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Homomorphisms of Intensionally Complemented Distributive Lattices *

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§ 1

In BELNAP and SPENCER, 1964, there was introduced the notion of an *intensionally complemented distributive lattice with truth-filter (icdl w/t-f)* as a quadruple $(A, \leqq, N, T)$, where $(A, \leqq)$ is a distributive lattice, where the operation of intensional complementation N is a function on A that is antitone and of period two (an involution), and where T is a truth filter, i.e., a filter that for every $a \in A$, contains exactly one of a and Na.

An icdl w/t-f is a generalization of a Boolean algebra, the truth filter T corresponding to a designated maximal filter of the Boolean algebra. Icdl's w/t-f differ perhaps most markedly from Boolean algebras in that it is in general not true that for elements a and b that $a \wedge Na \leqq b$ (nor that $a \leqq b \vee Nb$). This makes them ideally suited for semantic investigations of intensional logics, such as ANDERSON and BELNAP's system E of entailment, which do not contain the "paradoxes of implication," for example, that a contradiction implies anything, or that anything implies a tautology. The principal results along these lines are to be found in ANDERSON and BELNAP, 1962, 1963, and in BELNAP, 1967. In the latter, semantic notions from the previous papers are algebraized so that a model for a certain fragment of E is formulated in terms of icdl's w/t-f, and appropriate completeness and consistency theorems are forthcoming.

In these semantic applications of icdl's w/t-f, a certain especially simple icdl w/t-f plays a fundamental role similar to that played by the two-element Boolean algebra in the semantics of extensional (classical) logic. Thus, corresponding to the well-known fact that any formula of the extensional propositional logic that can be falsified in some Boolean algebra can also be falsified in the two-element Boolean algebra, we have the fact that any first degree entailment of E (a formula of the form $A \rightarrow B$, where A and B are truth functional) that can be falsified in some icdl w/t-f can also be falsified in this especially simple icdl w/t-f (cf. esp. ANDERSON and BELNAP, 1962). This icdl w/t-f, which we shall call M_0, has the following diagram, where $N(\pm a) = \mp a$, and $T = \{+0, +1, +2, +3\}$ (the labeling derives from the matrix in BELNAP, 1960).

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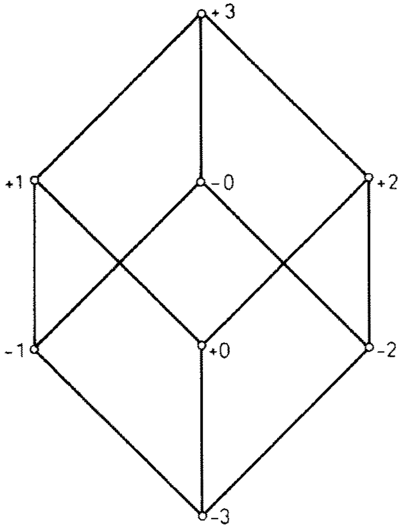
* This research was supported in part by National Science Foundation Grant GS-689 (History and Philosophy of Science). The terminology used in this paper, unless otherwise indicated, is from BIRKHOFF 1948, except that we use "filter" for what is there called a "dual ideal".



M_0

Fig. 1

The main import of the present paper is that M_0 plays a similarly fundamental role in the algebraic study of icdl's w/t-f. Thus, corresponding to the well-known theorem that every Boolean algebra has a homomorphism into the two-element Boolean algebra, we shall prove that every icdl w/t-f has a homomorphism into M_0. Other results will also indicate appropriate analogies to the Boolean situation. Finally, we shall investigate how far these results are capable of generalization to a complete analogue of the notion of an icdl w/t-f.

§ 2

We define a *homomorphism* between icdl's w/t-f $(A, \leqq, N, T)$ and $(A', \leqq', N', T')$ as a familiar lattice homomorphism h (preserving meets and joins) that has the further property that $h(Na) = N(ha)$. We note that the condition that h preserves joins follows from the other conditions. We define a *T-preserving homomorphism* or a *T-homomorphism* by adding on to the above the condition that if $a \in T$, then $h(a) \in T'$. Note that it trivially follows for a T-homomorphism h, that if $a \in \overline{T}$, then $h(a) \in \overline{T'}$. We define a *T-isomorphism* by adding to the definition of a T-homomorphism the requirement that h be one-to-one. We can now state

Theorem 2.1. *Every prime filter P of an icdl w/t-f $(A, \leqq, N, T)$ determines a T-homomorphism h of A into M_0 satisfying the following conditions (where F_i is the principle filter generated by i in M_0):*

i) $h(a) \in F_{-1}$ *iff* $a \in P$,

ii) $h(a) \in F_{-2}$ *iff* $Na \in \overline{P}$,

iii) $h(a) \in F_{+0}$ *iff* $a \in T$.

