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TWENTY-EIGHTH ANNUAL MEETING OF THE ASSOCIATION FOR SYMBOLIC LOGIC

The twenty-eighth annual meeting of the Association for Symbolic Logic was held in conjunction with the annual meeting of the Eastern Division of the American Philosophical Association at the Statler-Hilton Hotel, Washington, D.C., on Friday and Saturday, December 27–28, 1963.

Professors Alan Ross Anderson and Hugues Leblanc presided over the Friday morning session for contributed papers, Professor Leon Henkin over the Friday afternoon session which included Martin Davis' invited address, *First Order, Second-Order, and Higher Order Logic*, followed by contributed papers. The Council of the Association met at lunch.

The Saturday morning joint session with APA featured a symposium on Frege with Rulon Wells, commentary by James Bartlett, and with James F. Thomson in the chair. Professor S. C. Kleene presided over the Saturday afternoon session for contributed papers. The first seventeen papers below were presented in person, the remaining four by title.

CAROL R. KARP

N. D. BELNAP, JR., H. LEBLANC and R. THOMASON. *On not strengthening intuitionistic logic.*

Limiting ourselves to sequents of the kind $A_1, A_2, \dots, A_n \rightarrow B$, where A_1, A_2, \dots, A_n ($n \geq 0$), and B are wffs of the propositional calculus, we reexamine — in the wake of R. E. Vesley's "On strengthening intuitionistic logic" (*Notre Dame Journal of Formal Logic*, vol. IV, no. 1) — the problem of converting so-called structural and intelim rules for PC_I , Gentzen's intuitionistic calculus of sequents, into rules for PC_C , Gentzen's classical calculus of sequents. It has been known since 1935, 1960, and 1962, respectively, that one can pass from PC_I to PC_C by strengthening the intelim rules for ' \sim ', or those for ' \supset ', or those for ' \equiv '. Whether, however, one can do so by strengthening the intelim rules for ' \vee ' or those for '&' has so far remained an open question. Sharpening here the concept of an intelim rule for ' \vee ' or '&', we establish that: (1) any intelim rule for ' \vee ' or '&' which is provable in PC_C is also provable in PC_I , and (2) any intelim rule for ' \vee ' or '&' which is closed under substitution and admissible in PC_C is also admissible in PC_I . The answer to the above question is thus negative. (Received Oct. 10, 1963.)

WILLIAM T. PARRY. *Categorical forms in many-sorted logic with identity.*

Smiley (JSL XXVII 58) applies many-sorted logic to the traditional syllogism. He briefly indicates how a system MSI, adding identity to basic many-sorted logic, expresses 8 doubly quantified forms, corresponding to "quantification of the predicate." These forms merit further consideration.

Since the quantifiers have non-empty ranges, four forms express Aristotelian categoricals: $\Pi a \Sigma b(a = b)$ is **A**, Every a is (some) b ; $\Pi a \Pi b(a \neq b)$, **E**; $\Sigma a \Sigma b(a = b)$, **I**; $\Sigma a \Pi b(a \neq b)$, **O**. *Distribution* now has a simple definition: distributed terms are those having the universal quantifier. The scholastic doctrine of distribution as logical descent follows. Non-scholastic "explanations" seldom explain.

The other four forms express non-standard categoricals discussed since Aristotle, such as "Every {No, Some, Not every} a is every b ," which we assign Greek vowels. $\Pi a \Pi b(a = b)$ is *alpha*; $\Pi a \Sigma b(a \neq b)$, *eta*; $\Sigma a \Pi b(a = b)$, *iota*; $\Sigma a \Sigma b(a \neq b)$, *omega*. They mean: Every {No, Some, Not-every} a is the only b . The *eight* forms directly express De Morgan's "exemplar" forms, 'Any (some) one a is (is not) any (some) one b .' Ockham was apparently first to understand that alpha is doubly singular.

He and Pseudo-Scotus understood that *iota's* predicate is singular. But De Morgan misinterpreted *snp* as *pAs* (its subaltern), and *snp* as *pOs* (its superaltern). A sign for unit classes permits Venri-type diagrams.

Other points: (1) axb iff $(aib.bia)$. (2) With *iota* and *eta*, rules of distribution and quantity change. (3) Reichenbach's rule for use of 'any' is amended. (4) Interpretation of alpha divides "traditions" on quantification of the predicate into three: Aristotelian, alpha always false; Ockham-De Morgan, alpha doubly singular; Hamiltonian (collective), terms co-extensive. MSI gives a symbolic version of the second. (Received July 28, 1963.)

