MEETING OF THE ASSOCIATION FOR SYMBOLIC LOGIC

The annual meeting of the Association for Symbolic Logic was held at the Rice Hotel, Houston, Texas, on January 23 and 24, 1967, in conjunction with the annual meetings of the American Mathematical Society and the Mathematical Association of America. There were four sessions under the respective chairmanships of Professors Paul Benacerraf, Nuel D. Belnap, Jr., Dana Scott and Norman M. Martin. Three invited one-hour addresses were given: A survey of proof theory by Professor Georg Kreisel, Recent developments in and applications of the theory of degrees by Professor Gerald E. Sacks, and Large cardinals by Professor Robert M. Solovay. Thirty-four papers were contributed, eleven of which were presented by title only (they are the last eleven printed below), the rest being presented by their authors in lectures of twenty minutes' duration. A social session, at which refreshments were served, was held from 5:00 to 7:00 p.m. on January 23. The Council met on January 23 from 12:00 to 2:00 p.m. and from 8:00 to 10:00 p.m.

The Program Committee

J. A. ROBINSON, CHAIRMAN
N. M. MARTIN

P. ACZEL. Normal functors on linear orderings.

Let $\mathcal{L}$ be the category of (reflexive) linear orderings with one-one order preserving maps. If $A \in \mathcal{L}$ and $x$ is in the field of $A$ let

$$A[x = \{ \langle u, v \rangle \in A \mid \langle v, x \rangle \in A \& v \neq x \}].$$

Let $A \sim B$ if $A$ and $B$ are similar linear orderings. Each functor $F: \mathcal{L} \to \mathcal{L}$ induces a function $F/ \sim$ on linear order-types.

The functor $F: \mathcal{L} \to \mathcal{L}$ is normal if it preserves inclusion maps and directed limits and there is a natural transformation $\eta: 1 \to F$ such that $F(A[x) = F(A)[\eta_A(x)]$.

Using a natural category theoretic construction we can associate with each normal functor $F$ a normal functor $F^*$ such that:

1. There is a natural equivalence $F(F^*(A)) \sim F^*(A)$.
2. If $A$ is well-ordered and $A \sim F(A)$ then there is a $B \subseteq A$ such that $A \sim F^*(B)$.
3. If $F$ maps well-orderings to well-orderings then $F/ \sim$ is a continuous increasing function on the ordinals, $F^*$ maps well-orderings to well-orderings and $F^*/ \sim$ is the derived function of $F/ \sim$.

The constructive analogue of these results generalizes several theorems on constructive order types. (Received November 4, 1966).

PETER B. ANDREWS. On simplifying the matrix of a wff.

In her thesis (Harvard, 1964) and in the Journal of the Association for Computing Machinery, vol. 10 (1963), pp. 1–24 and 348–356, Joyce Friedman formulated and investigated certain rules which constitute a semi-decision procedure for wffs of first order predicate calculus in closed prenex normal form with prefixes of the form $\forall x_1 \cdots \forall x_k \exists y_1 \cdots \exists y_m \forall z_1 \cdots \forall z_n$. Given such a wff $QM$, where $Q$ is the prefix and $M$ is the matrix in conjunctive normal form, Friedman's rules can be used, in effect, to construct a matrix $M^*$ which is obtained from $M$ by deleting certain conjuncts of $M$. Obviously $\vdash QM \supset QM^*$. Using the Herbrand-Gödel Theorem for first order predicate calculus, Friedman showed that $\vdash QM$ if and only if $\vdash QM^*$. Clearly if $M^*$ is the empty conjunct (i.e., a tautology), $\vdash QM^*$ so $\vdash QM$. Friedman also showed that for certain classes of wffs, such as those in which $m \leq 2$ or $n = 0$ in the prefix above, $\vdash QM$ if and only if $M^*$ is the empty conjunct. Hence for such classes of wffs the rules constitute a decision procedure. Computer implementation of the procedure has shown it to be quite efficient by present standards.

We present two theorems which generalize and unify Friedman's rules, and can be applied to wffs of higher order logics as well as first order logic. They have the following form. Let
QM be a wff (which may contain free variables of any type) in prenex normal form, where the prefix Q is arbitrary and the matrix M is in full disjunctive normal form. Let N, N₁, ⋯, Nₙ be appropriately chosen wffs, each of which is a disjunction of some of the disjuncts of M, or the empty disjunction (i.e., the propositional constant F denoting falsehood).

Reduction Theorem. ‡QM ≡ QN.

Splitting Theorem. ‡QM = [QN₁ ∨ ⋯ ∨ QNₙ]. (Received December 5, 1966.)

JON BARWISE. α-finite derivations of α-finite theorems.

The class of infinitary formulas is the least class containing the finite atomic formulas and closed under negation, (finite) quantification, and arbitrary infinite conjunctions and disjunctions. Usual rules of inference for these logical operations in the finite case are extended in a natural way to the infinitary case. By a derivation D of a formula φ from a set of axioms Φ we mean a set which encodes a derivation tree in the usual sense.

Using the notions of recursion theory on the ordinals [Kripke, this JOURNAL, vol. 29 (1964), pp. 161–162], or on sets, one can isolate important subclasses of formulas and derivations. In particular, if α is a recursively regular (i.e., admissible) ordinal, then we consider the language of α-finite formulas. That is, all conjunctions are required to be α-finite. α-finite derivations are defined similarly.

Completeness results for α-finite formulas which are true in all models of an α-r.e. set Φ of axioms have been obtained for arbitrary recursively regular α < ω₁, and for α = ω₁ when Φ is empty [Barwise, Notices of the American Mathematical Society, vol. 13 (1966), pp. 730, 855]. A basic theorem used in these arguments is the following which holds for arbitrary regular α.

Theorem. Let Φ be an α-r.e. set of α-finite sentences. If the α-finite formula φ has any derivation from Φ, it has an α-finite derivation from Φ.

Let ω₁ = the least nonrecursive ordinal. Using (i) translations of the notions of ω₁-recursion theory into the theory of hyperarithmetic and Π₁¹ sets, (ii) the familiar treatment of the ω-rule in infinitary terms, and (iii) a reduction of hyperarithmetic derivations to recursive ones due to S. Feferman, the above theorem leads to a solution of the open problem in Shoenfield [On a restricted ω-rule, Bulletin de l'Academie Polonaise des Sciences, (1959), pp. 405–407].

Theorem. Let Φ be a Π₁¹ set of axioms for analysis. If φ is derivable from Φ by the full ω-rule, then it is derivable from Φ by the restricted (recursive) ω-rule. (Feferman has since pointed out that a direct proof of this can be obtained by means of the standard theory of hyperarithmetic sets.) In terms of the function-quantifier hierarchy, this result is the best possible. (Received January 23, 1967.)

NUEL D. BELNAP, JR. Special cases of the decision problem for entailment and relevant implication.

There are as yet no decision procedures for the calculuses E of entailment and R of relevant implication. (See Phil. Stud., vol. 13, for the former; the latter is E plus A → A → A.) Special cases of the decision problem with known solutions are as follows, where both calculuses are intended if none is mentioned, and where "implication" and its cognates refer ambiguously to both. (1) Which truth-functions are provable? (2) Which implication-negation formulas are provable from the implication-negation axioms? (3) Which formulas of R are provable without use of distribution? (4) Which truth-functions imply which truth-functions? (5) Which truth-functions of implications between truth-functions are provable? (6) Which truth-functions imply which implications between truth-functions? (Answer: none.) (7) Which implications between truth-functions imply which truth-functions? (8) Which implications between truth-functions imply which implications between truth-functions?

Attention now turns to the question (9). Which conjunctions of implications between truth-functions imply which implications between truth-functions? Specializing to the negation-free case for convenience, it has been shown that

(*) (A₁ → B₁ & ⋯ & Aₙ → Bₙ) → (C → D)

is provable only if so too is

(**) (C → ∨ xeX₁ Aₓ) & (yeY₂ Bₑ → ∨ xeX₂ Aₓ) & ⋯ & (yeYₚ₋₁ Bₑ → ∨ xeXₚ₋₁ Aₓ) & (yeYₚ Bₑ → D)
John Thomas Canty. Leśniewski’s ontology and Gödel’s incompleteness theorem.

Within Leśniewski’s ontology, a numerical epsilon (ε) can be defined, e.g.,

\[ \text{Df}\ e_1 (\Phi, \phi) \vdash e_1 (\Phi, \phi) \equiv [\exists a]. \Phi(a). \phi(a):
\]

\[ \varnothing (a, b): \Phi(a). \Phi(b). \supset \bigwedge (a, b) \vdash \bigwedge (a, b) \vdash \phi(b) \]

where the numerical epsilon is a proposition forming functor for two arguments, each of which is a proposition forming functor for one nominal argument, and ‘\( \bigwedge \)’ is the symbol for equinumerosity.

One can prove for the numerical epsilon a formula analogous to the single axiom of ontology, thus establishing an ontological model for the numerical epsilon. Moreover, by employing this defined term, it is possible to settle the question of the applicability of Gödel’s incompleteness theorem to ontology.

E. W. Beth raises the question of applicability in The Foundations of Mathematics, §80. The question arises because of a basic difference in the rules for ontology and the other systems surveyed by Beth: Whitehead and Russell’s, Quine’s, Zermelo’s, etc. Unlike these systems, the rules for ontology are defined by Leśniewski only in the context of a given development of ontology and then change as the exposition of the system progresses.

However, on the basis of definitions for zero, (0) successor, (S), finite, (Fin) and numerical identity (=), one can, if ontology is extended by an axiom of infinity, represent recursive functions (predicates) in the system by theorems. In particular, Peano’s five axioms for arithmetic can be derived, while for, say, addition the following are theorems:

\[ [\Phi]: e_1 (\Phi, \text{Fin}) \supset + (\Phi, 0) =_{1} \Phi, \]
\[ (\Phi, \Psi): e_1 (\Phi, \text{Fin}) \supset e_1 (\Psi, \text{Fin}) \supset + (\Phi, S(\Psi)) =_{1} S(\Phi, \Psi). \]

Most importantly, all of Leśniewski’s terminological explanations can be expressed recursively by the numerical epsilon, and thus, derived in the system as theorems. Thus, the system of ontology extended by the axiom of infinity can be shown to be incomplete by means of Gödel’s method of proof.

(Received December 10, 1966.)

Donald J. Collins. Recursively enumerable degrees and the conjugacy problem.

Main Result. Given any Thue system \( \mathcal{X} \), there exists a uniform explicit method to construct a finitely presented group \( G(\mathcal{X}) \) whose word problem is recursively solvable and whose conjugacy problem is Turing equivalent to the word problem for \( \mathcal{X} \).

Corollary. Given any r.e. degree \( a \), there exists a finitely presented group whose word problem is recursively solvable and whose conjugacy problem is of degree \( a \).

Boone (Annals of Mathematics, vol. 83 (1966), pp. 520–571) and Shepherdson (Zeitschrift für mathematische Logik und Grundlagen der Mathematik, vol. 11 (1965), pp. 149–175) have shown that for each r.e. degree \( a \) there is a Thue system whose word problem has degree \( a \).

Given \( \mathcal{X} \) we make sequence of constructions to yield a Thue system \( \mathcal{X} \) specified as

\[ \mathcal{X}: s_1, s_2, \ldots, s_M, q, \quad U_1: F_i = qK_i, \quad i = 1, 2, \ldots, P \]

where \( F_i, K_i \) are words on the s-symbols. \( \mathcal{X} \) satisfies

1. For each \( i \), \( F_i \) and \( K_i \) are nonempty;
2. If \( i \neq j \), then \( F_i \) is distinct from \( F_j \) and \( K_i \) is distinct from \( K_j \);
3. If \( \Delta, \Gamma, \Psi, \Xi \) are arbitrary words on the s-symbols then to determine whether or not \( \Delta K_i = \Xi \Psi q \Xi = \) Turing equivalent to the word problem for \( \mathcal{X} \).
G(\mathfrak{X}) is presented as

\[ S: s_1, s_2, \ldots, s_M, q, t, x, r_1, \]
\[ D: xx_b = s_b x^2, \quad r_is_b = s_b x r_1 x, \]
\[ tx = xt, \quad tr_1 = r_1 t, \]
\[ F_rq = r_1^{-1} q K_r t \]

where \( b = 1, 2, \ldots, M, i = 1, 2, \ldots, P \) and \( F_i \) is the word obtained from \( F_i \) by replacing each symbol \( s_b \) by \( s_b^{-1} \).

Boone (Annals of Mathematics, vol. 84 (1966), pp. 49–84) shows that the word problem for a certain group \( G_1 \) is recursively solvable. \( G(V) \) is very similar to \( G_1 \) and a virtual paraphrase of Boone's argument proves that \( G(\mathfrak{X}) \) has solvable word problem.

Let \( \mathfrak{X} \) have word problem of degree \( a \); let \( b \) be the degree of the conjugacy problem for \( G(\mathfrak{X}) \). To show \( b \leq a \) requires a fairly lengthy argument. Of note in the proof is the following lemma which may regarded as a conjugacy analogue of Lemma 4 of J. L. Britton (Annals of Mathematics, vol. 77 (1963), pp. 16–32). Call a word \( V \) a circular variant of a word \( U \) if there exist words \( W_1, W_2 \) such that \( U = W_1 W_2 \) and \( V = W_2 W_1 \). Using the terminology of §2 of Boone (Annals of Mathematics, vol. 84 (1966), pp. 49–84) call a word \( W \) of \( E^* p \)-contracted if every circular variant of \( W \) is \( p \)-reduced.

**GENERAL LEMMA.** Suppose \( \text{Cond}_{\text{l}(b)}(E^*, E, p_0) \) holds and let \( U, V \) be \( p \)-contracted words of \( E^* \) not both \( p \)-free. If \((3 W) W^{-1} U W = e. V \), then either

1. there exist words \( U_0, V_0, A \) such that
   a. \( U_0, V_0 \) are circular variants of \( U, V \) respectively,
   b. \( U_0, V_0 \) both have \( p_0^{-1} \) as their final symbol,
   c. \( A^{-1} U_0 A = e. V_0 \),

2. or there exist words \( U_0, V_0, B \) such that
   a. \( U_0, V_0 \) are circular variants of \( U, V \), respectively,
   b. \( U_0, V_0 \) both have \( p_0 \) as their final symbol,
   c. \( B^{-1} U_0 B = e. V_0 \).

This is a corollary of Lemmas 3 and 4 of Britton and, in a very real sense, Britton’s two lemmas are the only tools employed throughout our entire argument. The crucial application of the General Lemma occurs with \( q \) taken to be a stable letter for the whole group \( G(\mathfrak{X}) \) over a certain sub-group.

The following theorem shows \( a \leq b \).

**THEOREM.** \( \Delta q \Gamma = e. V \Gamma \Xi \) if and only if

\[ \frac{(3 W) W^{-1} \Gamma^{-1} (\Delta q \Gamma)^{-1} \Delta q \Gamma W = e \Gamma \Xi, \Gamma^{-1} (\Upsilon \Gamma \Xi)^{-1} \Upsilon \Gamma \Xi} \]

where \( \Delta, \Upsilon \) are obtained from \( \Delta, \Upsilon \) by replacing each \( s_b \)-symbol by \( s_b^{-1} \).

The proof employs one application of the General Lemma and, thereafter, an argument similar to that given for Britton’s Lemmas 7 and 8.

Using his own methods, partial results along these lines have been obtained by A. A. Fridman. Their exact nature is not known to us. (Received December 10, 1966.)

**J. Michael Dunn. The effective equivalence of certain propositions about de Morgan lattices.**

It is well known that certain basic propositions in Stone’s representation theory of Boolean algebras are **effectively equivalent** in the sense that their equivalence may be shown without using any equivalent of the axiom of choice, although all known proofs of these propositions use such an equivalent. We observe that a similar situation arises with certain basic propositions about a generalization of a Boolean algebra called a **de Morgan lattice**, which is a distributive lattice together with an involution.

Kalman (Transactions of the American Mathematical Society, vol. 87 (1958), pp. 485–491) has shown that every de Morgan lattice is embeddable in a direct product of a de Morgan lattice that he calls \( \mathcal{O} \). Bialynicki-Birula and Rasiowa (Bulletin de l’Academie Polonaise des Sciences, vol. 5 (1957), pp. 259–261) have shown that every de Morgan lattice is isomorphic to what they call a **quasi-field of sets**. We have shown that every de Morgan lattice is embeddable in a 2-product of a field of sets, where in general a 2-product of a Boolean algebra \( B \) is defined on the set of all pairs of elements of \( B \), defining meet and join component-wise, and 10—J.S.L.
defining the involution $N$ so that $N(b_1, b_2) = (b_2, b_1)$ where $-$ is Boolean complementation. And as a specialization of a well-known result of Stone, we have that any two distinct elements of a de Morgan lattice are separable by a prime filter.

The known proofs of each of these propositions use some equivalent of the axiom of choice, and yet they may be shown effectively equivalent. This is done for the first three by the construction of embeddings between the appropriate structures. The last proposition is then related to Kalman's embedding theorem by remarking a connection between prime filters and homorphisms into $\mathcal{O}$ that was reported by us and Nuel D. Belnap, Jr. in a previous abstract. (Received November 14, 1966.)

R. O. GANDY. Relations between analysis and set theory.

By analysis is meant second-order arithmetic with full comprehension axioms. Let $ZF^-$ be Zermelo-Frankel set theory without the power-set axiom. It is known that by using the axiom of choice one can interpret $ZF^-$ in analysis. However the indispensability of the axiom of choice is shown by the following theorem: $ZF^-$ is not a conservative extension of analysis. To prove the theorem we consider a sentence $S$ which implies that there is a countable sequence of countable well-orderings whose sum is not countable. Levy has shown that $S$ is consistent relative to $ZF$. In $ZF^-$ we show that $S$ implies the existence of a countable well-founded model for analysis + $S$, and this suffices to prove the theorem. (Received December 10, 1966.)

STEPHEN J. GARLAND. Second-order cardinal characterizeability.

For any positive $n$ in $\omega$ let $V_n^1 (\Lambda_3^\omega)$ be the set of prenex sentences of pure second-order logic with equality, all of whose relation quantifiers precede any individual quantifier, and which have $n - 1$ alternations of relation quantifiers, the first quantifier being existential (universal). A sentence $\phi$ with no parameters characterizes a cardinal $\kappa$ if for any universe $A$, $\phi$ is true in $A$ if and only if $\kappa = \kappa$. For any set $\Gamma$ of sentences, let $\Psi(\Gamma)$ be the set of cardinals characterized by a sentence of $\Gamma$. Let $\Psi = \bigcup \{\Psi(V_n^1) : 0 < n < \omega\}$. Löwenheim-Skolem and compactness arguments show that $\Psi(V_1^1) = \Psi(\Lambda_3^+).$ On the other hand, Zykov [AMS Translations, ser. 2, vol. 3] shows that $\Psi$ implies $2^\kappa \in \Psi(V_2^1).$ These results lend interest to $\Psi(\Lambda_3^+)$ and $\Psi = \Psi(V_1^1) \cap \Psi(\Lambda_3^+)$ as classes which contain infinite cardinals but which may not be cofinal with $\Psi$.

Let $\mu_n$ be the least ordinal which is not the order type of a $\Delta_3^1$ well-ordering of $\omega$. Let $\mu(\omega) = \sup(\mu_n : 0 < n < \omega)$. Let $\Omega_\alpha$ be the least ordinal of cardinality $\aleph_\alpha$. Define $f$ recursively by $f(\alpha, 0) = \Omega_\alpha$, $f(\alpha, \beta + 1) = \Omega_{f(\alpha, \beta)}$, and $f(\alpha, \lambda) = \sup(f(\alpha, \beta) = \beta < \lambda) + 1$ for limit ordinals $\lambda$.

**Theorem 1.** If $\beta < \mu_2$ then $n = \alpha \in \Psi$.

**Theorem 2.** If $\beta < \mu_2$ and $\alpha \in \Psi$, then $f(\alpha, \beta) \in \Psi$.

**Corollary.** $\Psi$ is not an initial segment of the sequence of cardinals.

The proofs of these theorems require only that $\beta < \mu_2^*$ (the least ordinal which is not the order type of both $\Sigma_3^1$ and $\Pi_3^1$ well-orderings of subsets of $\omega$). However $\mu_2^* = \mu_2$, so no strengthening is obtainable in this manner.

**Theorem 3.** If $\beta < \mu(\omega)$ and $2^{\aleph_0} < \aleph_\beta$, then $n = \alpha \in \Psi$.

**Theorem 4.** If $\beta < \mu(\omega)$, $\aleph_\alpha \in \Psi$, and $2^{\aleph_0} < f(\alpha, \beta)$, then $f(\alpha, \beta) \in \Psi$.

**Corollary.** The assertions " $\mu(\alpha) = \aleph_\alpha$" and " $\sup \Psi > f(0, \mu_2)$" are consistent relative to $ZF$.

Are they independent?

**Theorem 5.** The cardinal successor of $2^{\aleph_0}$ is in $\Psi$.

Hence it follows from Cohen's work that, loosely speaking, if $\alpha$ is any ordinal arithmetical in $\lambda(\Omega_\alpha)$, then the assertion " $\sup \Psi > \aleph_\alpha$" is consistent relative to $ZF$.

**Questions.** (i) Does $n = \alpha \in \Psi$ imply $\aleph_\alpha \in \Psi$? (ii) Is the assertion " $2^{\aleph_0} \in \Psi$" independent of $ZF$? A "yes" answer to (i) implies a "no" answer to (ii). (Received December 10, 1966.)

P. C. GILMORE. Extensionality in positive set theory.

Positive set theory with a comprehension rule ($PST^*$) is formalized within classical first order logic with identity and has two primitive relations $\in$ and $\phi$. A wff of $PST^*$ is positive if it is either atomic or uses only conjunction, disjunction or universal or existential quantification.
For any positive wff $P$ and $Q$ and any variable $x$, $(x: P, Q)$ is a term of PST* and is intended to denote the set with membership determined by $P$ and with complement membership determined by $Q$. The extra-logical axioms of PST* are:

\[(N) \quad (x)(y) \sim (x \in y \& x \notin y),\]

and the consequences of a comprehension rule: for any positive wff $P$ and $Q$ for which either $(u)(x) \sim (P \& Q)$ or $(u)(Ex)(P \& Q)$ is provable in PST*.

\[(C) \quad (u)(x)[(x \in x: P, Q) \equiv (x) \sim (P \& Q) \& P] \&(x \notin x: P, Q) \equiv (x) \sim (P \& Q) \& Q])\]

is provable in PST*. Here $(u)$ is a string of universal quantifiers one for each free variable, other than $x$, occurring in $P$ and $Q$.

$M(PST*)$ is the semi-model with domain the constant terms of PST* for which an atomic sentence is true if and only if it is provable in PST*.

**Theorem.** $M(PST*)$ is a model of PST*.

Extensional identity $x = \gamma y$ is defined to be $(u)[(u \in x \equiv u \in y) \&(u \notin x \equiv u \notin y)]$. The extensional axiom $x = \gamma y \Rightarrow x = y$ is inconsistent with PST*. The following extensionality rule, however, is possibly consistent with PST*: if $(Qu)(s = \gamma t)$ is provable for any term $s$ and $t$ and string of quantifiers $(Qu)$ then $(Qu)(s = t)$ is provable.

The theory PST$\dagger$ with axiom $(N)$ and axiom scheme $(C)$ is inconsistent. On the other hand PST with axiom scheme

\[(CA) \quad (x) \sim (P \& Q) \Rightarrow [(x \in x: P, Q) \equiv P] \&(x \notin x: P, Q) \equiv Q]\]

is possibly consistent. (Received December 7, 1966.)

**MATTHEW L. HASSETT.** Recursive equivalence types of groups.

Let $\epsilon$ and $\Lambda$ denote the collections of nonnegative integers and isols respectively. An *r.e. group* is a pair $\langle a, p \rangle$, where $a$ is an r.e. set of nonnegative integers and $p$ is a partial recursive group multiplication for $a$. An *$\omega$-group* is a subgroup of an r.e. group. If $G = \langle \beta, p \rangle$ is an $\omega$-group, we call $\text{Req}(\beta)$ the order of $G$ $\text{[}o(G)\text{]}$. $G$ is said to be isolated (immune, regressive) if $\beta$ is isolated (immune, regressive). Let $\alpha \subseteq \epsilon$. A fundamental example of an $\omega$-group is the group $P(\alpha)$ consisting of the set of all Godel numbers of finite permutations of $\alpha$ under the multiplication induced by composition of functions. Here $o(P(\alpha)) = A^1$, where $A = \text{Req}(\alpha)$.

A subgroup $H = \langle \beta, p \rangle$ of an $\omega$-group $G = \langle a, p \rangle$ is called recursive if $\beta | a - \beta$ and good choice (g.c.) if the left (or right) coset decomposition of $G/H$ is a good choice decomposition. In the latter case, the index of $H$ in $G$ $\text{[}i(G/H)\text{]}$ is defined to be the RET of the left (or equivalently, right) coset decomposition. **Proposition.** Every recursive subgroup of a regressive group is g.c. **Proposition.** Let $\beta$ be a subset of $\alpha$. Then the following conditions are equivalent:

1. $\beta | a - \beta$;  
2. $P(\beta)$ is a recursive subgroup of $P(a)$; 
3. $P(\beta)$ is a g.c. subgroup of $P(a)$. **Theorem** (Lagrange). Let $G$ be an $\omega$-group with g.c. subgroup $H$. Then $o(G) = o(H) \cdot i(G/H)$. **Corollary.** There exists an isol (regressive isol) such that $T \not= o(G)$ for any isolic group $G$. Two other results of interest are: (1) The analogue of Cayley's theorem fails to hold; i.e., there exists an isolic (Abelian) group which is not isomorphic to any subgroup of $P(\epsilon)$. (2) Any isolic Abelian group can be effectively decomposed into its Sylow subgroups $S(p)$. If $S(p)$ is regressive, $o(S(p)) = p^\epsilon$ for some regressive isol $T$. (Received October 23, 1966.)

**J. J. LE TOURNEAU.** Decision results concerning the notion of operation.

For any class $\mathcal{K}$ of similar structures let $T_\mathcal{K}$, $T_\mathcal{M}$, $T_\mathcal{W}$ and $T_{\mathcal{U}}$ be the elementary, the monadic second-order, the weak second-order (set variables restricted to range over finite sets), and universal monadic second-order theories of $\mathcal{K}$, respectively. (More fully, $T_\mathcal{U} = T_\mathcal{M} \& \{Q\phi: Q$ is a block of universal set quantifiers and $\phi$ is elementary$\})$. Let "U" range over nonempty sets. Let $Op = \langle U, f \rangle: f$ is a unary operation on $U$, $FiOp = \{\mathcal{U}\epsilon Op: \mathcal{U}\epsilon \omega\}$, and $InOp = Op - FiOp$. For any $m, n, \omega$ let $T_\mathcal{M}m,n = \langle U, f, g, \phi: f, g$ are permutations of $U \& (x \epsilon U)[(f(x)x = x \& \cdots \& f^n(x)x = x) \& (g(x)x = x \& \cdots \& g^n(x)x = x)] \rangle \& \langle (m, n)R = \langle U, R\rangle: R \subseteq U \times U \& (x \epsilon U)[(y, x) \epsilon R], (y, y, x) \epsilon R\rangle$ have cardinal $\leq m, n$, respectively$\rangle$. Let $N_2 = \{0, 1\}^n: n \epsilon \omega$ (the set of finite sequences of 0's and 1's), $r_0 = \langle s^?\langle 0\rangle: s \epsilon N_2\rangle$ and $r_1 = \langle s^?\langle 1\rangle: s \epsilon N_2\rangle$.

Ehrenfeucht has announced (Notices of the American Mathematical Society, vol. 6 (1959),
abstract 556–37) that (A) $T_0\text{Op}$ is decidable; Doner has announced (Notices of the American Mathematical Society, vol. 12 (1965), Abstract 65T-468) that (B) $T_\omega \langle N_2, r_0, r_1 \rangle$ is decidable.

THEOREM 1. $T_M\text{FiOp} (= T_\omega \text{FiOp})$ is decidable.


THEOREM 2. $T_0\text{InOp}$ is decidable.

The proof uses (A), the decidability of $T_\omega\text{FiOp}$, and a generalized product argument.

THEOREM 3. $T_\omega\text{TwoPm}(2, 2)$ and $T_\omega(2, 2)RI$ are faithfully interpretable into finite extensions of each other (in the sense of this JOURNAL, vol. 24 (1959), p. 282).

THEOREM 4. $T_\omega(2, 2)RI$ is undecidable (and hence $T_\omega\text{TwoPm}(2, 2)$ is undecidable, although $T_\omega(2, 1)RI$ and $T_\omega\text{TwoPm}(2, 2)$ are decidable). (Received December 10, 1966.)

KENNETH LOEWEN. Some results of a modification of strong reduction in combinatorial logic.

Curry and Feys in their Combinatory logic define strong reduction as a sequence of steps of three types. Type I being the basic rules for the combinators $\hat{S}$, $\hat{K}$ and $\hat{I}$. Type II introduces bound variables with the binding indicated by a $\lambda$-prefix. Type III is an algorithm for removing the $\lambda$'s introduced by type II steps.

In the original definition no $\lambda$'s introduced by type II steps could occur in arguments of type I and II steps. In a paper forthcoming in the Notre Dame journal of formal logic the author shows that this restriction is unnecessary. This answers a question raised by Curry and Feys.

On the basis of this definition the author has subsequently proved that for each strong reduction $X \vdash Y$, there is an associated standard reduction (a reduction in which type I and II steps are done first and from left to right). That is there is a $Z$ such that $X \vdash Z$ in a standard reduction and $Y \vdash Z$.

Also the author has constructed a direct proof of the Church-Rosser theorem for strong reduction. That means if $X = Y$ in combinatorial logic with equality, then there is a $Z$ such that $X \vdash Z$ and $Y \vdash Z$. (Received November 7, 1966.)

KUNO LORENZ. The game-theoretic completeness and independence of the intuitionistically primitive logical connectives.

As an alternative to the usual set-theoretic semantics of logical calculi it is possible to develop a game-theoretic approach to logic by using dialogues (i.e. the plays of an open two-person zero-sum game with two values, each proposition being an initial position—put forward by the proponent—of the game).

The finitely many moves of each play serve either to attack a prior move of the partner or to defend a move of one’s own against any such attack.

A theory of truth can now be established by defining:

$A$ is true if there exists a winning-strategy for the proponent of $A$.

Assuming now that a certain class $K$ of such dialogue-definite propositions has been defined it is natural to consider a new proposition $A$ logically composed out of the propositions of $K$ if the scheme of the attacks against $A$ as well as the defenses of $A$ against any such attacks consists essentially of just the propositions of $K$ themselves.

By an easy combinatorial procedure the different simple $n$-placed connectives, including infinite compositions, i.e. quantifiers, may be determined. Using the concept of logical equivalence to be defined relative to the rules of the game, any set of connectives sufficient to represent equivalently the remaining ones is called a base. The main result is:

THEOREM. The four junctors $\preceq$ (non), $\land$ (et), $\lor$ (vel), and $\rightarrow$ (sub) and the two quantifiers $\forall$ (for all) and $\exists$ (for some) form the only primitive and minimal base for any dialogue-game on propositions built up with arbitrary $n$-placed junctors as well as quantifiers out of prime propositions.

In view of the fact that the most general definition of a dialogue-game leads to intuitionistic logic this theorem yields a new proof of the well known independence of the intuitionistic propositional connectives $\preceq \land \lor$ and $\rightarrow$. (Received December 8, 1966.)
Michael Morley. The Hanf number for $\omega$-logic.

By $\omega$-logic we mean a two-sorted predicate calculus where the variables of one sort always range over the natural numbers. Scott (Notices of the American Mathematical Society, vol. 6 (1958), p. 778) has shown that for each recursive ordinal $\alpha$ there is a sentence in $\omega$-logic which has a model $\models_\alpha$ but none of any larger power. We show conversely:

**Theorem 1.** If $P$ is a sentence of $\omega$-logic which does not have models in every infinite power then there is a recursive ordinal $\alpha$ such that $P$ has no models of power greater than $\models_\alpha$.

Suppose $T$ is a set of sentences in a countable first order language (of the usual one-sorted kind) and $\Sigma$ a set of formulas with one free variable. A model $A$ of $T$ omits $\Sigma$ if no element of $A$ satisfies all the formulas of $\Sigma$. Theorem 1 follows immediately from:

**Theorem 2.** If $T$ and $\Sigma$ are recursive (or even hyperarithmetic) then either (i) $T$ has models of every infinite power which omit $\Sigma$, or (ii) there is a recursive ordinal $\alpha$ such that no model of $T$ of power $> \models_\alpha$ omits $\Sigma$. (Received December 5, 1966.)


Let $\mathcal{M}_0$ be Gödel-Bernays set theory with the local axiom of choice (every set can be well-ordered) substituted for the global axiom (there is a choice class for the whole universe). We obtain $\mathcal{M}_1$ by adding to the axioms of $\mathcal{M}_0$ the axiom schemata

\[(\Delta_1) \quad (x)\exists A (A, x) \equiv (\exists A)\exists (A, x) \rightarrow (\exists B) (x)(x \in B \equiv (\exists A)\phi(A, x))\]

\[(CC) \quad (x)(\exists A)\psi(x, A) \equiv (\exists A)(x)(\exists y)\psi(x, A, y)\]

of $\Delta_1$-comprehension and (weak) class choice (here $\phi$ and $\psi$ have no bound class variables and $A_y = \{x \mid \langle y, x \rangle \in A\}$). $\mathcal{M}_1$ is the weakest "impredicative" extension of $\mathcal{M}_0$, the first step towards the full Morse theory $\mathcal{M}_\infty$.

Let $\theta$ be a fixed inaccessible ordinal, let $R_\theta$ be the set of all sets of rank less than $\theta$. A $\theta$-model of $\mathcal{M}_1$ is a subset $\mathcal{P}$ of $R_{\theta+1}$ such that if we interpret "set" by "member of $R_\theta$" and "class" by "member of $\mathcal{P}$" we obtain a model of $\mathcal{M}_1$.

**Theorem 1.** There exists a subset $\mathcal{H}(\theta)$ of $R_{\theta+1}$ which is a $\theta$-model of $\mathcal{M}_1$ and which is contained in every $\theta$-model of $\mathcal{M}_0$.

A subset $A$ of $R_\theta$ is $\Sigma_1$-absolute (relative to $\mathcal{M}_1$) if there exists a formula $\phi(B, x)$ with no class variables other than $B$ but possibly with constants from $R_\theta$ such that for every $\theta$-model $\mathcal{P}$ of $\mathcal{M}_1$, all $x$,

\[x \in A \equiv (\exists B)\phi(B, x)\]

A set $A \subseteq R_\theta$ is $\Delta_1$-absolute (relative to $\mathcal{M}_1$) if both $A$ and $R_\theta - A$ are $\Sigma_1$-absolute.

**Theorem 2.** $\mathcal{H}(\theta)$ consists of all sets which are $\Delta_1$-absolute relative to $\mathcal{M}_1$.

The set $\mathcal{H}(\theta)$ of hyperprojective subsets of $R_\theta$ is the analog for set theory of the set of hyperarithmetical sets of integers in arithmetic, and the theorems above follow from an abstract construction of the hyperarithmetical hierarchy that we give in Abstract first order computability (to appear). The definition of hyperprojective sets given in that paper is quite similar to the definition of the hyperarithmetical sets of integers and is evidently predicative. Theorems 1 and 2 give one analogous of Kleene's theorem that the hyperarithmetical sets of integers are exactly the $\Delta_1$ sets of integers.

The hyperprojective subsets of $R_\theta$ fall in a natural hierarchy, similar to the hyperarithmetical hierarchy. We list here some results that can be proved using this hierarchy.

A subset $\mathcal{P}$ of $R_{\theta+1}$ is $\Sigma_1$-absolute (relative to $\mathcal{M}_1$) if there exists a formula $\phi(B, A)$ with no class variables other than $B$ and $A$ but possibly with constants from $R_\theta$ such that for every $\theta$-model $\mathcal{P}$ of $\mathcal{M}_1$ that contains $A$,

\[A \in \mathcal{P} \equiv (\exists B)\phi(B, A)\]

**Theorem 3.** (a) The set $\mathcal{H}(\theta)$ is $\Sigma_1$-absolute relative to $\mathcal{M}_1$.

(b) $\mathcal{H}(\theta)$ is $\Delta_1$-definable over $R_\theta$, i.e. for suitable $\phi(B, A)$, $\psi(B, A)$,

\[A \in \mathcal{H}(\theta) \equiv (\exists B)R_{\theta+1}\phi(B, A) \equiv (B)R_{\theta+1}\psi(B, A)\]

(Similarly, each member of $\mathcal{H}(\theta)$ is $\Delta_1$-definable over $R_\theta$.)

(c) If $\mathcal{P}$ and $\mathcal{Z}$ are $\Sigma_1$-absolute subsets of $R_{\theta+1}$ (or $R_\theta$), then there exist $\Sigma_1$-absolute sets $\mathcal{P}$, $\mathcal{Z}_1$ such that
The theory of hyperprojective subclasses of the universe in a model of $\mathcal{M}_1$ can be developed within $\mathcal{M}_1$. Using this and (c) of Theorem 3 we obtain the following interpolation theorem for $\mathcal{M}_1$:

**Theorem 4.** Let $\phi(A)$, $\psi(A)$ have no bound class variables assume $\forall \mathcal{M}_1(A) \phi(A) \rightarrow (\exists A)\psi(A)$. There exist $\phi_1(A)$, $\psi_1(A)$ with no bound class variables such that $\forall \mathcal{M}_1(A) \phi_1(A) \equiv (\exists A)\psi_1(A)$, $\forall \mathcal{M}_1(A) \phi_1(A) \rightarrow (A)\phi_1(A) \rightarrow (\exists A)\psi_1(A)$. (Received December 5, 1966.)

**John Myhill.** *A hierarchy of languages with infinitely long formulas.*

This is an extension of the Takeuti-Kino paper on predicates with constructive infinitely long expressions, *Journal of the Mathematical Society of Japan*, vol. 15 (1963), pp. 176–190, in which they constructed a language with recursive infinitely long expressions built up from atomic formulae of the form $t_1 - t_2$, where $t_1$ and $t_2$ are numerical constants or variables.

In that paper they proved that all the predicates in Kleene analytic hierarchy are represented by a small class of formulae of their language. We define a hierarchy of languages $L_a$ where $L_0$ is the Takeuti-Kino language, and $L_a$ is defined like $L_0$ except that the infinite conjunctions and infinite strings of quantifiers, instead of being required to be recursive, are required to be definable in some language $L_a$ with $a < \beta$. This hierarchy extends far into the nonconstructive ordinals; indeed we have

**Theorem 1.** If a well-ordering of ordinal $\gamma$ is definable in any of the language $L_a$, then there are predicates definable in $L_\gamma$ but not in any earlier $L$.

**Theorem 2.** Let $\phi(a)$ be the least ordinal not definable in $L_a$, and let $\tau$ be the least fixed point of $\phi$. Then for some $\tau' > \tau$, there are predicates definable in $L_{\tau'}$, but not before.

It is evident that all predicates definable in the language $L_a$ belong to $\Delta^2_0$; the converse is open, but we conjecture it to be false. (Received December 10, 1966.)

**Jim Owings.** *\Pi^1_1* sets, \omega sets, and metacompleteness.

All definitions not herein stated may be found in *Metarecursive sets*, G. Kreisel and G. E. Sacks, this Journal, vol. 30 (1965), pp. 318–338. $L$ is the set of recursive ordinals. If $A \subseteq L$, $A$ is unbounded, and $A$ has order type $\omega$, $A$ is called an \omega set. $A \leq \omega B$ means $A$ is metarecursive in $B$, $A \leq \omega B$ means $A$ is weakly metarecursive in $B$. If $C \subseteq L$ and $A \leq \omega C$ whenever $A$ is metarecursively enumerable (meta-r.e.), $C$ is called metacomplete.

**Theorem 1.** If $A$ is a meta-r.e. \omega set, there exists a metacomplete $\Pi^1_1$ set $C$ such that $C \leq \omega A$.

**Theorem 2.** If $B$ is a $\Pi^1_1 = \Sigma^1_1$ set, there exists a meta-r.e. \omega set $A$ such that $B \equiv_M A$.

**Corollary 1.** If $B$ is a $\Pi^1_1 = \Sigma^1_1$ set, there exists a metacomplete $\Pi^1_1$ set $C$ such that $C \leq \omega B$.

**Corollary 2.** $\leq_x$ is not a transitive relation on the $\Pi^1_1$ sets.

**Corollary 3.** A metadegree contains a $\Pi^1_1 = \Sigma^1_1$ set iff it contains a meta-r.e. \omega set.

**Corollary 2** requires the use of Theorem 2 of Post's problem, admissible ordinals, and regularity, G. E. Sacks, *Transactions of the American Mathematical Society*, vol. 124 (1966), p. 8, which states the existence of $\Pi^1_1$ sets $A$ and $B$ such that $A \not\equiv \omega B$ and $B \not\equiv \omega A$, while Corollary 3 requires the use of Theorem 5 of Metarecursion theory, G. E. Sacks, *Leicester Symposium Volume* (to appear) which states that if $A$ is any nonregular meta-r.e. set, there is a $\Pi^1_1 = \Sigma^1_1$ set $B$ such that $A \equiv_M B$. (Received December 8, 1966.)

**Michael D. Resnik.** *A set theoretic approach to the simple theory of types.*

Quine's one sorted formulation of the simple theory of types has diminished the contrast between set theory and type theory. (See Quine's *Set theory and its logic*, Sections 36–37. Earlier results on the present topic were announced by the present author in the *Notices of the American Mathematical Society*, vol. 13 (1966), p. 351.) The contrast may be further diminished by observing that a hierarchy of simple, non-cumulative types is a sequence—order type $\omega$ for non-negative types, order type $\omega^* + \omega$ for negative types—of equivalence classes of objects which are of the same type. The successor of each equivalence class in such a sequence is its power.
set, and one object in the hierarchy belongs to another only if the former exactly precedes the latter in type. By applying these observations it is possible to formulate the simple theory of types along lines that closely resemble the standard set theories.

Following Quine exact precedence in type is defined by \( PTxy \) for \( (\exists z)(x \equiv z, y \equiv z) \). Then the other basic concepts of type theory are given the following definitions:

\[
\begin{align*}
'STx'y' & \text{ for } (\exists z)(PTxz \equiv PTzy) \quad [x \text{ and } y \text{ are of the same type}]; \\
'Tx' \text{ for } (\exists z)(z \equiv x \equiv STyz) \quad [x \text{ is a type}]; \\
'T_{xy}' \text{ for } -(\exists z)PTxy' \quad [x \text{ is an individual}]; \\
'Sxy' \text{ for } 'Tx' \cdot Ty \cdot x \equiv y' \quad [y \text{ is the successor of type } x].
\end{align*}
\]

Next a system \( T \) of nonnegative type theory is developed from the following axioms:

\[
\begin{align*}
\text{Ax. 1. } & y \in x \cdot y \in w \Rightarrow (\exists l)(x, w \in l); \\
\text{Ax. 2. } & u, v \in x, v \in w \Rightarrow (\exists l)(y, u, v \in l), \\
\text{Ax. 3. } & y \in x \cdot v \in x, w \in z \Rightarrow (\exists l)(y, v \in l), \\
\text{Ax. 4. } & (\exists y)(x)(x \in y \equiv STxz), \\
\text{Ax. 5. } & T_{xy} \Rightarrow (\exists z)(z \in x \equiv z \in y)' \Rightarrow x = y, \\
\text{Ax. 6. } & (\exists l)(l)(z \in y) \Rightarrow (x)(x \equiv z \Rightarrow x \equiv y \Rightarrow STxz), \\
\text{Ax. 7. } & (\exists x)T_{xy}.
\end{align*}
\]

A system \( T' \) of negative type theory is easily obtained from \( T \) by dropping the condition '-' \( T_{xy} \) from Ax. 5 and by replacing Ax. 7 by \(-(\exists x)T_{xy}'\). Some of the theorems of both \( T \) and \( T' \) which have not been proved in previous formulations of simple type theory are now listed:

\[
\begin{align*}
x \equiv y \Rightarrow PTxy; & \quad STxy \equiv (\exists z)(x, y \equiv z); \\
& \quad (x)(3y)(Ty \cdot x \equiv y); \\
& \quad y \equiv x \Rightarrow (\exists y)(x)(x \equiv y \equiv z \equiv x \equiv \neg Fx); \\
& \quad (3y)(x)(x \equiv y \equiv z \equiv x \equiv \neg Fx); \\
& \quad STxy \Rightarrow (\exists x)(x \equiv y \equiv z \equiv x \equiv \neg Fx).
\end{align*}
\]

Unlike previous systems of the type theory \( T \) has no axiom schema which use schematic type indices; however, the other formulations of the simple theory of types may be translated into \( T \). \( T \) is consistent relative to these systems and continues to be when axioms of infinity are added to \( T \) and these systems. \( T \) with or without an axiom of infinity has a model in Zermelo's system and \( T \) may be consistently extended to include negative types. (Received November 7, 1966.)

Gonzalo E. Reyes. A Baire space for relations.


Assume that \( \kappa = \kappa' \) and that \( \mathcal{M} \) is a class of structures \( \mathcal{M} = \langle A, R \rangle \) such that \( R \subseteq A \times A \). \( \mathcal{M} \) has \( G \) iff \( \mathcal{M} \neq \emptyset \) and every structure in \( \mathcal{M} \) has a proper extension in \( \mathcal{M} \). Let \( \tau^{\mathcal{M}}[\kappa] = \langle R \rangle(\kappa, R) \in \mathcal{M} \) be the topological space induced by the \( \kappa \)-topology \( \tau^R[\kappa] \) on \( \mathcal{E}(\kappa \times \kappa) \) (i.e., \( \mathcal{M} \) is a basis of \( \tau[R] \) iff \( \mathcal{M} = \{ R \in \kappa \times \kappa : W^+ \subseteq R \land W^- \cap R = \emptyset \) for some \( W^+ \subseteq \kappa \times \kappa \) such that \( W^+ \cup W^- < \kappa \). \( \mathcal{M} \subseteq \tau^{\mathcal{M}}[\kappa] \) is \( k \)-meager iff \( \mathcal{M} \) is a union of \( k \) nowhere dense sets. All \( R \in \tau^{\mathcal{M}}[\kappa] \) have property \( \mathbb{B} \) iff the set of relations in \( \tau^{\mathcal{M}}[\kappa] \) not having \( \mathbb{B} \) is \( k \)-meager.

\( \tau^{\mathcal{M}}[\kappa] \) is a Baire space iff the only \( k \)-meager open subset of \( \tau^{\mathcal{M}}[\kappa] \) is \( \emptyset \).

Lemma. Assume \( \mathcal{M} \) has I, II, V, VI, and \( \mathcal{M} \), then \( \tau^{\mathcal{M}}[\kappa] \) is a Baire space.

Theorem 1. Assume \( \mathcal{M} \) has I, II, VI, and \( \mathcal{M} \), then:

(1) \( \mathcal{M} \) has III iff almost all \( R \in \tau^{\mathcal{M}}[\kappa] \) satisfy \( \mathcal{M} \subseteq \mathcal{E}(\kappa, R) \),

(2) \( \mathcal{M} \) has IV iff almost all \( R(\kappa, R) \) is \( \mathcal{M} \)-homogeneous.

We define a forcing relation between (the diagrams of) structures in \( \mathcal{M} \cap \langle B, S \rangle : B \subseteq \kappa \) and formulas of \( L_{\infty}(\kappa) \) (the infinitary language having expressions of length \( < \kappa \) and a name for each \( y \in \kappa \). \( R \) is \( \mathcal{M} \)-generic if \( R \in \tau^{\mathcal{M}}[\kappa] \) and \( \langle \kappa, R \rangle \) is the union of a chain of structures in \( \tau^{\mathcal{M}}[\kappa] \) which is complete (i.e., such that every formula of \( L_{\infty}(\kappa) \) or its negation is forced by some element of the chain).

Theorem 2. Assume \( \mathcal{M} \) has II, V, and \( \mathcal{M} \), then:

(1) There are \( \mathcal{M} \)-generic relations.

(2) \( \mathcal{M} \) has III iff for every \( \mathcal{M} \)-generic \( R \), \( \mathcal{M} \subseteq \mathcal{E}(\kappa, R) \).
Theorem 3. Under the hypothesis of Theorem 1, almost all $R \in \mathcal{M}[\kappa]$ are $\mathcal{M}$-generic.

(Received December 10, 1966.)

P. Axt and W. E. Singletary. On general combinatorial decision problems and monogenic normal systems.

The general halting problem for a class $\mathcal{C}$ of combinatorial systems is the problem to determine given a system $\Sigma \in \mathcal{C}$ and a word $W$ of $\Sigma$ whether all deductions in $\Sigma$ from $W$ halt. The general halting problem for classes $\mathcal{C}_1$ and $\mathcal{C}_2$ of combinatorial systems are strongly equivalent iff there are effective maps $\mathcal{C}_1 : \mathcal{C}_1 \to \mathcal{C}_2$ and $\mathcal{C}_2 : \mathcal{C}_2 \to \mathcal{C}_1$ such that for any $\Sigma_1 \in \mathcal{C}_1$ and any $\Sigma_2 \in \mathcal{C}_2$ the halting problems for $\Sigma_1$ and $\Sigma_2$ are equivalent, as are the halting problems for $\Sigma_2$ and $\Sigma_2(\Sigma_2)$. One can describe similarly the general derivability and general decision problems, as well as strong equivalence of any pair of general problems. There is a constructive proof of the following results for monogenic normal systems (see Abstract 640-49, Notices of the American Mathematical Society for definition).

Theorem 1. The general halting, derivability and decision problems for monogenic normal systems are strongly equivalent.

Theorem 2. Each of the problems of Theorem 1 is strongly equivalent to the general halting and derivability problems for Turing machines, the general derivability and decision problems for semi-Thue systems and for Post's canonical systems, the general Post correspondence problem and the general decision problem for propositional calculi. (Received as revised, February 9, 1967.)

Vladeta D. Vuckovic. Recursive word-functions over infinite alphabets.

The author considers the recursive notions in the set $\Omega(A)$ of words over some (finite or denumerable infinite) alphabet $A = \{a_0, a_1, \ldots\}$, using enumerations $\alpha : D_\alpha \rightarrow \Omega(A)$ of $\Omega(A)$. $(\alpha$ is a map of some subset $D_\alpha$ of the set $N$ of nonnegative integers onto $\Omega(A)$.) A word-function $f$ is called an $\alpha$-R-function (where $R$ stands for primitive recursive, recursive, resp. partially recursive) iff there is a numerical $R$-function $f^*$ satisfying $f(\alpha(x_1), \ldots, \alpha(x_p)) \simeq \alpha(f^*(x_1, \ldots, x_p))$, for all $\langle x_1, \ldots, x_p \rangle \in D_\alpha^p$. The author studies various possibilities of defining recursive notions in $\Omega(A)$. He introduces some special enumerations, called codings, and proves that under any coding $\alpha$ all $\alpha$-primitive recursive word-functions can be obtained from some fundamental word-functions by finite many substitutions and primitive recursions of the form $f(X, 0) = \chi(X)$, $f(x, a, Y) = \psi(\alpha(i), X, \alpha(Y), f(X, Y))$, where $\chi$ is the empty word and where $v : N \rightarrow \Omega(\alpha(0))$ is defined by $v(0) = 0$ and $v(n + 1) = \alpha_0 v(n)$. Similar results are proved for $\alpha$-(partially) recursive word-function, using a minimalization over "numerals" $v(n)$ only. Those results explain the possibility of "absolute" definitions of recursive notions in $\Omega(A)$ (i.e. definitions without reference to any enumeration). The author gives examples of codings and proves that any two codings are primitively recursively isomorphic; moreover, if any simple enumeration $\alpha : N \rightarrow \Omega(A)$ of $\Omega(A)$ is primitively recursively isomorphic to a coding, it is a coding. (Received October 7, 1966.)

Charles F. Miller, III. Conjugacy and word problems in finitely generated groups.

Let $D_1$ and $D_2$ be arbitrary degrees of unsolvability with $D_1$ a recursively enumerable (r.e.) degree.

Theorem 1. There is a finitely generated (f.g.) group $G$ whose word problem has degree $D_2$ and whose conjugacy (or transformation) problem has degree lub $(D_1, D_2)$. If $D_2$ is an r.e. degree, $G$ can be chosen recursively presented.

The group is constructed as follows: Let $g(i)$ be a one-one total recursive function with infinite range $g(N)$, $0 \notin g(N)$, and $g(N) \in D_1$. Define $f(i)$ by $f(2i) = g(i)$, $f(2i + 1) = g(i) + 1$. Then $f(i)$ has the properties listed for $g(i)$. Let $B = \langle \varepsilon, b, e \rangle$ be a free group and $H$ its infinitely generated subgroup

$$H = \langle e^{-1}c^{-1}f^{(i)}d^{(i)}e, i > 0 \rangle.$$  

Lemma. The problem of deciding, for arbitrary $b \in B$ whether $b$ is conjugate to an element of $H$ is equivalent to the membership problem for $f(N)$. Let $B = \langle \varepsilon, d, e \rangle$ be a distinct copy of $B$
with \( H \) the copy of \( H \) in \( \mathcal{B} \). Now form \( T = *(B, \mathcal{B}; H = H) \) the free product of \( B \) and \( \mathcal{B} \) amalgamating \( H \) and \( \mathcal{H} \) under the obvious isomorphism.

**Theorem 2.** \( T \) is a recursively presented group with solvable word problem but with conjugacy problem of degree \( D_1 (= \text{degree } f(N)) \).

Now let \( S_2 \) be an infinite subset of \( N \) with decision problem of degree \( D_2 \) and \( 0 \notin S_2 \). Repeat the above with

\[
H_1 = \langle e^{-m}c^{-m}dc^m, m \in S_2 \rangle
\]

to obtain another group \( R = *(B, \mathcal{B}; H_1 = H_1) \) whose word problem and conjugacy problem both have degree \( D_2 \). Finally, the desired group is defined as the ordinary free product of \( T \) and \( R, G = T \ast R \). (Received December 10, 1966.)

**Charles F. Miller, III.** *Characterization of r.e. degrees by conjugacy and word problems for f.g. groups.*

Theorem 1 of the preceding abstract yields the following group-theoretic characterization of Turing reducibility:

**Corollary.** Let \( S_1 \) and \( S_2 \) be r.e. sets of natural numbers. A necessary and sufficient condition that \( S_2 \) be Turing reducible to \( S_1 \) is that there exist a f.g. recursively presented group \( G \) whose word problem has degree \( = \text{degree } S_2 \) and whose conjugacy problem has degree \( = \text{degree } S_1 \).

This result answers for f.g. recursively presented groups a question suggested by W. W. Boone for finitely presented groups: can the relationships between conjugacy and word problems be used to characterize some other kind of reducibility—say for example reducibility by unbounded truth tables? The above result and its method of proof suggest that full Turing reducibility is the natural reducibility even for finitely presented groups. The question arose from two observations: (i) the word problem is always many-one reducible to the conjugacy problem; (ii) many proofs of the unsolvability of the word problem use only reducibility by unbounded truth tables. Also, Boone's proof (Word problems and r.e. degrees of unsolvability, *Annals of Mathematics*, vol. 83 (1966), pp. 520-571 and vol. 84, pp. 49-84) of the existence of Thue systems with word problems of arbitrary r.e. degree uses only reducibility by unbounded truth tables. However, his proof of the corresponding result for finitely presented groups would seem to require full Turing reducibility (via Britton's lemma). (Received December 10, 1966.)

**John Corcoran and George Weaver.** *Logical consequence in modal logic.*

This paper presents a (modal, sentential) language system \( \text{PN} \) which may be thought of as a partial explication of the semantic and deductive properties of physical necessity \((\Box)\). The semantics of \( \text{PN} \) consists in interpretations of its *entire* language \( L \). An interpretation of \( L \) is an ordered pair \( \langle A, P \rangle \) where \( A \) (the actual world) is an interpretation of the nonmodal part of \( L \) and \( P \) (the physically possible worlds) is a set of such interpretations with \( A \in P \).

For \( S \subseteq L \) and \( p \in L \) we define: \( S \vdash p \) (\( p \) is a logical consequence of \( S \)) iff every interpretation making all sentences in \( S \) true also makes \( p \) true. The deductive system defines \( S \vdash p \) (\( p \) is provable from \( S \)) for arbitrary \( S \) (possibly containing modal sentences). This system is shown to be sound \((S \vdash p \text{ implies } S \vdash p)\) and strongly complete \((S \vdash p \text{ implies } S \vdash p)\).

Let \( M \) and \( N \) be respectively a set of modal sentences and a set of nonmodal sentences and let \( m \) and \( n \) be respectively a modal sentence and a nonmodal sentence \((m = \Box n \text{ for some } n)\). As corollaries to completeness and soundness we have (i) \( N \vdash n \text{ iff } N \vdash n \), (ii) \( M \vdash n \text{ iff } M \vdash n \) (and, therefore \( K n \text{ iff } t n \) (iii) if \( N \) is consistent then \( N \vdash m \text{ iff } m \), and (iv) if \( M \cup N \) is consistent then \( M \cup N \vdash n \text{ iff } M \vdash n \).

By dint of (iv) a system of this sort may be used to maintain in an axiomatic theory the distinction between physically necessary theorems and merely factual ones provided that such a distinction is made in the axioms by writing \( \Box n \text{ if } n \text{ is taken as a "law" and by writing } n \) (simply) if \( n \) is taken as a mere fact. Moreover, (iv) also permits the maintaining of the distinction in an axiomatization of a science between the theorems of the underlying mathematics and the properly empirical theorems. Similarly, in view of (i), (ii) and (iii), it is possible to distinguish in set theory (e.g.) between the theorems which are laws of logic and those which are properly theorems of set theory. In cases like this where there are no modal axioms \( \Box \) may be thought of as logical necessity.
In PN there are no nested modalities because of the nature of the intended interpretations. As can be seen from the above theorems the main force of the modal deductive system is to permit inference from □φ to φ anywhere and from φ to □φ in proofs having nonmodal assumptions. The authors do not know of any other modal system which treats logical consequences of arbitrary sets of sentences although Curry (Foundations of mathematical logic, New York (1963)) considered the deductive side of the question. (Received November 23, 1966.)

GEORG KREISEL. Relative recursiveness in metarecursion theory.

For basic notions, see pp. 190–205 of: Theory of models (TM), Amsterdam (1965), Fraisse (F), pp. 323–328 of: Infinitistic methods, Warsaw (1961) and Lacombe (L) [C.R. Acad. Sci. Paris, vol. 258 (1964), pp. 3141–43, 3410–13]. As in (F) and (L), the main tools are uniform implicit invariant definitions (u.i.i.d.); in contrast to (F) and (L) not all general models are considered, but only suitable e-extensions in the sense of Feferman et al. [Bulletin of the American Mathematical Society, vol. 72 (1966), pp. 480–485]. Specifically the class ω₁ of orderings possessing an initial segment isomorphic to the recursive ordinals (ω₁) in order of magnitude, i.e., e-extensions of the structure (ω₁, <), where the realization ε of ε is <. (Alternatively: e-extensions of (L₀, ε), where L₀ is the collection of hereditarily hyperarithmetic sets and ε = ε)

Definition 1. (TM), p. 198. The formula α with the (binary relation) symbol ε, the relations symbols X, X₁, ..., Xₙ, Y (having p₀, p₁, ..., pₙ₊₁ arguments) is a u.i.i.d. of X₀ from Y₀ (X₀ < ω₀, Y₀ < ω₀ⁿ₊₁) provided (a) α is satisfied by some W = (A, ε, X, X₁, ..., Xₙ, Y) with domain A, ω₀ⁿ₊₁ ∩ Y = Y₀, (A, ε) ∈ ω₁ or (a*) in addition, A = ω₁, ε = <; and (b) ∀W(((A, ε) ∈ ω₁) ∧ ω₀ⁿ₊₁ ∩ Y = Y₀) → X = X₀).

Definition 2. α is a u.i.i.d. of X₀ from Y₀ on ω₁ provided (a), resp. (a*) and ∀W(((A, ε) ∈ ω₁) ∧ ω₀ⁿ₊₁ ∩ Y = Y₀) → X ∩ ω₀ⁿ₊₁ = X₀. Condition (a*) corresponds to treating Y₀ as a property, (TM), p. 197, (a) as a set: this elegant formulation of the distinction intended in (TM), is due to (L). The following results are independent of whether one takes (a) or (a*): stability.

Theorem 1. There is a u.i.i.d. of X₀ from ∅ (on ω₁) iff X₀ is metadeterminate (metarecursives).

Theorem 2. There is a u.i.i.d. of X₀ from Y₀ on ω₁ if X₀ is invariably definable from the set Y₀ on ω₁ (in the sense of (TM)).

Corollary. For X₀ < ω₀, Y₀ < ω₀ⁿ₊₁: There is a u.i.i.d. of X₀ from Y₀ iff X₀ is hyperarithmetic in Y₀.

The results extend to infinite formulæ α provided one restricts α as in (TM), p. 192, since then the satisfaction relation is implicitly definable. Advantages of (i) implicit over explicit definitions, and (ii) ω₁ over general models: ad (i), the relations (X₀, Y₀) of Definitions 1 and 2 are not sensitive to choice of language nor to the distinction between definability from a property, resp. a set [cf. (TM), p. 204]; ad (i) and (ii), the only structure needed on ω₁ is < [cf. (TM), p. 195 (c) for finite explicit definitions; not all metarecursive sets are 'recursive' in (ω₁, <) in the sense of (F)]. The relation of Definition 1 will be considered the basic reducibility notion in metarecursion theory; evidently there are others corresponding to bounded truth table, many-one reducibility, etc.

The theorems above are special cases of much more general results of K. Kunen. (Received January 8, 1967.)

GEORG KREISEL. Relative recursive enumerability in metarecursion theory.

Notation as in preceding abstract. The following definition is implicit in work of Kunen. X₀ is ω₁-r.e. in Y₀ provided there is some α such that

X₀ = ω₀ⁿ₊₁ ∩ (X: W ⊨ α, Y ∩ ω₀ⁿ₊₁ = Y₀, (A, ε) ∈ ω₁).

Theorem 1. If Y₀ = ∅, X₀ is the range of some metarecursive function on ω₁, i.e., X₀ is Σ₁ over ω₁.

Theorem 2. If X₀ ⊆ ω₀ⁿ⁺₁, Y₀ ⊆ ω₀ⁿ⁺₁, then X₀ is ω₁-r.e. in Y₀ iff X₀ is Π₁ in Y₀.

Corollary. There are X₀ and X₀, such that X₀ is ω₁-r.e. in Y₀ but X₀ is not the range of any
function whose domain is $\omega_1$ and whose graph is metarecursive relative to $Y_0$ (in the sense of the abstract above).

If one wants $X_0$ in ‘$\Sigma_1$-form’ one must take as domain the sets that are metafinite relative to $Y_0$, i.e. which can be defined from $Y_0$ according to Definition 1 (i.e., not only on $\omega_1$). Thus (meta) recursive enumerability in $Y_0$ does not refer to ‘enumerations’ from the domain of the recursion theory but from the collection $\mathcal{MF}(Y_0)$ of sets that are metafinite relative to $Y_0$. (In the classical case, i.e., $\omega$ in place of $\omega_1$, this is independent of $Y_0$; (TM), p. 198, Lemma 1.) Sacks has analyzed for what $Y_0$ there is a suitable correspondence between $\omega_1$ and $\mathcal{M}(Y_0).

**Theorem 1.** If $X_0$ and $\omega_1^X - X_0$ are $\omega_1$-r.e. in $Y_0$, then $X_0$ is metarecursive relative to $Y_0$.

**Theorem 3.1 of (TM), p. 201** reads now: If a set $\Phi$ of (finite) formulae is $\omega_1$-r.e. in $Y_0$, and every subset $\Phi_1 \subseteq \Phi$ which is metafinite relative to $Y_0$, has an $\omega$-model then $\Phi$ has an $\omega$-model. (This seems to be incomparable with the result of Barwise [Notices of the American Mathematical Society, vol. 13 (1966), pp. 855–856] in terms of the notion: $\omega$-r.e. of Kripke [this JOURNAL, vol. 29 (1964), p. 162], since there are $Y_0$ such that not all sets $\omega_1$-r.e. in $Y_0$ are $\omega_1^Y$-r.e. Note that the answer to the question after Theorem 3.1 of (TM), p. 201 is negative for $X_0 = \{\text{kalee's} O\}.

**Theorem 4.** There is a set $\Phi$ of formulae in second order arithmetic which is $\omega_1$-r.e. in $O$ and has no $\omega$-model, but every subset of $\Phi$ definable from the property $O$ (in the sense of (TM)) does have an $\omega$-model.

**Proof.** Take $\Phi_0 = \text{diagram of } O \text{ together with the usual axioms of second order arithmetic, } D \text{ a set hyperarithmetic in } O \text{ and not definable from the property } O \text{ (cf. (TM), p. 200, 1. 11–13), and } D \text{ a definition of } D \text{ invariant for all } \omega \text{-models of } \Phi_0. \text{ If } a \text{ is a new constant, put } \Phi = \Phi_0 \cup \{D\} \cup \{a \neq n : n \in D\}$. (Received January 8, 1967.)

**Storrs McCall.** *Connexive class logic.*

In Boolean algebra, the null class is an exception to the general rule that no class can be included in its own complement. But by modifying the Boolean axioms a system of class logic is obtainable in which $a \neq a' \neq a \neq b = a \neq b' \neq a'$. asserting, but in which $0 \neq a$ is not. The axioms of this system CA of *connexive algebra* (so-called because of the analogy with the laws $\neg (p \supset \neg p)$ and $(p \supset q) \supset \neg p \supset \neg q)$ of connexive propositional logic) are as follows, substitution and *modus ponens* being the sole rules of inference:

1. Axiom-schemata for two-valued propositional logic
2. $(b \neq b = c) \supset \neg a = c$
3. $a \neq b \supset ac \neq cb$
4. $a(b \neq c) \neq ab \neq ac$
5. $a(b \neq c) \neq (a(b \neq c))$
6. $(a \neq b) \neq (a \neq b)\neq (a \neq b)\neq (a \neq b)\neq (a \neq b)$
7. $b \neq a \neq b$

The consistency of CA is demonstrable using an algebra with only two elements 0 and 1; the operations of intersection, union and complementation, and the relation of inclusion, being defined as follows:

<table>
<thead>
<tr>
<th>$\setminus$</th>
<th>0</th>
<th>1</th>
<th>$\cup$</th>
<th>0</th>
<th>1</th>
<th>$\subseteq$</th>
<th>0</th>
<th>1</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
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<td>0</td>
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</tr>
</tbody>
</table>

The regular two-valued tables for the propositional connectives ‘$\supset$’ and ‘$\neg$’ apply. Finally, the completeness of CA is shown to be of the same type as that of Boolean Algebra. That is, any consistent extension of CA is finite in the sense of containing theses true only of a finite algebra (e.g. the thesis $a = 0 \vee a = 1$). Since CA itself contains no such theses we say that CA, like Boolean algebra, is *saturated with respect to the property of being infinite*. (Received December 10, 1966.)

**Richard Montague.** *A generalization of recursion theory.*

In connection with any model $\mathcal{A}$ and cardinal $\alpha$ we consider object variables of type $n$
(ranging over $T(n, \mathcal{U}, \alpha)$, where $T(0, \mathcal{U}, \alpha)$ is the universe of $\mathcal{U}$ and $T(k + 1, \mathcal{U}, \alpha)$ is the set of subsets of $T(k, \mathcal{U}, \alpha)$ having power $< \alpha$) and predicate variables of type $<n_0, \ldots, n_r$ (ranging over the set of $(k + 1)$-place relations between members of $T(n_0, \mathcal{U}, \alpha), \ldots, T(n_r, \mathcal{U}, \alpha)$). $\Sigma_n = \Pi_0$ is the smallest set of $\Gamma$ such that $P_\alpha \ldots u_0$ is in $\Gamma$ whenever $P$ is a predicate constant or predicate variable and $u_0, \ldots, u_n$ are object variables of appropriate types, $\Gamma$ is closed under sentential connectives, and $\forall u (u \in v \land \phi)$ is in $\Gamma$ whenever $\phi$ is in $\Gamma$ and $u, v$ are object variables of successive types; $\Sigma_{n+1} = \Sigma_n$. $\Pi_n$ is the set of existential quantifications with respect to object variables of formulas in $\Pi_n$; dually for $\Pi_{n+1}$. $\Sigma_n(\mathcal{U}, \alpha, K)$ is the set of relations $R$ definable in $A$, in terms of relations in $K$, by a formula in $\Sigma_n$; this formula is to contain predicate variables representing fixed members of $K$ and object variables representing the relata of $R$. $\Pi_n(\mathcal{U}, \alpha, K)$ is analogous, and $\Delta_n(\mathcal{U}, \alpha, K) = \Sigma_n(\mathcal{U}, \alpha, K) \cap \Pi_n(\mathcal{U}, \alpha, K)$.

**Theorem.** Let $\mathcal{U}$ be the standard model of arithmetic and $K \cup (R)$ be a set of relations among natural numbers. Then if $n \neq 0$, $R$ is $\Sigma_n$ (or $\Pi_n$) in the Kleene hierarchy if and only if $R$ is in $\Sigma_n(\mathcal{U}, \mathcal{K}, \Lambda)$ (or $\Pi_n(\mathcal{U}, \mathcal{K}, \Lambda)$); $R$ is recursive in members of $K$ if and only if $R \in \Delta_1(\mathcal{U}, \mathcal{K}, \Lambda)$.

**Generalized Post's Theorem.** If $a \geq 3$, then $\Delta_1(\mathcal{U}, \alpha, \Sigma_0(\mathcal{U}, \alpha, \Lambda) \cup \Pi_0(\mathcal{U}, \alpha, \Lambda)) = \Delta_{n+1}(\mathcal{U}, \alpha, \Lambda)$.

**Generalized Kleene Hierarchy Theorem.** Suppose that $n \neq 0$, $\mathcal{U}$ has finite similarity type, $F$ is a 1-1 mapping of $T(1, \mathcal{U}, \alpha)$ into $T(0, \mathcal{U}, \alpha)$, and both $F$ and $\alpha$ are in $\Sigma_n(\mathcal{U}, \alpha, \Lambda)$. If $\alpha = \mathcal{K}_0$, then both $\Sigma_n(\mathcal{U}, \alpha, \Lambda) \subseteq \Pi_n(\mathcal{U}, \alpha, \Lambda)$ and $\Pi_n(\mathcal{U}, \alpha, \Lambda) \subseteq \Sigma_n(\mathcal{U}, \alpha, \Lambda)$ contain relations among individuals of $\mathcal{U}$ if $\alpha$ is any regular cardinal and $(\langle x, y \rangle : y \in \mathrm{rng} F$ and $x \in F^{-1}(y))$ is a well-founded relation, then the same conclusion holds.

(Previous public presentations: Stockholm, March 11, 1966 (for $a = \mathcal{K}_0$); Los Angeles, October 14, 1966 (the general case).) (Received December 7, 1966.)

**RICHARD MONTAGUE.** General formulations of Gödel's second undecidability theorem.

For terminology see Tarski, Mostowski, Robinson, Undecidable theories. $\Sigma_0$-formulas (also, $\Pi_0$-formulas) are formulas of $P$ (Peano's arithmetic) built up from atomic formulas by sentential connectives and quantifications $\forall x (x \leq \xi \land \phi)$, where $\xi$ is a term not containing $x$; $\Sigma_{n+1}$-formulas are existential quantifications of $\Pi_n$-formulas; dually for $\Pi_{n+1}$-formulas. A set $A$ of sentences is $\Pi_n$-correct (or relatively $\Pi_n$-correct) if every $\Pi_n$-sentence $\phi$ such that $\forall x \phi$ (or such that $\forall \forall \phi(x)$, where $N$ is a particularly one-place predicate) is true in $\mathcal{U}$ (the standard model of $P$). With any formula $\alpha$ of $P$ we associate sentences $\Pi_n$-Cor$_\alpha$ and $\Pi_n$-Cor'$_\alpha$ of $P$ which express in the natural way (using bounded truth definitions; cf. Kreisel, Wang, Fundamenta Mathematicae, vol. 42 (1955), pp. 101–110) that the set defined by $\alpha$ in $\mathcal{U}$ (i.e., the set of sentences whose Gödel numbers satisfy $\alpha$ in $\mathcal{U}$) is respectively $\Pi_n$-correct or relatively $\Pi_n$-correct.

**Theorem 1.** Suppose that (1) $T$ is a theory axiomatized by a set $A$, (2) $n \neq 0$, (3) $\alpha$ is a $\Sigma_n$-formula defining $A$ in $\mathcal{U}$, (4) $T$ is an extension of $P$, (5) $A$ is $\Pi_n$-correct. Then not $\vdash_T \Pi_n$-Cor$_\alpha$.

**Theorem 2.** Assume (1)–(3), that $T$ is an extension of $P^{(n)}$, and that $A$ is relatively $\Pi_n$-correct. Then not $\vdash_T \Pi_n$-Cor'$_\alpha$.

In view of the fact that if the axioms of Robinson's arithmetic are derivable from $A$, then $A$ is consistent if and only if $\Pi_n$-correct, together with the additional fact that this is provable in $P$, the case of Theorem 1 in which $n = 1$ corresponds to Feferman's general formulation of Gödel's theory on non demonstrable consistency (Feferman, Fundamenta Mathematicae, vol. 49 (1960), pp. 35–92, Theorem 5.6); a similar connection exists between Theorem 2 and Feferman's Theorem 8.4. Theorems 1 and 2 have the advantage of being applicable not just to recursively enumerable theories, but to arbitrary theories definable in $\mathcal{U}$. (Presented before ASL April 30, 1965; the term ' $\Pi_n$-correct' was suggested by Putnam.) (Received December 7, 1966.)

**A. A. MULLIN.** Theorem-proving by finite automata.

The essence of this short paper is a complete detailed program of moderate logical power and moderate machine efficiency written in the LISP language for proving theorems in the Principia Mathematica of B. Russell and A. N. Whitehead. The output format is in a form readily accessible to applied logicians rather than having a form digestible only by other automata. The program is a second generation modification of E. Stefferud's program [The
logic theory machine: A model heuristic program, Rand Corp. (1963), 187 pp., but it avoids his minor and easily correctable error (see the author's review of Steffrud's monograph in Computing Reviews, vol. 7 (1966)) which generates false "proofs" of established theorems and "proofs" for nontheorems. Frequent use is made of LISP's logical power for handling tree structures and for applying recursion techniques. The author's methods avoid Steffrud's "substitution", but use his " detachment", "forward-chaining", and "backward-chaining", among various tree-searching schemes developed by R. E. Millstein. Efficiency of the program is measured in terms of (1) the number of subproblems generated, (2) the number of "cons" operations (concatenations peculiar to the LISP language) used, and the time for the compiled machine run on the theorem. The known theorems 2.01 to 4.43 of Principia were run in one batch on a two-hour run with an IBM 7094 at Lawrence Radiation Laboratory. Then "difficult" theorems were run individually until either a complete proof was given or else a "garbage collect" (GC-2) error in LISP was obtained. Typical output statistics: theorem 2.01, 2 subproblems, 1012 cons, 513 milliseconds; theorem 2.14, 14 subproblems, 17,418 cons, 9022 milliseconds; theorem 2.49, 12 subproblems, 11,710 cons, 6218 milliseconds; and theorem 3.40, 19 subproblems, 18,195 cons, 11,781 milliseconds. (This research conducted under the auspices of the U.S. Atomic Energy Commission and the U.S. Army (Pacific).) (Received October 11, 1966.)

Sergiu Rudeanu. Axiomatization of certain problems of minimization.

The problem of simplifying the disjunctive normal form of a Boolean function is one of the basic problems in switching theory. There are many algorithms which carry out the simplification in two steps: (1) determination of the prime implicants; (2) determination of a solution made up of prime implicants.

Gr. C. Moisil has used an axiomatic approach in order to show that the above method also applies to functions expressed in terms of n-ary Sheffer functions, as well as to two-valued functions with three-valued or five-valued arguments; such functions occur in switching theory too.

The present paper consists of two parts.

In Part I an abstract analogue of the minimization problem is defined so that, under hypotheses as general as possible, this problem may be solved in the above indicated steps (1) and (2).

Part II establishes the relationship among the above minimization problems and other set-theoretical and graph-theoretical problems; it is shown that they are particular cases of the general problem defined in Part I.

Some conclusions are drawn for the practical solution of the discussed problems.

Paul Strauss. Number-theoretic set theory.

The following extension of classical elementary number theory avoids Gödel's incompleteness theorem. First, adjoin as primitive symbols $T_1, T_2, \cdots \cdots$ and stipulate that if $a$ is a term, then $T_n(a)$ is a formula. We identify formulas with their Gödel numbers. We write $\nu(n)$ or $\bar{n}$ for the $n$th numeral in our system. We say that the degree of a formula $A$ is $n$, if $T_n$ occurs in $A$ but no $T_m$ occurs for $m > n$. If no $T_n$ occurs in $A$, we say that its degree is 0. Next, we adjoin the following axioms to our system. If $A$ is a sentence of degree $\leq n$, then $T_{n+1}(A) \equiv A$ is an axiom.

We further extend our system by adjoining the restricted $\omega$-rule of J. R. Shoenfield's article On a restricted $\omega$-rule, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, vol. 7 (1959), pp. 405-407. Roughly speaking, this rule states that if $f$ is a recursive function such that for every $n, f(n)$ is a proof of $A(\bar{n})$, then infer $(x)A(x)$.

Let $x_1, x_2, \cdots$ be the individual variables of our system. Define $\delta^*(m_1, \cdots, m_n, n) = \{ \text{Sub in } n^*: m_1 \text{ for } x_1, \ldots, m_n \text{ for } x_n \}$ where $n^*$ is an effectively chosen alphabetic variant of $n$ so that for every $m_n, m_i$ is free for $x_i$ in $n^*$. If $f$ is a recursive function, we write $f$ for its representing function in our system. Now we write $S_{2k+1}(a_1, \cdots, a_k, b)$ for $T_{n+1}(\delta^*(\bar{a_1}, \cdots, \bar{a_k}, b))$. In particular, we interpret $S_{2k+1}(x, y)$ as $x \in y$.

In dealing with this system, it appears to be desirable to use a set-theoretic notion of equality rather than the primitive number-theoretic one. So, define $\theta(m, n) = [(x_1) \cdot m \equiv n]$. Then
write $E_{n+1}(a, b)$ for $T_{n+1}(\theta(a, b))$. If $a$ and $b$ are terms representing sets, then $E_{n+1}(a, b)$ states that they have the same members, hence are set-theoretically equal.

We now briefly state some of the deductive properties of our system. First, there are sets of degree $= n + 1$ which are not set-theoretically equal to any sets of degree $\leq n$. Secondly, all sets are countable. Finally, a very general version of the axiom of choice is provable.

The similarity of our system with that of H. Wang in *The formalization of mathematics*, this JOURNAL, vol. 19 (1954), pp. 241–266 should be noted. (Received September 21, 1966.)

H. C. WASSERMAN. *A discussion of some completeness proofs for propositional logic.*

In this paper, we classify all existing proofs of the completeness of the propositional calculus. Five types of proof are discussed. These are: (1) The normal-form type proof of Post, (2) A proof due to Łukasiewicz, (3) A proof due to Kalmár, (4) Proofs of Łos and Hiż, and (5) The "Lindenbaum-type" proof.

The types (1) and (2) are similar but distinct, and the same is true for types (4) and (5). The proofs due to Post and Łukasiewicz are designed especially for systems with a substitution rule of formulas for propositional variables, the Lindenbaum-type proof, actually a proof of strong completeness, is designed especially for systems without a substitution rule, and Kalmár's proof works equally well for either type of system.

Also included is an exposition of Post's method of constructing a complete axiomatization of any propositional logic based on any given functionally complete set of connectives. (Received November 17, 1966.)

J. MICHAEL DUNN and NUEL D. BELNAP, Jr. *Homomorphisms of intensionally complemented distributive lattices.*

A generalization of a Boolean algebra called an *intensionally complemented* distributive lattice (icdl), where *intensional complementation* is defined as a dual automorphism of period two with no fixed point, plays a role in the semantic investigations of certain fragments of the system $E$ of entailment that is analogous to that played by Boolean algebras for classical logic. We show that a certain icdl $M_0$, which plays a fundamental role in the semantics of $E$ (cf. A. R. Anderson and N. D. Belnap, Jr.'s *Tautological entailments*, Phil. St., vol. 13 (1962), pp. 9–24), plays a correspondingly fundamental role in the algebra of icdl's. Its special role is thus analogous to that of the two element Boolean algebra.

The principal results along these lines are that every icdl has a homomorphism into $M_0$, and that every icdl is embeddable into a product algebra of $M_0$. The analogy between $M_0$ and the two element Boolean algebra is reinforced by the fact that given plausible definitions of 'simplicity' and 'normality' for the Boolean and intensional cases, it emerges that just as the two element Boolean algebra is uniquely both simple and normal among Boolean algebras, so also is $M_0$ among icdl's.

Similar results are obtained for structures called *de Morgan lattices*, which are like icdl's except that the no-fixed-point requirement is dropped. In these results $M_0$ is replaced in its fundamental role by a smaller de Morgan lattice, which is not an icdl. The two sorts of structure are connected by observing that a de Morgan lattice is an icdl just in case it has a homomorphism into $M_0$.

All of these results generalize in a natural way (with appropriate restrictions) to complete analogues. (Received November 30, 1965.) [This abstract was presented at the annual meeting in New York, December 27–28, 1965.]