It is clear that these conditions uniquely determine a (single-valued) function h, for if we let μF_i be either F_i or $\overline{F_i}$, the set theoretic intersection of μF_{-1}, μF_{-2}, and μF_{+0} is always a unit set, for every such choice function μ.

We now prove that h is a T-homomorphism. We first show that $h(a \wedge b) = h(a) \wedge h(b)$. In view of our remarks above concerning the unique determination of h, it suffices to show that $h(a \wedge b) \in F_i$ iff $h(a) \wedge h(b) \in F_i$ ($i = -1, -2, +0$). The cases $i = -1$ and $i = +0$ may be treated together. Thus $h(a \wedge b) \in F_{-1}[F_{+0}]$, iff $a \wedge b \in P[T]$, iff $a, b \in P[T]$, iff $h(a), h(b) \in F_{-1}[F_{+0}]$, iff $h(a) \wedge h(b) \in F_{-1}[F_{+0}]$. We treat $i = -2$ by using the fact that $\overline{P}$ is a prime ideal (a well-known consequence of the fact that P is a prime filter), together with an immediate de Morgan property of N. Thus $h(a \wedge b) \in F_{-2}$, iff $N(a \wedge b) = Na \vee Nb \in \overline{P}$, iff $Na, Nb \in \overline{P}$, iff $h(a), h(b) \in F_{-2}$, iff $h(a) \wedge h(b) \in F_{-2}$.

We now show that $h(Na) = N(ha)$. Again it suffices to show that $h(Na) \in F_i$ iff $N(ha) \in F_i$ ($i = -1, -2, +0$). We first define $N(F_i)$, as the set of Nb such that $b \in F_i$, and observe that $F_{-1} = N(\overline{F_{-2}})$, $F_{-2} = N(\overline{F_{-1}})$, and $F_{+0} = N(\overline{F_{+0}})$. Then $h(Na) \in F_{-1}$, iff $Na \in P$, iff $h(a) \in \overline{F_{-2}}$, iff $N(ha) \in N(\overline{F_{-2}}) = F_{-1}$. And $h(Na) \in F_{-2}$, iff $NNa = a \in \overline{P}$, iff $h(a) \in \overline{F_{-1}}$, iff $N(ha) \in N(\overline{F_{-1}}) = F_{-2}$. And finally, $h(Na) \in F_{+0}$, iff $Na \in T$, iff $a \in \overline{T}$, iff $h(a) \in \overline{F_{+0}}$, iff $N(ha) \in N(\overline{F_{+0}}) = F_{+0}$.

We complete our proof by noting that condition iii) insures that the homomorphism h is T-preserving.

We also have the converse theorem,

Theorem 2.2. *Every T-homomorphism h of an icdl w/t-f $(A, \leq, N, T)$ into M_0 determines a prime filter P of $(A, \leq, N, T)$ in accord with conditions* i) *and* ii) *of Theorem 2.1.*

We first remark that, given a T-homomorphism h, condition i) by itself determines the set P, so it suffices to show that P is a prime filter and that it satisfies condition ii). That the set P is a prime filter follows immediately from the easily verified fact that if h is a homomorphism into a prime filter (in this case F_{-1}), then the inverse image of that prime filter under h (in this case P) is also a prime filter. We now demonstrate that P satisfies condition ii). Observing that $F_{-2} = N(\overline{F_{-1}})$, we have that $h(a) \in F_{-2}$, iff $h(a) \in N(\overline{F_{-1}})$, iff $N(ha) = h(Na) \in NN(\overline{F_{-1}}) = \overline{F_{-1}}$. But contraposing condition i), we have $h(Na) \in \overline{F_{-1}}$ iff $Na \in \overline{P}$. So any set P which satisfies condition i) also satisfies condtition ii), and our proof is complete.

We remark that by combining Theorems 2.1 and 2.2 we obtain a natural one-to-one correspondence between prime filters and T-homomorphisms into M_0. That no two different prime filters determine the same T-homomorphism is immediate, since they would differ at an element a, and hence the homomorphisms determined by them would differ in that one would send a into F_{-1} and the other would not. That no two different T-homomorphisms h and h' determine the same prime filter is shown as follows. Suppose for some element a, $h(a) \neq h'(a)$. We shall show that there is always some element b such that $h(b) \in F_{-1}$ but $h'(b) \notin F_{-1}$. The proof is by cases. Assume first that $h(a)$, $h'(a) \in F_{-1}$. Make the further assumption that $a \in T$. Then since h and h' preserve T, $h(a)$, $h'(a) \in \{+1, +3\}$. Assume without loss of generality that $h(a) = +1$ and $h'(a) = +3$. Then $h(Na) = -1 \in F_{-1}$, but $h'(Na) = -3 \notin F_{-1}$. On the assumption that $a \notin T$, it follows immediately from the fact that h and h' are T-preserving, that $h(a), h'(a) \in \{-1, -0\}$. Assume without loss of generality that $h(a) = -1$ and $h'(a) = -0$. Then $h(Na) = +1 \in F_{-1}$, but $h'(Na) = +0 \notin F_{-1}$. A glance at the diagram of M_0 should convince the reader that the remaining case, in which $h(a), h'(a) \notin F_{-1}$, may also be divided according as to whether a is or is not in T, and treated analogously.