DON D. ROBERTS. *Peirce's existential graphs and natural deduction.*

Late in 1896 Charles S. Peirce invented a system of logic diagrams which he called the "existential graphs" (EG). He meant for this system to be a deductive treatment of both non-relative and relative logic. The first part of the system, part Alpha, treats of the logic of propositions; the second part, Beta, treats of the logic of quantification.

Peirce's usual method of providing his system with rules of inference but no axioms is suggestive of the method characteristic of natural deduction systems. And the purpose of this paper is to make the following three points:

1. The idea basic to natural deduction systems — that of using the rule of conditionalization as a primitive rule of inference — did in fact occur to Peirce. In 1900 he stated this as a primitive rule for an algebraic system of notation which he modeled after EG. Although Peirce gives no explicit development of this idea, he nevertheless anticipates Gentzen and Jaśkowski by 34 years.

2. The Alpha part of EG, when augmented by the rule of conditionalization as a primitive rule of inference, is adequate to the entire logic of propositions. This is proved by establishing, as theorems in EG, the graphical equivalents of the axioms and rule of inference of an algebraic formulation of the propositional calculus which is known to be adequate. The strategy for a consistency proof for Alpha is outlined.

3. The Beta part of EG can be made adequate to the logic of quantification by adding to the Alpha rules (the propositional basis) appropriate formulations of the rules of universal instantiation, universal generalization, existential instantiation, and existential generalization.

The ease of operation characteristic of this system, and illustrated in the proofs for (2), suggests that certain heuristic values are associated with EG. (Received Oct. 14, 1963.)

GEORGE GOE. *First-order logic without existential import.*

The provability in first-order logical calculi of wffs not valid in the empty domain has been often taken exception to. In fact, assertions of the existence of objects cannot be regarded as logical truths. Though a non-empty domain is presupposed in all axiomatic systems outside pure logic, it is clearly desirable for the latter sharply to separate from all others, as theorems of its own, those wffs which are valid in every domain, while those valid in every non-empty domain should be derivable within the pure logic system from a special assumption formula.

The presence of the controversial theorems is in most traditional first-order calculi related to the use therein of free variables in the wffs that occur as lines in derivations, which has also been independently criticized as a duplicate way of expressing universality; but some systems which dispense with that use of free variables none the less are complete with respect to validity in every non-empty domain. Insights are gained in syntactically and semantically tracing the origin of the controversial theorems in both types of systems (note that such wffs as ' $(x)Fx \supset Fy$ ' or ' $Fx \supset (\exists y)Fy$ ' are valid in the empty domain if they are regarded as semantically equivalent to their closures).

A system of first-order logic wherein only closed wffs occur as lines in derivations, and wherein all and only wffs valid in every domain are theorems, can be formulated with five axiom schemata, and *modus ponens* as sole rule of derivation. Every wff valid in every non-empty domain can be derived in this system from $(\exists x)(Fx \vee \sim Fx)$, or from any prenex form opening with an existential quantifier, taken as an assumption formula. Thus the addition of a single axiom (not an infinity, as by a schema) transforms the system into one equivalent to traditional first-order logical systems. (Received Oct. 15, 1963.)

ALBERT A. MULLIN. *Studies in arithmetic.*

Circa 300 B.C. Euclid of Alexandria (*Elements*, Book IX, Proposition 14) stated the spirit, and if one interprets *thirdness* properly, the essence, of the Fundamental Theorem of Arithmetic (FTA), which is so useful for, e.g., (i) Gödel numbering wffs of axiomatics (or if the reader wishes, for approaching Leibniz' *Characteristica Universalis*), (ii) serving as the paradigm *par excellence* for no small part of mathematics (e.g., *algebraic* number theory à la Kummer-Kronecker-Hilbert); analytic number theory; combinatorial topology, especially graph theory, and less directly through the Fundamental Theorem of finitely generated abelian groups), and (iii) more applied matters of value, such as circuit *analysis*.

This monograph, whose syllabus is available from the author, introduces a new formulation of FTA, which for some good reason no doubt, has escaped the notice of the community, and then proceeds to classify all *effective* (multiplicative) models for FTA. The models for FTA are collected into three forms (classes of models), viz., (i) the *primordial* Euclidean form (the product of the first power of primes alone), (ii) the *mixed* form (in which a model is determined, in general, by both primes and composites as with Gauss' model, *circa* A.D. 1800), and (iii) the *pure* non-Euclidean form (in which a model is determined by primes alone).

The point of departure for the new method is to use an induction with the Gaussian model for FTA upon *itself*, i.e., upon the exponents of the Gaussian model for FTA. To this extent the method is analogous to Euclid's Algorithm for the GCD of two natural numbers in that it applies a given rule to a *portion of the object* obtained from a prior application of the given rule until the process terminates in a *finite number* of steps.

