*** decidable. Can the systems $\boldsymbol{TE}_{\overset{\sim}{\to}}$, $\boldsymbol{TR}_{\overset{\sim}{\to}}$ and $\boldsymbol{TRM}_{\overset{\sim}{\to}}$ be used to show $\boldsymbol{E}_{\overset{\sim}{\to}}$, $\boldsymbol{R}_{\overset{\sim}{\to}}$ and $\boldsymbol{RM}_{\overset{\sim}{\to}}$ decidable directly without translating them into the respective left handed Gentzen systems whose decidability is known (e.g., see Kripke [17] and Belnap and Wallace [2])?

5. What are the analytic tableau formulations of at least the implication/negation fragments of the weak relevant systems ***T***, ***T–W***, ***S*** and ***B***?

We close with a final observation. Until this paper, tableau systems have always been construed semantically; and even our given results, our tableau systems still have a strong semantical flavour. This fact taken together with the essential simplicity of operation of our tableau systems leads us to speculate that there may in fact be a simpler semantics for ***E***, ***R*** and ***RM*** than those given by Routley and Meyer in [27], which are the best results to date.

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