§ 3

In Belnap and Spencer, 1964, there was also introduced the notion of an *intensionally complemented distributive lattice (icdl)* as a triple $(A, \leqq, N)$ such that for some subset T of A, $(A, \leqq, N, T)$ is an icdl w/t-f. Structures like icdl's except that it is not required that there exist a truth filter T have been called *de Morgan lattices* in Monteiro, 1960. (They have also been called *quasi-boolean algebras* in Białnicki-Birula and Rasiowa, 1957, and *distributive i-lattices* in Kalman, 1958.) In Belnap and Spencer, 1964, it was shown that for an arbitrary de Morgan lattice, the condition that it is an icdl is equivalent to the condition that N have no fixed point, i.e., that for all elements a, $a \neq Na$.

It is an immediate corollary of Theorem 2.1 that for an arbitrary de Morgan lattice, the condition that it is an icdl, i.e., that it have a truth filter T, implies that it have a homomorphism into M_0 (letting T play a double role, both as the truth filter and as the prime filter). It also turns out that the converse is true. Thus, let $(A, \leqq, N)$ be a de Morgan lattice and h a homomorphism of it into M_0. Let T be the inverse image of F_{+0} under h. As has already been remarked, the inverse image of a prime filter under a homomorphism is a prime filter. So it only remains to prove that for all $a \in A$, T contains exactly one of a and Na. Suppose that both $a, Na \in T$. Then $h(a)$ and $h(Na) = N(ha)$ are both members of F_{+0}, but this is impossible since F_{+0} is a truth filter. Suppose that for some a, $a, Na \notin T$. Then neither $h(a)$ nor $h(Na) = N(ha)$ are members of F_{+0}, but again this is impossible since F_{+0} is a truth filter. So we have

Theorem 3. *A de Morgan lattice is an icdl iff it has a homomorphism into* M_0.

§ 4

In Belnap, 1967, there was introduced the notion of a *product* of icdl's w/t-f as follows: Let $\{(A_x, \leqq_x, N_x, T_x)\}_{x \in X}$ be an indexed set of icdl's w/t-f, and let the product $\Pi_{x \in X} A_x$ be that quadruple $(A, \leqq, N, T)$ such that $T = \bigtimes_{x \in X} T_x$, $\bar{T} = \bigtimes_{x \in X} \bar{T}_x$, $A = T \cup \bar{T}$, $\{a_x\}_{x \in X} \leqq \{b_x\}_{x \in X}$ if $a_x \leqq_x b_x$ for all $x \in X$, and $N\{a_x\}_{x \in X} = \{N_x a_x\}_{x \in X}$. Observe that the product of icdl's w/t-f is an icdl w/t-f.

Belnap then defined for any cardinal $\mathfrak{c}$, $M^{\mathfrak{c}} = \Pi_{x : x < \mathfrak{c}} A_x$, where each A_x is M_0, and showed utilizing proof-theoretical (syntactical) methods that every at most denumerable icdl w/t-f is T-isomorphic to a sublattice of $M^{\mathfrak{c}}$ for some $\mathfrak{c} \leqq \aleph_0$. We are now in a position to prove by algebraic means the following theorem, which both generalizes and at the same time sharpens this result.

Theorem 4. *Every icdl w/t-f* $(A, \leqq, N, T)$ *of cardinality* $\mathfrak{d}$ *is T-isomorphic to a sublattice of* $M^{\mathfrak{c}}$, *for some cardinal* $\mathfrak{c} \leqq \mathfrak{d}^2$.

The proof begins by taking the set of all pairs of elements (a, b) such that $a, b \in A$ and $a \nleqq b$, and indexing these pairs with some set X. Note that the cardinality of X, say $\mathfrak{c}$, is obviously such that $\mathfrak{c} \leqq \mathfrak{d}^2$, where $\mathfrak{d}$ is the cardinality of A. STONE, 1936, showed that if a lattice is distributive, then for any two elements a, b such that $a \nleqq b$, there exists a prime filter P such that $a \in P$ and $b \notin P$. So we may associate with each pair $(a, b)_x$ a prime filter P_x such that $a \in P_x$ and $b \notin P_x$. By Theorem 2.1, each pair $(a, b)_x$ determines through the prime filter P_x a T-homomorphism h_x into M_0.