In this mathematico-logical investigation it is shown that modest portions of mathematics (notably algebra, analysis, number theory, geometry, and topology) and mathematical logic (notably recursive function theory, the theory of mathematical induction, and intuitionistic analysis) can be related by a common method, among diverse methods. (Received June 23, 1963.)

BRUCE LERCHER. *Strong reduction and normal form in combinatory logic.*

Weak reduction, $U \geq V$, is generated by axiom schemes for the *atomic combinators* I, K, S: $IX \geq X$; $KXY \geq X$; $SXYZ \geq XZ(YZ)$. The left-hand sides are *weak redexes*. For *strong reduction*, $U \succ V$, change the " \geq " to " \succ " and add the rule: If $X \succ Y$, then $[x]X \succ [x]Y$. The abstraction $[x]X$ is defined by algorithm (abcf) of Curry and Feys **Combinatory Logic** (where strong reduction was first defined). X is in *normal form* iff (i) X is an indeterminate; (ii) $X \equiv xX_1 \dots X_n$, where X_1, \dots, X_n are in normal form; (iii) $X \equiv [x]Y$, where Y is in normal form. (This definition is due to L. E. Sanchis in his Penn State Ph.D. thesis.)

Theorem 1. If X is strongly irreducible, then it is in normal form.

This is the converse of the main theorem about strong reduction in **Combinatory Logic**. Examples of preliminary results follow. In each statement, x, x_1, \dots, x_k are indeterminates not occurring in X.

Lemma 1. Let $X \equiv [x]Y$. Then X is weakly irreducible iff every component of Y which is a weak redex contains an occurrence of x.

Lemma 2. If X is weakly irreducible, there is at most one weakly irreducible Y such that $X \equiv [x]Y$.

Theorem 2. If X is strongly irreducible, there is a weakly irreducible Y such that $X \equiv [x_1, \dots, x_k]Y$.

The proof here is by induction on the number of occurrences of atomic combinators in X . The induction step is handled by a secondary induction on k , where in that induction step, we assume $X \equiv [x_1, \dots, x_{k-1}]Z$ and consider cases for the possible forms of Z . All are trivial except $Z \equiv SUV$. We show there are Y_1 and Y_2 which are weakly irreducible such that $U \equiv [x_k]Y_1$, $V \equiv [x_k]Y_2$, and $Z \equiv [x_k]Y_1Y_2$. The assumption that Y_1Y_2 is weakly reducible then is shown to lead to contradiction.

Theorem 3. If X is weakly irreducible and there is a $k \geq 0$ and indeterminates y_1, \dots, y_k such that $[y_1, \dots, y_k]X$ is strongly irreducible, then X is in normal form. (Received Oct. 14, 1963.)

JOHN CORCORAN. *Generative structure of logics.*

In his *Two-valued Iterative Systems of Mathematical Logic* Post gives characterizations of all two-valued logics in such a way as to answer the question: given a set S of truth functions what is the logic which results using S as a set of "primitive functions"? We continue his work. We characterize all sets of truth functions in such a way as to answer the question: given a truth-function logic L what is the class of all sets S such that S is a primitive set for L ? In order to answer this question, we discover which two-valued iterative logics can be based on a single function. We call equivalence classes of such functions Sheffer sets.

In order to get a more detailed knowledge of the interrelations among two-valued iterative logics and Sheffer sets, we define a measure which assigns to certain sets of truth-functions numbers which can be thought of as the fractions of the set of all truth functions included in those sets. We show that all two-valued logics and all Sheffer sets are measurable, and we compute their measures. We find that a Sheffer set has non-zero measure iff the corresponding two-valued logic does as well. Moreover, we find that the Sheffer set of the complete logic has measure one-fourth — intuitively, that any one of a quarter of all truth functions can be used as a primitive on which to base a functionally complete logic.

Finally, we study the structure relations between two-valued iterative logics and can conclude that the only non-trivial isomorphism between two-valued iterative logics is duality — the function that maps each truth-function into its dual. (Received Oct. 4, 1963.)

