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INTENSIONALLY COMPLEMENTED DISTRIBUTIVE LATTICES*

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It is well known that plausible semantic completeness and consistency conditions for ordinary classical logic are possible in terms of boolean algebras ([11]), and that something similar is possible for both intuitionistic logic and for modal logics via topological structures ([13]). An analogous problem is the provision of a suitable semantics for the intensional system E of entailment ([1], [2], [3], [6], [7]). Although the general problem remains unsolved, it has nevertheless been possible to provide solutions for at least certain fragments of E ([4], [5], [8]). These partial solutions to the semantic problem implicitly utilize a certain type of lattice, as follows: an ordered quadruple $<A, \leq, N, T>$ (where A is a set, $\leq$ a relation on A, N a function on A, and T a subset of A) is said to be an *intensionally complemented distributive lattice with truth-filter (icdl w/t—f)* provided

(DL) A is a distributive lattice under $\leq$ (we use $\cap$ and $\cup$ for the lattice meet and join respectively, as in [10]);

(N_1) $NNa = a$, all $a \in A$;

(N_2) if $a \leq b$, then $Nb \leq Na$, all $a, b \in A$;

(T) T is a *truth-filter;* i. e., T satisfies these three conditions: (*i*) T is a *filter* ([15]): $(a \cap b) \in T$ if both $a \in T$ and $b \in T$, and $b \in T$ if both $a \leq b$ and $a \in T$; (*ii*) T is *consistent:* there is no a such that both $a \in T$ and $Na \in T$; and (*iii*) T is *exhaustive:* for every $a \in A$, either $a \in T$ or $Na \in T$.

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In the semantic applications to intensional logic, A is thought of as a set of propositions, $\leq$ as a relation of entailment, the lattice meet and join as propositional conjunction and disjunction respectively, and N as propositional negation. T is then thought of as representing the «true» propositions in the set A.

In these applications the truth-filter T is employed explicitly; however, certain of the semantic results can be seen to hold also where the notion of an *icdl w/t — f* is replaced by the more general notion of an *intensionally complemented distributive lattice,* i.e., a triple $<A, \leq, N>$ such that for some T, $<A, \leq, N, T>$ is an *icdl w/t — f*.

Separate study has also been given structures satisfying (DL), (N_1), and (N_2), which are *de Morgan Lattices* in the sense of [14]. (They have also been called *quasi-boolean algebras* in [9], and *distributive i-lattices* in [12].) Comparison of the two notions naturally leads to the request for a necessary and sufficient condition that a de Morgan lattice be an *icdl*. It then turns out that a quite simple condition will do, namely, that N have no fixed point:

(N_3) $\qquad a \neq Na$, all $a \in A$.

Two remarks are in order before turning to a proof of the equivalence of (N_3) and (T). (*i*) The equivalence gains interest from the extremely high intuitive plausibility of (N_3) and (T) as conditions on propositional negation. Indeed, it is hard to think of anything more absurd than supposing a proposition could be its own negation, and it is hardly less absurd to suppose that it should be impossible to exactly divide the propositions into the False and the True, a proposition being true iff its negation is false, and a conjunction being true iff each conjunct is true. (*ii*) In [14] a de Morgan lattice, A, is said to be *essentially irregular* if there is no set D such that $<A, D>$ is a regular characteristic matrix for the classical propositional calculus (see references in [14]). It is a corollary of the chief result of this paper, together with Theorem 4 of [14], that a necessary and sufficient condition for essential irregularity is the presence of some fixed point for N.

Since the equivalence of (N_3) and (T) is obvious in one direction, it suffices to prove the following

THEOREM. *If* A *is a de Morgan Lattice such that* N *has no fixed point, then some filter* T *of* A *is both consistent and exhaustive.*

Were A a boolean algebra, the theorem would amount to the existence of maximal proper filters; but where the complementation is intensional, not every proper filter is consistent nor is every exhaustive filter maximal.

Turning now to a proof of the theorem, we first remark that all the various forms of de MORGAN's law hold, a fact to which we refer below by means of «(deM)». Now let T_0 be the set of all elements q of A such that $q = (q \cup Nq)$, and let E be the set of all consistent filters in A containing T_0. We first show E non-empty by showing consistent the filter $F(T_0)$ generated by T_0. For suppose otherwise; then for some $p_1, \cdots p_n \in T_0$ and some $b \in A$, we would have, for $p = p_1 \cap \cdots \cap p_n$, both $p \leq b$ (hence $Nb \leq Np$) and $p \leq Nb$, so that

$$(1) \qquad Np = p \cup Np.$$

Now define a_n inductively as follows:

$$(2) \qquad a_1 = p_1; \quad a_{i+1} = p_{i+1} \cap (N p_{i+1} \cup a_i).$$

We first prove inductively that

$$(3) \qquad a_i = (p_1 \cap \cdots \cap p_i) \cup N a_i, \quad \text{all } i \quad (1 \leq i \leq n).$$

$a_1 = p_1 \cup N a_1$ is immediate, and we now assume for induction that $a_i = (p_1 \cap \cdots \cap p_i) \cup N a_i$. Since $p_{i+1} \in T_0$, we have $p_{i+1} = (p_{i+1} \cup N p_{i+1})$, hence by (DL), $p_{i+1} \cap (N p_{i+1} \cup a_i) = (p_{i+1} \cap a_i) \cup N p_{i+1}$; so that by (2),

$$(4) \qquad a_{i+1} = (p_{i+1} \cap a_i) \cup N p_{i+1}.$$

By the hypothesis of the induction, (4) yields

$$\begin{aligned}
(5) \qquad a_{i+1} &= (p_{i+1} \cap ((p_1 \cap \cdots \cap p_i) \cup N a_i)) \cup N p_{i+1} \\
&= (p_1 \cap \cdots \cap p_{i+1}) \cup (N p_{i+1} \cup (p_{i+1} \cap N a_i)) && \text{by (DL)} \\
&= (p_1 \cap \cdots \cap p_{i+1}) \cup N (p_{i+1} \cap (N p_{i+1} \cup a_i)) && \text{by (deM)} \\
&= (p_1 \cap \cdots \cap p_{i+1}) \cup N a_{i+1} && \text{by (2)}
\end{aligned}$$

Hence (3) holds, which can be used to show that $a_n = N a_n$ as follows:

(6)
$$\begin{aligned} a_n &= p \cup \mathrm{N}\, a_n && \text{by (3)}\\ &= p \cup \mathrm{N}(p \cup \mathrm{N}\, a_n) && \text{by (3)}\\ &= p \cup (\mathrm{N}\, p \cap a_n) && \text{by (deM)}\\ &= (p \cup \mathrm{N}\, p) \cap (p \cup a_n) && \text{by (DL)}\\ &= \mathrm{N}\, p \cap (p \cup a_n) && \text{by (1)}\\ &= \mathrm{N}\, p \cap (p \cup (p \cup \mathrm{N}\, a_n)) && \text{by (3)}\\ &= \mathrm{N}\, p \cap (p \cup \mathrm{N}\, a_n) && \text{by (DL)}\\ &= \mathrm{N}\, p \cap a_n && \text{by (3)}\\ &= \mathrm{N}(p \cup \mathrm{N}\, a_n) && \text{by (deM)}\\ &= \mathrm{N}\, a_n && \text{by (3)} \end{aligned}$$

But this contradicts (N_3), so that $\mathrm{F}(\mathrm{T}_0)$ must after all be consistent, and the set E of all consistent filters in A containing T_0 must be non-empty. Since it is easy to see that if a subset G of E is simply ordered by $\subseteq$, then $\bigcup \mathrm{G} \in \mathrm{E}$, it follows by a form of ZORN's Lemma that E has a maximal element T. T is obviously consistent; we proceed to show that it is exhaustive as well, which will complete the proof of the theorem.

We observe to begin with that

(7) $$(a \cup \mathrm{N}\, a) \in \mathrm{T}, \quad \text{all} \quad a \in \mathrm{A},$$

since $(a \cup \mathrm{N}\, a) = ((a \cup \mathrm{N}\, a) \cup \mathrm{N}(a \cup \mathrm{N}\, a))$ by (deM) and (DL); so that $(a \cup \mathrm{N}\, a) \in \mathrm{T}_0$. Furthermore, since T is a filter, it follows from (7) that $((c \cup \mathrm{N}\, c) \cup (d \cap \mathrm{N}\, d)) \in \mathrm{T}$ for all $c, d \in A$; hence, since T is consistent,

(8) $$\mathrm{N}((c \cup \mathrm{N}\, c) \cap (d \cup \mathrm{N} d)) \notin \mathrm{T}, \quad \text{all } c, d \in \mathrm{A}.$$

Now suppose for *reductio* that T is not exhaustive, so that for some b, neither $b \in \mathrm{T}$ nor $\mathrm{N}\, b \in \mathrm{T}$. Then the filters $\mathrm{F}(\mathrm{T}, b)$ and $\mathrm{F}(\mathrm{T}, \mathrm{N}\, b)$ generated by T with b and T with $\mathrm{N}\, b$ respectively each have T as a proper subset; and so since T is maximal in E, neither $\mathrm{F}(\mathrm{T}, b)$ nor $\mathrm{F}(\mathrm{T}, \mathrm{N}\, b)$ can be consistent. Consequently, for some c, both c and $\mathrm{N}\, c$ are in $\mathrm{F}(\mathrm{T}, b)$, and for some d, both d and $\mathrm{N}\, d$ are in $\mathrm{F}(\mathrm{T}, \mathrm{N}\, b)$; so since $\mathrm{F}(\mathrm{T}, b)$ is the filter generated by T with b and since $\mathrm{F}(\mathrm{T}, \mathrm{N}\, b)$ is the filter generated by T with $\mathrm{N}\, b$, it must be that for some $t_1, t_2, t_3, t_4 \in \mathrm{T}$,

(9) $$t_1 \cap b \leq c, \quad t_2 \cap b \leq \mathrm{N}\, c, \quad t_3 \cap \mathrm{N}\, b \leq d, \quad \text{and} \quad t_4 \cap \mathrm{N}\, b \leq \mathrm{N}\, d,$$

which leads by (D L) to

(10) $t_1 \cap t_2 \cap t_3 \cap t_4 \cap (b \cup \mathrm{N}\, b) \leqq (\mathrm{N}\, c \cap c) \cup (\mathrm{N}\, d \cap d)$.

Since $t_1, t_2, t_3, t_4 \in \mathrm{T}$, since also $(b \cup \mathrm{N}\, b) \in \mathrm{T}$ by (7), the fact that T is a filter implies that $(t_1 \cap t_2 \cap t_3 \cap t_4 \cap (b \cup \mathrm{N}\, b)) \in \mathrm{T}$, and accordingly, with (10), that $((\mathrm{N}\, c \cap c) \cup (\mathrm{N}\, d \cup d)) \in \mathrm{T}$. So by (deM)

(11) $\mathrm{N}((c \cup \mathrm{N}\, c) \cap (d \cup \mathrm{N}\, d) \in \mathrm{T}$.

But (11) contradicts (8), rendering absurd the supposition that T fails to be exhaustive.

This completes the proof of the theorem. We remark that it cannot be generalized to *complete* distributive lattices (where meets and joins of arbitrary subsets always exist) satisfying (N_1) — (N_3) and *complete* filters (filters closed under the meet of arbitrary subsets). The following is an example of a complete distributive lattice satisfying (N_1) — (N_3) which contains no complete truth-filter: let $< \mathrm{A}, \leqq, \mathrm{N} >$ be such that (i) the elements of A are $(1, \infty)$ and $(-1, -\infty)$ together with all ordered pairs (x, y) where $x = \pm 1$ and y is an integer; (ii) $(x_1, y_1) \leqq (x_2, y_2)$ iff $x_1 \leqq x_2$ and $y_1 \leqq y_2$; and (iii) $\mathrm{N}(x, y) = (-x, -y)$. It is easily shown that $< \mathrm{A}, \leqq, \mathrm{N} >$ is a complete distributive lattice satisfying (N_1) — (N_3). Suppose now there is a complete, consistent, and exhaustive filter T in A. Then since $(-1, 0 \leqq \mathrm{N}(-1, 0)$, we shall have to have $(-1, 0) \notin \mathrm{T}$, on pain of inconsistency of T; hence $\mathrm{N}(-1, 0) \in \mathrm{T}$, since T is exhaustive. However, for all $x \geqq 0$, $(-1, x) \cap \mathrm{N}(-1, 0) = (-1, 0)$, so that the filterhood of T implies that $(-1, x) \notin \mathrm{T}$ for all such x. But T is exhaustive, so the set X of all pairs $\mathrm{N}(-1, x)$ $[= (1, -x)]$, for $x \geqq 0$, is a subset of T; and since T is a complete filter, the meet of X is in T. But $(-1, -\infty)$ is evidently the meet of X so that $(-1, -\infty) \notin \mathrm{T}$ — from which it follows immediately that T is inconsistent, contrary to assumption.

We may also observe that the theorem cannot be strengthened to read «every consistent filter is a subset of some exhaustive consistent filter»; a counter-example is provided by $< \mathrm{B}, \leqq, \mathrm{N} >$, where $\mathrm{B} = \{(-1, -\infty), (1, \infty)\ (-1, 0), (1, 0), (-1, 1), (1, -1)\}$ and where $\leqq$ and N are as in the previous example: the filter $\{(-1, 1), (1, \infty)\}$ is consistent, but not extendible to an exhaustive consistent filter.

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