Then h, defined for $a \in A$ by $h(a) = \{h_x(a)\}_{x \in X}$, is a T-isomorphism of $(A, \leqq, N, T)$ into M^c. h is obviously defined over all of A, since that is true for each h_x. And h is into M^c and indeed is T-preserving, because each h_x is into M_0 and is T-preserving. And h is one-to-one, for suppose $a \neq b$, and assume without loss of generality that $a \nleqq b$. Then the prime filter P_x associated with $(a, b)_x$ determines the T-homomorphism h_x such that $h_x(a) \in F_{-1}$ and $h_x(b) \notin F_{-1}$. So $h_x(a) \neq h_x(b)$, and $h(a)$ differs from $h(b)$ in the x-th component.

It remains to show h a homomorphism; this follows immediately from the fact that operations in the product algebra were defined component-wise:

$$h(a \wedge b) = \{h_x(a \wedge b)\}_{x \in X} = \{h_x(a) \wedge h_x(b)\}_{x \in X} = h(a) \wedge h(b),$$

and

$$h(Na) = \{h_x(Na)\}_{x \in X} = \{N h_x(a)\}_{x \in X} = N(ha).$$

We mention that corresponding to this embedding theorem we have a sort of representation theorem in terms of a ring of sets with an intensional complement N. This theorem, however is not nearly as interesting as the representation theorem for Boolean algebras that corresponds to the embedding theorem of Boolean algebras into the product algebra of the two-element Boolean algebra, since the intensional complement on the ring of sets is not as natural an operation as might be desired. The representation for icdl's w/t-f follows trivially from the fact that every distributive lattice is isomorphic to a ring of sets under the mapping h which associates with each element a the set of all prime filters P such that $a \in P$ (cf. SZÁSZ, 1963). Defining $N(ha)$ to be the set of all prime filters P such that $Na \in P$, we trivially insure that h preserves N. Perhaps the only interest such a representation theorem has derives from the demonstrated one-to-one correspondence between prime filters and T-homomorphisms into M_0. From this it is easy to see that the mapping h which associates with each element a the set of T-homomorphisms that carry a into F_{-1} is an isomorphism onto a ring of sets that preserves intensional complementation when $N(ha)$ is defined as the set of T-homomorphisms that carry a into the ideal generated by -1, which is just $N(F_{-1})$.

§ 5

BELNAP, 1967, gave a complete generalization of an icdl w/t-f that is useful in the semantics of quantified intensional logics. An icdl w/t-f $(A, \leqq, N, T)$ is there said to be *complete*, *completely distributive*, and *with complete truth filter*, provided, respectively, the following conditions hold:

(C) A is complete in the sense that for every $B \subseteqq A$, A contains the glb, $\bigwedge B$, and the lub, $\bigvee B$.

(CD) $\bigwedge_{x \in X} \bigvee_{y \in Y} a_{x,y} = \bigvee_{f \in U} \bigwedge_{x \in X} a_{x,f(x)}$, and dually, where U is the set of all functions from X into Y.

(CT) The filter T is complete in the sense that whenever $B \subseteqq T$, $\wedge B \in T$. Note that the converse property, namely that $B \subseteqq T$ for every $\wedge B \in T$, follows immediately from T's being a filter. Since in the sequel we do not consider icdl's w/t-f which are complete in the sense of (C), but fail to satisfy (CD) and (CT), we shall permit ourselves to mean by a *complete icdl w/t-f* one satisfying all of (C), (CD), and (CT).

A *complete T-homomorphism* between complete icdl's w/t-f is then defined by modifying our definition in § 2 so that the generalized operations $\wedge$ and $\vee$ must be preserved as well. We remark again that the condition that $\vee$ is preserved follows from the other conditions. A *complete (T-) isomorphism* is then a complete (T-) homomorphism that is one-to-one.

One more definition, and we may state a theorem. A filter P is *completely prime* iff, if $\vee_{x \in X} a_x \in P$, then some $a_x \in P$. Note that if P is a complete and completely prime filter, then $\bar{P}$ is a complete and completely prime ideal (defined dually).

Theorem 5.1. *Every complete and completely prime filter P of a complete icdl w/t-f $(A, \leqq, N, T)$ determines a complete T-homomorphism h of A into M_0 that satisfies conditions* i), ii), *and* iii) *of Theorem 2.1.*

In view of Theorem 2.1 and the remarks concerning the strategy of its proof, it suffices to show that $h(\wedge_{x \in X} a_x) \in F_i$ iff $\wedge_{x \in X} h(a_x) \in F_i$ $(i = -1, -2, +0)$. Again we treat $i = -1$ and $i = +0$ together, observing first that F_{-1} $[F_{+0}]$ is complete (since M_0 is finite) and that $P[T]$ is complete. Thus, $h(\wedge_{x \in X} a_x) \in F_{-1}$ $[F_{+0}]$, iff $\wedge_{x \in X} a_x \in P$ $[T]$, iff $\{a_x\}_{x \in X} \subseteqq P$ $[T]$, iff $\{h(a_x)\}_{x \in X} \subseteqq F_{-1}$ $[F_{+0}]$, iff $\wedge_{x \in X} h(a_x) \in F_{-1}$ $[F_{+0}]$. We treat $i = -2$ by utilizing the fact that $\bar{P}$ is a complete and completely prime ideal, together with a de Morgan property of N over $\wedge$. Thus, $h(\wedge_{x \in X} a_x) \in F_{-2}$, iff $N \wedge_{x \in X} a_x = \vee_{x \in X} N a_x \in \bar{P}$, iff $\{a_x\}_{x \in X} \subseteqq \bar{P}$, iff $\{h(a_x)\}_{x \in X} \subseteqq F_{-2}$, iff $\wedge_{x \in X} h(a_x) \in F_{-2}$.

We again also have the converse theorem:

Theorem 5.2. *Every complete T-homomorphism h of a complete icdl w/t-f $(A, \leqq, N, T)$ into M_0 determines a complete and completely prime filter P of $(A, \leqq, N, T)$ in accord with conditions* i) *and* ii) *of Theorem 2.1.*

That condition i) determines P to be a complete and completely prime filter follows immediately from the easily verified fact that if h is a complete homomorphism into a complete and completely prime filter (in this case F_{-1}), then the inverse image of that complete and completely prime filter under h (in this case P) is a complete and completely prime filter also. The proof that P as determined by i) also satisfies ii) is the same as in Theorem 2.2.

We remark that by combining Theorems 5.1 and 5.2 we obtain a natural one-to-one correspondence between complete, completely prime filters and complete T-homomorphisms into M_0. The justification of this remark rests on the same argument as after Theorem 2.1.

§ 6

Let us mean by a *complete icdl* a triple $(A, \leqq, N)$ such that for some subset T of A, $(A, \leqq, N, T)$ is a complete icdl w/t-f. And let us say that a structure like a complete icdl, except that it is not required that there exist any truth filter, is a *complete de Morgan lattice.*

In BELNAP and SPENCER, 1964, it was shown that there are de Morgan lattices that satisfy (C) and such that N has no fixed point, and yet still have no truth filter T that satisfies (CT). In BELNAP, 1965, it was announced that adding (CD) does guarantee the existence of such a T. We now prove this result.

Theorem 6.1. *A necessary and sufficient condition for a complete de Morgan lattice* $(A, \leqq, N)$ *to be a complete icdl is that for all* $a \in A, a \neq Na$.

The necessity is obvious, because for all $a \in A$, T must contain exactly one of a and Na.

To prove sufficiency, we begin be defining Σ to be the collection of all selection sets S, where S is a selection set iff for every $a \in A$, S contains at least one of a and Na. To prove that A includes a complete truth filter T, it then suffices to show that for some $S \in \Sigma$, $\bigwedge S \nleqq N \bigwedge S$. For let S_0 be such an S, and let T be the principal filter generated by $\bigwedge S_0$. It is immediate that T, being principal, is complete. And T trivially contains for every $a \in A$, at least one of a and Na. Further, T contains at most one of a and Na, for suppose to the contrary that $\bigwedge S_0 \leqq a$ and $\bigwedge S_0 \leqq Na$. Since S_0 contains one of a and Na, consequently NS_0 contains one of Na and a. Thus $\bigwedge S_0 \leqq \bigvee NS_0$, but $\bigvee NS_0 = N \bigwedge S_0$, and thus $\bigwedge S_0 \leqq N \bigwedge S_0$, contrary to hypothesis.

The proof strategy then is to suppose for *reductio* that there is no such selection set S_0, i.e., that for all $S \in \Sigma$, $\bigwedge S \leqq N \bigwedge S$, and to derive under this hypothesis that N has a fixed point.

First well-order the elements of A, and let $a \prec b$ mean that a properly precedes b in the well-ordering. For $x \in A$, let $F(x)$ be $\{x \wedge Nx\} \cup \{y \vee Ny : y \in A \,\&\, y \prec x\}$, and let $G(x)$ be $\{x \vee Nx\} \cup \{y \wedge Ny : y \in A \,\&\, y \prec x\}$. Then define $a_0 = \bigvee_{x \in A} \bigwedge F(x)$, whence $Na_0 = \bigwedge_{x \in A} \bigvee G(x)$. We show that $a_0 = Na_0$.

To prove that $a_0 \leqq Na$ reduces to showing for arbitrary elements p and q that $\bigwedge F(p) \leqq \bigvee G(q)$. We have three cases according to the relative position of p and q in the well-ordering. If $p = q$, then $p \wedge Np \in F(p)$ and $p \vee Np \in G(q)$, but $p \wedge Np \leqq p \vee Np$. If $p \prec q$, then $p \wedge Np \in F(p)$ and $p \wedge Np \in G(q)$. And if $q \prec p$, then $q \vee Nq \in F(p)$ and $q \vee Nq \in G(q)$.

To show that $Na_0 \leqq a_0$, we use complete distribution. Thereby we find that $a_0 = \bigwedge_{f \in U} \bigvee_{x \in A} f(x)$, where U is the set of all functions f of A such that $f(x) \in F(x)$. And $Na_0 = \bigvee_{g \in V} \bigwedge_{x \in A} g(x)$, where V is the set of all functions g of A such that $g(x) \in G(x)$. To prove that $Na_0 \leqq a_0$, it suffices to show for some arbitrary $g \in V$ and $f \in U$, that $\bigwedge_{x \in A} g(x) \leqq \bigvee_{x \in A} f(x)$. Proof is by cases. First suppose that for some $y \in A$, there exists $p \in A$ such that $g(p) = y \wedge Ny$, and that for some $z \in A$, there exists $q \in A$ such that $f(q) = z \vee Nz$. Invoking the well-ordering principle, let y_0 be the first such element y, and let z_0 be the first such element z. If $y_0 = z_0$, then $y_0 \wedge Ny_0 \in \{g(x)\}_{x \in A}$ and $y_0 \vee Ny_0 \in \{f(x)\}_{x \in A}$, but $y_0 \wedge Ny_0 \leqq y_0 \vee Ny_0$. If $y_0 \prec z_0$, then either $f(y_0) = y_0 \wedge Ny_0$, in which case $y_0 \wedge Ny_0$ is contained in both $\{g(x)\}_{x \in A}$ and $\{f(x)\}_{x \in A}$, or $f(y_0) = y \vee Ny$ for some $y \prec y_0$. But this last is impossible, for then $y \prec y_0 \prec z_0$, contradicting that z_0 is first among the elements z such that for some $q \in A$, $f(q) = z \vee Nz$. An exactly parallel argument holds for the supposition that $z_0 \prec y_0$.

Next suppose that for all $p \in A$, $g(p) = p \vee Np$, and that for some $q \in A$, $f(q) = z \vee Nz$ for some $z \prec q$. Then for that z, $g(x) = z \vee Nz$, and thus $z \vee Nz$ is in both $\{g(x)\}_{x \in A}$ and $\{f(x)\}_{x \in A}$.

Finally, suppose that for all $p \in A$, $g(p) = p \vee Np$, and for all $q \in A$, $f(q) = q \wedge Nq$. Then $\{g(x)\}_{x \in A} = \{x \vee Nx\}_{x \in A}$, and $\{f(x)\}_{x \in A} = \{x \wedge Nx\}_{x \in A}$, so we must show $\bigwedge_{x \in A} x \vee Nx \leqq \bigvee_{x \in A} x \wedge Nx$. But this follows by complete distribution from $\bigvee_{S \in \Sigma} \bigwedge S \leqq \bigwedge_{S \in \Sigma} \bigvee S$, which in turn follows from our *reductio* supposition that for all $S \in \Sigma$, $\bigwedge S \leqq N \bigwedge S$. Thus, consider arbitrary $S_0, S_1 \in \Sigma$. If S_0 and S_1 share some element, then $\bigwedge S_0 \leqq \bigvee S_1$. Otherwise, $S_1 = NS_0$, and by our reductio assumption $\bigwedge S_0 \leqq N \bigwedge S_0 = \bigvee NS_0 = \bigvee S_1$.

Then $a_0 = Na_0$, contrary to the hypothesis that N has no fixed point; and so for some $S \in \Sigma$, $\bigwedge S$ must generate a complete truth filter.

We also have a complete generalization of Theorem 3.

Theorem 6.2. *A necessary and sufficient condition for a complete de Morgan lattice* $(A, \leqq, N)$ *to be a complete icdl is that there be a complete homomorphism h of A into* M_0.

The necessity of this condition is an immediate corollary of Theorem 5.1, with T playing the double role of the truth filter and the complete and completely prime filter. The sufficiency follows by letting T be the inverse image of F_{+0} under the assumed complete homomorphism h, and noting, as has already been remarked, that the inverse image of a complete, completely prime filter under a complete homomorphism is itself a complete, completely prime filter. Thus T so defined is complete, and that T is a truth filter follows from the proof of Theorem 3.

§ 7

At this point a proof of a complete generalization of the embedding Theorem 4 would be appropriate, since it may be easily proven that the product of complete icdl's w/t-f (as defined in § 4) is a complete icdl w/t-f and since the analogue holds in the Boolean case. But such a generalization cannot be proven without restriction. Let us look at the proof of Theorem 3 to see where it breaks down if we try to modify it by substituting a complete and completely prime filter in the role of the prime filter. The key assumption is then that for any two elements a, b such that $a \not\leqq b$, there exists a complete and completely prime filter P such that $a \in P$ and $b \notin P$. We find that this assumption is unjustified in the case of complete icdl's w/t-f. Thus, let A be the union of the two closed intervals $[0,1/3]$ and $[2/3,1]$ of the real line with the usual ordering. It is easily seen that this is a complete and completely distributive lattice, and that if Na is defined as $1-a$, then N is antitone and of period two, and $[2/3,1]$ becomes a complete truth filter. It is easy to see that there is no complete and completely prime filter that is a subset of the half-open interval $(2/3,1]$, for suppose there is such a filter P. Since P is complete, for some $a \in (2/3,1]$, P is the principal filter generated by a. But then consider $\bar{P}=\{b\in A: b\leqq a\}$. It is well-known from analysis that $a=\bigvee\bar{P}$, but since P and $\bar{P}$ are disjoint, P is not completely prime. Thus, no two points in (2/3,1] can be separated by a complete and completely prime filter.

However if we assume that a complete icdl w/t-f $(A, \leqq, N, T)$ is such that any two elements of A can be separated by a complete and completely prime filter P, then the proof of the complete generalization of Theorem 4 would go through as for Theorem 4, with the obvious association of a complete and completely prime filter P_x, with each pair $(a, b)_x$. The thereby associated complete T-homomorphisms $\{h_x\}_{x\in X}$ would then define a T-isomorphism h of $(A, \leqq, N, T)$ into $M^{\mathfrak{c}}$, just as before. And obviously h preserves $\wedge$, since each h_x preserves $\wedge$, and $\wedge$ in $M^{\mathfrak{c}}$ is defined componentwise. Thus we have

Lemma 7.1. *Every complete icdl w/t-f $(A, \leqq, N, T)$ of cardinality $\mathfrak{d}$, and such that for every pair of elements a, b with $a\nleqq b$, there exists a complete and completely prime filter P such that $a\in P$ and $b\notin P$, is completely T-isomorphic to a complete sublattice of $M^{\mathfrak{c}}$ for some cardinal $\mathfrak{c}\leqq\mathfrak{d}^2$.*

We also have a converse form of Lemma 7.1, namely,

Lemma 7.2. *Every complete icdl w/t-f $(A, \leqq, N, T)$ which is completely T-isomorphic to a complete sublattice of $M^{\mathfrak{c}}$ for some cardinal $\mathfrak{c}$, is such that for every pair of elements a, b with $a\nleqq b$, there exists a complete and completely prime filter P such that $a\in P$ and $b\notin P$.*

In view of the given T-isomorphism of A onto a complete sublattice A' of $M^{\mathfrak{c}}$, it suffices to show that for $a', b'\in A'$ with $a'\nleqq b'$, there exists a complete and completely prime filter P' such that $a'\in P'$ and $b'\notin P'$. And to show this, it suffices to show that for $\{a_x\}_{x<\mathfrak{c}}, \{b_x\}_{x<\mathfrak{c}}\in M^{\mathfrak{c}}$ with $\{a_x\}_{x<\mathfrak{c}}\nleqq\{b_x\}_{x<\mathfrak{c}}$, there exists a complete and completely prime filter Q of $M^{\mathfrak{c}}$ such that $\{a_x\}_{x<\mathfrak{c}}\in Q$ and $\{b_x\}_{x<\mathfrak{c}}\notin Q$. For let Q be a complete and completely prime filter of $M^{\mathfrak{c}}$ such that $a'\in Q$ and $b'\notin Q$. Then it is easy to see that $Q\cap Q'$ is a complete and completely prime filter of A' such that $a'\in Q\cap A'$ and $b'\notin Q\cap A'$. So consider $\{a_x\}_{x<\mathfrak{c}}, \{b_x\}_{x<\mathfrak{c}}\in M^{\mathfrak{c}}$ with $\{a_x\}_{x<\mathfrak{c}}\nleqq\{b_x\}_{x<\mathfrak{c}}$. Then for some $x_0<\mathfrak{c}$, $a_{x_0}\nleqq b_{x_0}$. Then there exists a complete and completely prime filter Q_{x_0} of M_{x_0} such that $a_{x_0}\in Q_{x_0}$ and $b_{x_0}\notin Q_{x_0}$, as inspection of M_0 shows. Now define functions f and g on $\{x: x<\mathfrak{c}\}$ such that $f(x_0)=Q_{x_0}\cap F_{+0}$ and for $x\neq x_0$, $g(x)=\overline{F_{+0}}$, and such that $g(x_0)=Q_{x_0}\cap\overline{F_{+0}}$ and for $x\neq x_0$, $f(x)=F_{+0}$. Then let $Q=\bigtimes_{x<\mathfrak{c}}f(x)\cup\bigtimes_{x<\mathfrak{c}}g(x)$. Componentwise considerations now show that Q is the desired complete and completely prime filter of $M^{\mathfrak{c}}$.

It is interesting to note that the condition that every pair of distinct elements be separated by a complete and completely prime filter is equivalent to a rather simple condition on the elements.

Lemma 7.3. *Given a complete and completely distributive lattice $(A, \leqq)$, a necessary and sufficient condition for any pair of distinct elements being separated by a complete and completely prime filter is that every element be a generalized join of completely join-irreducible elements, where an element a is said to be completely join-irreducible if for no set $\{a_x\}_{x\in X}$ such that each $a_x\leqq a$, does $a=\bigvee_{x\in X}a_x$.*

The sufficiency is easy, for suppose that $a \nleqq b$, and say $a = \bigvee_{x \in X} a_x$, where each a_x is completely join-irreducible. Now choose some $a_x \nleqq b$. (There must be one, for if all $a_x \leqq b$, then $a = \bigvee_{x \in X} a_x \leqq b$, contrary to assumption.) Since $a_x \nleqq b$, the principal filter P generated by that a_x contains a, but does not contain b. And it is trivial that P is complete, as are all principal filters in a complete lattice. It only remains to show P completely prime. But this follows immediately from a complete generalization of a lemma in BIRKHOFF, 1948, namely that in a complete and completely distributive lattice, if an element p is completely join-irreducible, then $p \leqq \bigvee_{x \in X} q_x$ implies $p \leqq q_x$ for some $x \in X$. For proof of the generalization, note that $p \leqq \bigvee_{x \in X} q_x$ implies $p = p \wedge (\bigvee_{x \in X} q_x) = \bigvee_{x \in X} p \wedge q_x$. But then since p is completely join-irreducible, $p = p \wedge q_x$, i.e., $p \leqq q_x$, for some $x \in X$.

The necessity follows by letting $\{p_x\}_{x \in X}$ be the set of all completely join-irreducible elements p_x such that $p_x \leqq a$ for some arbitrary element a. This set is non-empty since $\bigwedge A$ is completely join-irreducible. Now if $a = \bigvee_{x \in X} p_x$, we are through. So assume $a \nleqq \bigvee_{x \in X} p_x$. Then $a \nleqq \bigvee_{x \in X} p_x$. But then there is by assumption a complete and completely prime filter P that contains a and does not contain $\bigvee_{x \in X} p_x$. But $\bigwedge P$ is completely join-irreducible, for otherwise P, being completely prime, would contain some element $q < \bigwedge P$. Thus $\bigwedge P = p_x$ for some $x \in X$, and $\bigwedge P \leqq \bigvee_{x \in X} p_x$, contrary to the hypothesis that P does not contain $\bigvee_{x \in X} p_x$.

We remark in passing that Lemma 7.1, although handy for our purposes, is not stated in the best possible form, for in its proof we make use of the condition of complete distributivity only in the proof of the complete generalization of Birkhoff's lemma, and there only in a weakened form. Further we could do without any condition on distributivity by defining a *completely join-prime* element as one satisfying the consequent of the generalization of Birkhoff's lemma, and formulating the necessary and sufficient condition of Lemma 7.3 in terms of completely-join-primeness instead of completely-join irreducibility.

We now collect our lemmas together in the following theorem:

Theorem 7. *For a complete icdl w/t-f* $(A, \leqq, N, T)$, *the following conditions are equivalent:*

i) *There exists a complete T-isomorphism of A into* $M^{\mathfrak{c}}$, *for some cardinal* $\mathfrak{c}$.

ii) *For any pair of elements a, b with* $a \nleqq b$ *there exists a complete and completely prime filter P such that* $a \in P$ *and* $b \notin P$.

iii) *Every element a is a generalized join of completely join-irreducible elements.*

We remark in closing then that for a complete icdl w/t-f that satisfies one of these conditions (i)–(iii), we get the same sort of trivial representation theorem as was described in § 4, this time mapping each element into the set of complete and completely prime filters that contain it, or into the set of complete T-homomorphisms that carry it into F_{+0} of M_0.

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