LEON HENKIN. *A class of non-normal models for classical sentential logic.*

A system $\mathcal{L} = \langle L, +, \cdot, - \rangle$ is an *A-lattice* if $\mathcal{L}_1 = \langle L, +, \cdot \rangle$ is a distributive lattice and $-$ is an anti-automorphism of \mathcal{L}_1 (i.e., a one-one mapping of L onto itself such that $-(x + y) = -x \cdot -y$ for all $x, y \in L$). Let T be the filter of \mathcal{L}_1 generated by all elements $x + -^q x$, where $q = 1, 3, 5, \dots$ and $x \in L$. *Theorem:* If $T \neq L$ then $\langle \mathcal{L}, T \rangle$ is a characteristic model for classical sentential logic with primitives \vee, \wedge, \neg . (I.e., among sentential formulas the classical tautologies, and only these, assume values lying entirely in T for arbitrary assignments of L -elements to variables.) This generalizes the corresponding theorem for De Morgan lattices (i.e., *A-lattices* in which $-x = x$ for all $x \in L$), due to Monteiro (Academia Brasileira de Ciências, 1960, pp. 1-7). A representation theorem for *A-lattices* is obtained, generalizing the corresponding result for De Morgan lattices due to Białynicki-Birula and Rasiowa (Bulletin de l'Académie Polonaise des Sciences, 1957, pp. 259-261). Finite *A-lattices* are constructed which are not homomorphic to any De Morgan lattice, and are hence non-normal in the sense of Church (Boletín de la Sociedad Matemática Mexicana, 1953, pp. 41-52). (Received Oct. 25, 1963.)

WILLIAM CRAIG. *An implicit definition of satisfaction in n-th order theory.*

Let $2 \leq n < \omega$, $0 \leq p < \omega$, and let R_0, \dots, R_{p-1} be variables of order $\leq n$. For each type $\tau' = (\tau'_0, \dots, \tau'_{r-1})$ let $\tau'_0, \dots, \tau'_{r-1}$ be the immediate predecessors of τ' . Let $Q = \{\tau_0, \dots, \tau_{q-1}\}$ consist of the predecessors of the types of R_0, \dots, R_{p-1} , the n monadic types of order $< n$ and the two types corresponding to order 1 and rank 2 or 3 respectively. Let L consist of those formulas of [arithmetized] pure simple type theory in which only variables of order $\leq n$ ($< n$) occur free (bound), and L' (L'') of those ψ in L (in L') such that if a variable occurs free (bound) in ψ then either its type is in Q or it is one of R_0, \dots, R_{p-1} (its type is in Q). Elaboration of Vaught's argument in XXIII 306, 307 shows:

Lemma 1. There are ϑ_i in L' whose free variables are N, X, V, Z, u, W_i of appropriate type, $i < q$, which yield, for any h mapping an infinite A onto $A^{(\omega)}$ and any N, X, V, f satisfying (iv) and (v) below, a mapping h^* as follows: (i) the domain of h^* consists of the monadic $(n-1)$ -th order sets Z on A ; (ii) each $h^*(Z)$ is a sequence whose $(f(u) \cdot q + i)$ -th term is an object W_i of type τ_i on A , $u \in N$; and (iii) every such sequence having eventually period q is in the range of h^* . *Lemma 2.* There is a recursive g which for any ψ in L' yields a $g(\psi)$ in L'' having the same infinite models. *Theorem 1.* Some $\varphi \in L$ defines implicitly, for any infinite \mathfrak{A} of the appropriate kind, the satisfaction relation between sequences and formulas $\psi \in L'$. [Indeed, the definition is by induction on the length of $g(\psi)$ and hence "predicative".] More precisely, we construct a $\varphi \in L$ such that: (1) The only variables free in φ are R_0, \dots, R_{p-1} and S ; and (2) $\langle A, R_0, \dots, R_{p-1}, S \rangle$ is a [standard] model of φ if and only if S consists of exactly those $\langle N, X, V, Z, y \rangle$ of the appropriate kind such that for some h of A onto $A^{(\omega)}$ and some f : (iv) f maps $\langle N, X \rangle$ isomorphically onto $\langle \omega, Sc \rangle$ where Sc is the successor relation; (v) V is a function such that $V(z, u) = (h(z))(f(u))$ for each $z \in A$ and $u \in N$; (vi) $f(y) \in L'$; and (vii) $h^*(Z)$ satisfies $g(f(y))$ in $\langle A, R_0, \dots, R_{p-1} \rangle$. *Theorem 2.* Consider any infinite $\mathfrak{A} = \langle A, R_0, \dots, R_{p-1} \rangle$ of the appropriate kind and the unique S such that φ is true for (\mathfrak{A}, S) ; then for each $\chi \in L'$ which contains free only R_0, \dots, R_{p-1} and variables N, X, V, Z, y appropriate as arguments for S , $\forall N \dots \forall y [S(N, \dots, y) \equiv \chi]$ is false for (\mathfrak{A}, S) . For proof, let T be the theory whose set W of valid sentences consists of those ψ in L which are true in \mathfrak{A} , and as on p. 47 of *Undecidable Theories* prove: *Lemma 3.* Let $\pi(R_0, \dots, R_{p-1}, N, X, V, u)$ define a set $P \subseteq \omega$ in T [relative to formulas $\Delta_n(N, X, u)$ and $\lambda(N, X, V)$] if and only if, for each $n \in \omega$, $\forall N \forall X \forall V [\lambda \rightarrow \forall u [\Delta_n \rightarrow \pi]]$ or $\forall N \forall X \forall V [\lambda \rightarrow \forall u [\Delta_n \rightarrow \neg \pi]]$ is valid in T depending on whether $n \in P$ or $n \notin P$; for functions modify the notion of UT of definability in T similarly; let D be a function from ω into ω such that, if $n = \forall N \forall X \forall V [\lambda \rightarrow \psi] \in L'$, then $D(n) = \forall N \forall X \forall V [\lambda \rightarrow \forall u [\Delta_n \rightarrow \psi]]$; assume that T is consistent and that $\exists N \exists X \exists V \lambda$ and all $\forall N \forall X \forall V [\lambda \rightarrow \exists u \Delta_n]$ are valid in T ; then D and W are not both definable in T . *Corollary.* Beth's (and hence Craig's) theorem for first order theories (languages) has no [literal] analogue for T (L' or T). (Received Oct. 16, 1963.)

ROBERT MCNAUGHTON. *A solvability algorithm for Büchi's sequential calculus.*

Using the theory of finite automata, J. R. Büchi has settled the basic questions about the sequential calculus (SC), which is a semantically interpreted system of second-order logic (without axioms and rules of inference) having quantifiable variables ranging over natural numbers and over monadic predicates of natural numbers. ("On a decision method in restricted second-order arithmetic," *Logic, methodology and philosophy of science: proceedings of the 1960 international congress*, Stanford University Press, 1962.) That is to say, he has shown the precise limits of definability in SC, and has presented a decision procedure for it.

One of the open questions that are listed at the end of Büchi's paper concerns the possibility of a solvability algorithm for SC: namely, is there an effective manner

of determining whether or not there exists a finite automaton satisfying a given condition expressed in SC? The present paper answers this question in the affirmative, using Büchi's theorem 1 (the main theorem of his paper, from which the above results follow), to the effect that every formula of SC without free individual variables is effectively equivalent to a formula having a certain form, and thus saying something rather complicated about the infinite history of a finite automaton. My algorithm uses the concept of a game in which two players play in turn infinitely often, and where the winning and losing depends on whether or not a certain event has occurred infinitely often. If there is a finite automaton satisfying a condition expressed in SC, then the algorithm continues beyond the affirmative answer to the synthesis of it. If there is none, the algorithm produces, in addition to the negative answer, a further algorithm that can generate, for a given finite automaton, an infinite input history that falsifies the original condition. (Received Oct. 1, 1963.)

J. A. ROBINSON. *A machine-oriented first-order logic.*

A formulation of first-order logic was made as the basis for an efficient proof procedure to be realized in a program for a digital computer. The formalism is quantifier-free, and contains symbols for no truth-functional connectives except negation. *Atomic formulae* are formed from predicate symbols, function symbols, individual variables, and individual constants in the usual way; these, and negations of these, are *literals*. A finite set of literals (including the empty set) is a *clause*; the semantics of the system construes clauses as disjunctions. Finite sets of clauses (including the empty set) are construed as conjunctions of their elements. The empty clause is always false, the empty conjunction always true. Individual variables are construed as universally quantified with scope the entire clause in which they occur. Existential quantification is achieved via function symbols.

Only one principle of inference, the Resolution Principle, is used; there are no axioms or axiom schemata. The Resolution Principle is a two-premise rule, with clauses as the premises. The system is proved to be complete, and it is shown that the problem of *instantiation*, hitherto a barrier to efficient proof-procedures, is completely solved; no choices are "creatively" made of terms with which to instantiate free variables — the Resolution Principle makes the choice automatically.

The system is compared, via examples, with earlier systems, and we show that in our system proofs are shorter and are discovered more quickly.

The Resolution Principle, as a price of its power, is complex and machine-oriented. The human cannot in general determine in one gestalt that the principle is applicable to a pair of given clauses, nor what the result of applying it would be. This must be determined by the application of certain algorithms to the two clauses as inputs. The Resolution Principle is in fact defined by these algorithms. (Received Oct. 14, 1963.)

WILLIAM BRUCE PITT. *Mathematical methods for family resemblances.*

Many problems of philosophy arise in the context of attempts to maintain that some particular linguistic expressions are univocal. Solutions which reject these expressions as not proper or which reject the original intuition of univocity, and which go no further, fail to take into account certain unusual relationships between the instances of use of the expressions. For example it may be difficult to draw less than arbitrary distinctions separating pairs of usages which, if identified, lead to contradiction. The claim is made that more may be said about linguistic expressions which have various incompatible usages than is said in merely noting these usages. The properties of these family resemblances are made precise in a number of ways by means of the imposition of various mathematical structures. The geometric-like invariants assigned to representations of linguistic phenomena provide potentially useful tools for classification and analysis. These structures are applicable to a number of variants of well-known formal languages. In von Neumann-Robinson class

theory the addition of an axiom is suggested which, although it arises from philosophical considerations, resembles the crucial axiom of the theory. A "spiral" type theory (of function variables) is exhibited in which the collection of the various types is not disjoint or nested but has more complicated intersection and order properties. Some light is shed, belatedly, on the puzzles of modal logic by pointing out that the intuition of univocity about possibility may have caused them, by noticing that several features of Hintikka's solutions are amenable to this analysis, and by suggesting that a further generalization of Hintikka's results to systems of modal systems and so on should be undertaken. Some methods of approximation for the empirical determination of these invariants provide a possible but as yet impractical method for the analysis of rational discourse in ordinary language. (Received Oct. 14, 1963.)

R. SOLOVAY. 2^{\aleph_0} can be anything it ought to be.

The methods of Cohen are used to study the possible values of the function 2^{\aleph_α} in various models for Z - F set theory. (In all models considered, the axiom of choice is valid.) For example, the following theorem is proved:

If Z - F is consistent, it remains consistent if the axiom of choice and the assertions

$$2^{\aleph_0} = \aleph_2, 2^{\aleph_1} = \aleph_5 \text{ and } 2^{\aleph_2} = \aleph_{10}$$

are added as axioms. (Received Oct. 14, 1963.) (At the meeting this paper was introduced by A. Church.)

W. A. HOWARD. Bar induction, bar recursion and the Σ_1^1 axiom of choice.

Let Z_2 denote classical second order arithmetic with no comprehension axiom but with primitive recursion on type level 1 and certain trivial functionals of type level 2. Let H_2 be the corresponding system based on intuitionistic predicate logic. Let (BI) denote the schema of bar induction of type 0 and let (BR) denote the schema of bar recursion of type 0 [3]. For any formula A of Z_2 let A' be Gödel's translation of A into the system T of primitive recursive functionals of finite type given in [1].

Theorem 1. For any formula A, if A is provable in $H_2 + (BI)$ then A' is provable in $T + (BR)$.

Let (SP-BI) denote (BI) in which P(c) in the hypothesis $\bigwedge x \forall x P(\bar{a}(x))$ of (BI) is quantifier free. For any formula A, let A^- denote the "negative" version of A obtained by prefixing every disjunction and existential quantifier by double negation.

Theorem 2. If A is provable in $Z_2 + (SP-BI)$ then $(A^-)'$ is provable in $T + (BR)$.

Theorem 3. The Σ_1^1 axiom of choice (see [2] and [3]) is provable in $Z_2 + (SP-BI)$.

Theorem 4. The comparability of well-orderings is provable in $Z_2 + (SP-BI)$.

Theorems 2 and 3 provide a constructive justification, which is simpler than that provided by [3], of the use of the Σ_1^1 axiom of choice in Z_2 . Indeed, we have justified the stronger system $Z_2 + (SP-BI)$.

Finally, we show that (BI) of type 1 is reducible to (BI) of type 0 and that (BR) of type 1 is reducible to (BR) of type 0.

[1] K. Gödel, *Dialectica*, vol. 12 (1958), pp. 280-287.

[2] G. Kreisel, *K. N. Akad. van Wetenschappen*, vol. LXV (1962), pp. 136-146.

[3] C. Spector, *Proc. Symposia in Pure Mathematics*, vol. 5 (1962), pp. 1-27, American Math. Soc.

(Received Dec. 2, 1963.)

RAYMOND M. SMULLYAN. Superinductive classes and ordinal numbers.

We think of the ordinal numbers as just those sets which can be obtained, starting with the empty set 0, and by taking the successor x^+ (i.e. $x \cup \{x\}$) of any set x already obtained, and by taking the union $\cup y$ of any set y of sets already obtained. This idea is best formalized in a class-set theory such as VNB. Define a class A to be super-

inductive iff: (i) $0 \in A$; (ii) for every set x , $x \in A \Rightarrow x^+ \in A$; (iii) for every set y , $y \subseteq A \Rightarrow \cup y \in A$. Now we define $\text{On}(x)$ [read ‘ x is an ordinal number’] iff x belongs to every superinductive class. To verify that a given property P (defined by a normal formula) holds for every ordinal number, it obviously suffices to show that the class of all x ’s having the property is superinductive. By this method, one easily establishes all the usual properties of the ordinal numbers.

Since the formula $\text{On}(x)$ is itself not normal, we do not immediately have the class On of all ordinal numbers. [We would have this class at once, if we were working in impredicative class-set theory rather than VNB]. To establish in VNB the existence of On , one proves the equivalence of $\text{On}(x)$ with any of the usual definitions (by normal formulas), or one can use either Th. 1 or Th. 2 below.

The following modification is applicable to ZF as well as VNB. For any set A and any subset B of A , define B to be superinductive *relative to* A iff (i) $0 \in A \Rightarrow 0 \in B$; (ii) for every $x \in B$, $x^+ \in A \Rightarrow x^+ \in B$; (iii) for every subset x of B , $\cup x \in A \Rightarrow \cup x \in B$. Let A^* be the intersection of all subsets of A which are superinductive relative to A . *Theorem 1.* x is an ordinal number iff there exists a set A such that $x \in A^*$. A more elegant characterization is given by: *Theorem 2.* x is an ordinal number iff $x = x^*$. (Received Dec. 27, 1963.)

G. KREISEL and G. E. SACKS. *Metarecursive sets I.*

In preparation for abstract recursion theory we analyse classical recursion theory in terms of the following notions. A *universe* F (of formal objects) including ω ; a collection \mathfrak{F} of *sequences* of elements from F (computations), including the set \mathfrak{B} of finite sequences of some distinguished object in F , say 1; a collection \mathcal{R}^1 of *functions*: $\mathfrak{F} \rightarrow \mathfrak{F}$, and pairing functions $\mathfrak{F}^2 \leftrightarrow \mathfrak{F}$ which induce \mathcal{R}^n : $\mathfrak{F}^n \leftrightarrow \mathfrak{F}$. Derived notions are the collections \mathfrak{F}^* of (unordered) sets in \mathfrak{F} , \mathcal{R}_0 of *subsets* of \mathfrak{F} whose *characteristic functions* are in \mathcal{R}^1 ; $\mathcal{R}\mathcal{E}$ of *subsets* of \mathfrak{F} which are the *ranges* of functions in \mathcal{R}^1 ; \mathcal{B} of the field of subsets of \mathfrak{F} generated by $\mathcal{R}\mathcal{E}$ and included in some set $\in \mathcal{F}^*$, finally, for $X \subset F$, $Y \subset F$ we mean by: X is \mathcal{R} in Y that for some $f \in \mathcal{R}^3$ (a ranging over F , a over \mathfrak{F} , $\langle \rangle$ denoting the empty sequence):
 $a \in X \leftrightarrow \forall a \forall a_1 [a^* \subset Y \wedge a_1^* \subset \mathcal{C}Y \wedge f(a, a, a_1) = 1]$,
 $a \notin X \leftrightarrow \forall a \forall a_1 [a^* \subset Y \wedge a_1^* \subset \mathcal{C}Y \wedge f(a, a, a_1) = \langle 1, 1 \rangle]$, and
 $\bigwedge a \bigwedge a' \bigwedge a_1 \bigwedge a_1' \{ [f(a, a, a_1) \neq \langle \rangle \wedge a^* \subset a'^* \subset Y \wedge a_1^* \subset a_1'^* \subset \mathcal{C}Y] \rightarrow f(a, a', a_1') = f(a, a, a_1) \}$ where a^* denotes the (unordered) set consisting of the elements of a . In classical recursion theory F itself is ω , \mathfrak{F} are ordered (and thus well-ordered) finite sequences of natural numbers, \mathcal{R}^1 recursive functions, \mathcal{F}^* unordered finite sets, \mathcal{R}_0 recursive sets, $\mathcal{R}\mathcal{E}$ recursively enumerable, \mathcal{B} bounded (and thus $\mathcal{B} = \mathcal{F}^*$), X is \mathcal{R} in Y if X is recursive in Y . These objects may be construed as *model theoretic invariants* of the general models of certain large classes of formal systems of arithmetic or concatenation theory. The same invariants of ω -models for analogous systems yield for F the recursive ordinals. These may be considered either as transfinite sequences of 1 of length $< \omega_1$ (first non-recursive ordinal), or, more conventionally, as the initial well-ordered segment W of length ω_1 of some hyperarithmetic ordering of the natural numbers (cf. e.g. Gandy, *Bull. Acad. Pol. Sc.* 8 (1960) 571–575). Any two such W, W' are isomorphic by mappings which are $\Pi_1^1 \cap \Sigma_1^1$ on W and W' . \mathfrak{F} are the hyperarithmetic subsequences of W , which we call *metafinite*, \mathcal{R}^1 the functions which are $\Pi_1^1 \cap \Sigma_1^1$ on \mathfrak{F} with values in \mathfrak{F} which we call *metarecursive*, \mathcal{F}^* hyperarithmetic subsets of W i.e. absolutely $\Pi_1^1 \cap \Sigma_1^1$, \mathcal{R}_0 are sets that are $\Pi_1^1 \cap \Sigma_1^1$ on \mathfrak{F} , $\mathcal{R}\mathcal{E}$ are Π_1^1 subsets of \mathfrak{F} which we call *metarecursively enumerable*. $\mathcal{B} \supset \mathcal{F}^*$. If X is metarecursive in Y then X is hyperarithmetic in Y , but, in general, not conversely; e.g. the sets of natural numbers which are metarecursive in O can be enumerated by a function hyperarithmetic in O (and not metarecursive in O).

$\mathcal{R} \cap \mathcal{R}\mathcal{E}$ consists of exactly those $\mathcal{R}\mathcal{E}$ sets whose members contain only ordinals $<$ some $\alpha < \omega_1$. (Received Nov. 11, 1963.)

G. KREISEL and G. E. SACKS. *Metarecursive Sets II.*

For notation see preceding abstract. For $\mathcal{M} \subset \mathfrak{P}(X)$, $X \subset F$, we call M_0 *maximal simple* for $\langle \mathcal{M}, X \rangle$ if (i) $X - M_0$ is infinite, (ii) $(\bigwedge M \in \mathcal{M}) [M \supset M_0 \rightarrow (M - M_0 \text{ is finite or } X - M \text{ is finite})]$; *metamaximal simple* if (i)' $X - M_0$ is unbounded, (ii)' $(\bigwedge M \in \mathcal{M}) [M \supset M_0 \rightarrow (M - M_0 \text{ is bounded or } X - M \text{ is bounded})]$; and *strongly (meta)maximal simple* if (i)' is unchanged, but 'bounded' is replaced by '(meta) finite'.

Theorem 1 (Generalisation of Friedberg XXV 165). *For* $X = \omega$, $\mathcal{M} = \mathcal{R}\mathcal{E}$ *there is a maximal simple set.* [The proof is uniform for both interpretations. Besides some elementary closure conditions on \mathcal{R}^1 such as $\exists \epsilon \in \mathcal{R}_0$, it uses essentially the facts: the collection $\mathcal{R}\mathcal{E}$ is metarecursively enumerable over ω , i.e. there is $f_\epsilon \in \mathcal{R}^3$ such that for each $X \in \mathcal{R}\mathcal{E}$, there is an $n \in \mathfrak{N} : X = \{\alpha : \forall b [f_\epsilon(n, \alpha, b) = 0]\}$; and a finite collection of (meta) finite sets is (meta) finite.] *Covollary to the Theorem.* For $X = F$, $\mathcal{M} = \mathcal{R}\mathcal{E}$ there is a maximal simple set. By a different proof, again uniformly, we get *Theorem 2. There is a strongly maximal simple set.* For the analogue of Post's problem one needs additional closure conditions on F , \mathfrak{F} and \mathcal{R} , in particular, that (a) a metafinite collection of metafinite sets is metafinite, (b) F is not (meta)finite, (c) F is well-ordered by a metarecursive relation, (d) \mathcal{R}^1 closed under definition by recursion on that well-ordering, and (e) there is a metarecursive isomorphism between F and \mathfrak{F} .

Theorem 3 (Generalisation of Friedberg's: XXIII, 225). *There are two (meta) recursively enumerable subsets of F which are (meta)recursively incomparable.* Analysis of the proof gives: *Theorem 4: There are two unbounded metarecursively enumerable subsets of F which are metarecursively incomparable.* A different proof yields such subsets of ω (instead of F) in the metarecursive case (e.g. by (b)).

By Spector [XXI, 412] any two bounded $\mathcal{R}\mathcal{E}$ subsets of F are hyperarithmetically comparable. For the sake of easy comparison with the literature we formulate here our results for subsets of F instead of subsets of \mathfrak{F} ; but in an abstract theory it is best to consider an alphabet F containing at least two letters, and for \mathfrak{F} (meta)finite sequences [cf. Smullyan, *Theory of formal systems*, Princeton 1961, esp. pp. 88-89, and also G. Higman, *Proc. Roy. