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MODALITIES IN ACKERMANN'S "RIGOROUS IMPLICATION"

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Following a suggestion of Feys, we use "rigorous implication" as a translation of Ackermann's *strengte Implikation* ([1]). Interest in Ackermann's system stems in part from the fact that it formalizes the properties of a strong, natural sort of implication which provably avoids standard implicational paradoxes, and which is consequently a good candidate for a formalization of *entailment* (considered as a narrower relation than that of strict implication). Our present purpose will not be to defend this suggestion, but rather to present some information about rigorous implication. In particular, we show first that the structure of modalities (in the sense of Parry [4]) in Ackermann's system is identical with the structure of modalities in Lewis's S4, and secondly that (Ackermann's apparent conjecture to the contrary notwithstanding) it is possible to define modalities with the help of rigorous implication.

Familiarity with [1] will be presupposed.

1. Structure of modalities in Π'' . By Π' we mean the system consisting of Ackermann's axioms (1)–(15) and rules (α)–(δ), and by Π'' we mean the system Π' together with axioms (16)–(17) and rule (ϵ). In proofs, we refer to axioms and rules by Ackermann's designations (including his derived rules (a)–(g), [1], pp. 121–122; we also use (h) to refer to his derived rule, p. 125, that in Π'' , if $\vdash A$, then $\vdash \bar{A} \rightarrow \lambda$).

We first prove some theorems.

By axioms (3), (12), and rule (a), we have

$$1. \quad \vdash (A \rightarrow \lambda) \rightarrow (\overline{A \rightarrow \lambda} \rightarrow A \rightarrow A),$$

and by (1) and rule (h), we have

$$2. \quad \vdash \overline{A \rightarrow A} \rightarrow \lambda,$$

whence by rule (d),

$$3. \quad \vdash (A \rightarrow \lambda) \rightarrow (\overline{A \rightarrow \lambda} \rightarrow \lambda).$$

Ackermann defines NA ("it is necessary that A ") as $\bar{A} \rightarrow \lambda$, from which we see that 3, taking A as \bar{A} , comes to

$$3'. \quad \vdash NA \rightarrow NNA,$$

the characteristic axiom of S4.¹

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¹ Thus Ackermann's remark that $NA \rightarrow NNA$ is not provable ([1], p. 126, and repeated in the review [2]) requires correction, as does the further remark in [2] that the structure of modalities in Ackermann's systems is like that of von Wright's M ([6]).

Now by 3 and rule (f) we have

$$4. \vdash (\mathbf{A} \rightarrow \wedge) \rightarrow ((\overline{\wedge} \rightarrow \overline{\mathbf{A} \rightarrow \wedge}) \rightarrow (\overline{\wedge} \rightarrow \wedge)),$$

whence by (12), (14), (15), etc., together with Ackermann's *Ersetzungstheorem* (stated for Π' in [1], p. 123; it is trivially extendible to Π''), we have

$$5. \vdash (\mathbf{A} \rightarrow \wedge) \rightarrow (((\mathbf{A} \rightarrow \wedge) \rightarrow \wedge) \rightarrow (\overline{\wedge} \rightarrow \wedge)).$$

By (16), (15), and rule (a), we have

$$6. \vdash (\overline{\wedge} \rightarrow \wedge) \rightarrow \wedge,$$

and 5 and 6 together with rule (d) yield (omitting parentheses in accord with a convention of association to the left),

$$7. \vdash (\mathbf{A} \rightarrow \wedge) \rightarrow (\mathbf{A} \rightarrow \wedge \rightarrow \wedge \rightarrow \wedge).$$

Then from 7 by rule (c) we get

$$8. \vdash (\mathbf{A} \rightarrow \wedge \rightarrow \wedge \rightarrow \wedge \rightarrow \wedge) \rightarrow (\mathbf{A} \rightarrow \wedge \rightarrow \wedge),$$

and also from 7, taking \mathbf{A} as $\mathbf{A} \rightarrow \wedge$, we get

$$9. \vdash (\mathbf{A} \rightarrow \wedge \rightarrow \wedge) \rightarrow (\mathbf{A} \rightarrow \wedge \rightarrow \wedge \rightarrow \wedge \rightarrow \wedge).$$

Then 8 and 9 give the following (with " \leftrightarrow " defined in the obvious way):

$$10. \vdash (\mathbf{A} \rightarrow \wedge \rightarrow \wedge) \leftrightarrow (\mathbf{A} \rightarrow \wedge \rightarrow \wedge \rightarrow \wedge \rightarrow \wedge).$$

Now by judicious use of (14), (15), and the *Ersetzungstheorem*, 10 can be made to yield

$$11. \vdash N \sim NA \leftrightarrow N \sim N \sim N \sim NA$$

(where for clarity we use the curl rather than the bar), and it follows immediately by an argument like Parry's ([4], pp. 148–149), that Π'' distinguishes no more than fourteen modalities. That Π'' distinguishes just fourteen modalities, in fact the same fourteen distinguished in S4, follows from the fact (easily proven) that under an obvious transformation, Π'' is a subsystem of S4. The transformation consists of replacing \wedge , in wffs of Π'' , by some standard contradiction in S4 (say $p \sim p$), and replacing \rightarrow by strict implication, leaving other connectives unchanged.

2. Reduction of modalities in Π'' to modalities in Π' . Ackermann remarks that the definitions $\overline{\mathbf{A}} \rightarrow \mathbf{A}$ (of NA) and $\mathbf{A} \rightarrow \overline{\mathbf{A}}$ (of UA : "it is impossible that \mathbf{A} ") are not satisfactory for the system Π' , since e.g. we would wish to regard \mathbf{A} as impossible if $\mathbf{A} \rightarrow \mathbf{B} \ \& \ \overline{\mathbf{B}}$ were a theorem; but from this $\mathbf{A} \rightarrow \overline{\mathbf{A}}$ does not follow in Π' , the required step $\mathbf{B} \ \& \ \overline{\mathbf{B}} \rightarrow \overline{\mathbf{A}}$ being "paradoxical," and by Ackermann's matrix unprovable. He is therefore led, adapting an idea of Johansson [3], to introduce a propositional constant \wedge for "das Absurde," and to define UA as $\mathbf{A} \rightarrow \wedge$, NA as $U\overline{\mathbf{A}}$, and MA as $U\overline{\mathbf{A}}$, and to add the axioms

$$(16) \quad (\mathbf{A} \rightarrow \wedge) \rightarrow \overline{\mathbf{A}}, \text{ and}$$

$$(17) \quad \mathbf{A} \ \& \ \overline{\mathbf{A}} \rightarrow \wedge,$$

together with the rule

$$(e) \quad \text{from } \mathbf{A} \rightarrow \mathbf{B} \text{ and } (\mathbf{A} \rightarrow \mathbf{B}) \ \& \ \mathbf{C} \rightarrow \wedge \text{ to infer } \mathbf{C} \rightarrow \wedge.$$

But we now show that the additional axioms and rule are redundant, since UA (hence also the other modalities) may be defined in Π' in such a way as to obtain in Π' a theory of modalities equivalent to that of Π'' . Since the principal motivation for introducing (16), (17), and (ε), was simply to generate a theory of modalities, the point of extending Π' is thus lost.²

In Π' it is somewhat more natural to take N as the fundamental modality, since the definition of N takes the form

$$NA =_{\text{df}} (A \rightarrow A) \rightarrow A.^3$$

It will be convenient to refer to those wffs of Π'' in which \wedge occurs, if at all, only in contexts of the form $(A \rightarrow \wedge)$ (so that \wedge is always eliminable in favor of U) as " U -formulas" of Π'' . If A^U is any U -formula of Π'' , then the " N -transform" of A^U is the formula A^N got by replacing every wf part of A^U of the form $B \rightarrow \wedge$ by $(\bar{B} \rightarrow \bar{B}) \rightarrow \bar{B}$. Evidently if A^U is a U -formula of Π'' , then A^N will be a wff of Π' (as well as of Π''), and we can now state the result in the following form.

THEOREM. If A^U is a U -formula of Π'' , and A^N is its N -transform, then $\vdash A^U$ in Π'' if and only if $\vdash A^N$ in Π' .

We require the following lemmas.

LEMMA 1. $\vdash A^U$ in Π'' if and only if $\vdash A^N$ in Π'' .

Proof. The lemma will follow from $(B \rightarrow \wedge) \leftrightarrow ((B \rightarrow B) \rightarrow B)$ together with the *Ersetzungstheorem*.

By axioms (12), (14), (15), etc.,

$$I2. \quad \vdash ((\bar{B} \rightarrow \bar{B}) \rightarrow \bar{B}) \rightarrow (B \rightarrow \overline{\bar{B} \rightarrow \bar{B}}),$$

whence by 2 (above) and rule (d),

$$I3. \quad \vdash ((\bar{B} \rightarrow \bar{B}) \rightarrow \bar{B}) \rightarrow (B \rightarrow \wedge).$$

To prove the converse, we use axiom (3):

$$I4. \quad \vdash (B \rightarrow \wedge) \rightarrow ((B \rightarrow B) \rightarrow (B \rightarrow \wedge)).$$

Since $B \rightarrow B$ is easily shown rigorously equivalent to $\bar{B} \rightarrow \bar{B}$ (by axiom (12), etc.), the *Ersetzungstheorem* then gives us

$$I5. \quad \vdash (B \rightarrow \wedge) \rightarrow ((\bar{B} \rightarrow \bar{B}) \rightarrow (B \rightarrow \wedge)).$$

Then using axiom (16) and rule (d), we have

$$I6. \quad \vdash (B \rightarrow \wedge) \rightarrow ((\bar{B} \rightarrow \bar{B}) \rightarrow \bar{B}).$$

And $I3$ and $I6$ give us the rigorous equivalence from which the lemma follows.

² In this connection we quote the following from Robinson's review [5] of Ackermann's [1]: "It appears to the reviewer that the introduction of such a symbol [as \wedge] is not quite in keeping with the general spirit of the paper."

³ We remark in passing that necessity is thus definable in the pure theory of rigorous implication. The same definition of necessity appears also in *Calculi of pure strict implication*, by E. J. Lemmon, C. A. Meredith, D. Meredith, A. N. Prior, and I. Thomas (mimeographed).

LEMMA 2. If $\vdash A$ in Π'' , and if A is \wedge -free (i.e., A contains no occurrences of \wedge when written in primitive notation), then $\vdash A$ in Π' .

Proof. We shall show how to replace any proof of a \wedge -free formula in Π'' by a \wedge -free proof of that formula, which will therefore be a theorem of Π' . The leading idea is that although \wedge cannot be replaced by the same \wedge -free formula in every proof, it is still possible to find for each proof of a \wedge -free formula, a particular \wedge -free formula which may replace \wedge throughout that proof.

Let A_1, \dots, A_n ($A_n = A$) be a proof of A in Π'' , and let B_1, \dots, B_m be a list of all propositional variables occurring in the proof A_1, \dots, A_n . Then for this proof of A , we define \wedge' as

$$\overline{(B_1 \rightarrow B_1) \& \dots \& (B_m \rightarrow B_m)}.$$

Let A_i' be the result of replacing \wedge throughout A_i by \wedge' . Since A is \wedge -free, $A_n' = A_n = A$. After two sublemmas, we show inductively that A_1', \dots, A_n' is a proof of A in Π' .

SUBLEMMA 1. In Π' , $\vdash \overline{\wedge'}$.

Proof. Obvious from (1), rule (β), axiom (14), and (α).

SUBLEMMA 2. If C is a wf proper or improper part of any A_i' , then in Π' , $\vdash \overline{C \rightarrow C} \rightarrow \wedge'$.

Proof. Let D_1, \dots, D_k be a complete list of variables in C . Now in Π' we have

$$\begin{aligned} (D \rightarrow D) &\rightarrow (D \rightarrow D), \\ (D \rightarrow D) &\rightarrow (\overline{D} \rightarrow \overline{D}), \\ (D \rightarrow D) \& (E \rightarrow E) &\rightarrow (D \& E \rightarrow D \& E), \\ (D \rightarrow D) \& (E \rightarrow E) &\rightarrow (D \vee E \rightarrow D \vee E), \text{ and} \\ (D \rightarrow D) \& (E \rightarrow E) &\rightarrow ((D \rightarrow E) \rightarrow (D \rightarrow E)). \end{aligned}$$

An induction on the length of C then suffices to show that in Π' ,

$$\vdash (D_1 \rightarrow D_1) \& \dots \& (D_k \rightarrow D_k) \rightarrow (C \rightarrow C).$$

Since all the D_1, \dots, D_k are among the B_1, \dots, B_m , by axiom (6) and the associative and commutative properties of conjunction (stated by Ackermann for Π') we have

$$\vdash (B_1 \rightarrow B_1) \& \dots \& (B_m \rightarrow B_m) \rightarrow (D_1 \rightarrow D_1) \& \dots \& (D_k \rightarrow D_k),$$

and from the last two formulas the sublemma follows by rule (a), axiom (12), and rule (α).

Returning now to the proof of lemma 2, we distinguish two cases:

Case 1. A_i is an axiom of Π'' . (i) If A_i is one of the axioms (1)–(15) of Π'' , then $\vdash A_i'$ in Π' by the same axiom. (ii) If A_i is axiom (16) of Π'' , then A_i' is of the form $(C \rightarrow \wedge') \rightarrow \overline{C}$. By axiom (12) we have in Π'

$\vdash (\mathbf{C} \rightarrow \wedge') \rightarrow (\overline{\wedge'} \rightarrow \overline{\mathbf{C}})$, and by sublemma 1 and rule (δ) , $\vdash \mathbf{A}_i'$ in Π' .
 (iii) If \mathbf{A}_i is axiom (17) of Π'' , then \mathbf{A}_i' is of the form $\mathbf{C} \& \overline{\mathbf{C}} \rightarrow \wedge'$. By sublemma 2, $\vdash \overline{\mathbf{C}} \rightarrow \mathbf{C} \rightarrow \wedge'$ in Π' , and by axiom (13) and rule (a), $\vdash \mathbf{A}_i'$ in Π' .

Case 2. \mathbf{A}_i is a conclusion of a rule, where lemma 2 is true for the premisses. (i) If \mathbf{A}_i is a conclusion from premisses \mathbf{A}_j and \mathbf{A}_k by (α) , (β) , (γ) , or (δ) , then $\vdash \mathbf{A}_j'$ and $\vdash \mathbf{A}_k'$ in Π' by the inductive hypothesis, and $\vdash \mathbf{A}_i'$ in Π' by the same rule. (ii) If \mathbf{A}_i is the conclusion from \mathbf{A}_j and \mathbf{A}_k by rule (ε) in Π'' , then \mathbf{A}_k' (say) is of the form $\mathbf{A}_j' \& \mathbf{C} \rightarrow \wedge'$. By the hypothesis of the induction, (13), etc., we have in $\Pi' \vdash \overline{\mathbf{A}_j'} \& \overline{\mathbf{C}} \vee \wedge'$, and then by de Morgan's laws, double negation, and commutation, $\vdash \overline{\overline{\wedge'}} \vee \overline{\overline{\mathbf{A}_j'}} \vee \overline{\mathbf{C}}$. By sublemma 1, the inductive hypothesis for \mathbf{A}_j' , and (γ) twice, we have $\vdash \overline{\mathbf{C}}$ in Π' . Then by axiom (12) and rule (δ) we have $\vdash (\mathbf{C} \rightarrow \mathbf{C}) \rightarrow \overline{\mathbf{C}}$, and by axiom (12), rule (α) , axiom (14), and rule (a), $\vdash \mathbf{C} \rightarrow \overline{\mathbf{C}} \rightarrow \overline{\mathbf{C}}$. By sublemma 2 and rule (a) it follows that $\vdash \mathbf{A}_i'$ in Π' .

This completes the proof of lemma 2.

And from the two lemmas, the theorem is immediate. For since Π' is contained in Π'' , if $\vdash \mathbf{A}^N$ in Π' then $\vdash \mathbf{A}^N$ in Π'' , and by lemma 1, $\vdash \mathbf{A}^U$ in Π'' . And if $\vdash \mathbf{A}^U$ in Π'' , then by lemma 1 $\vdash \mathbf{A}^N$ in Π'' . But \mathbf{A}^N is \wedge -free, so by lemma 2, $\vdash \mathbf{A}^N$ in Π' .⁴

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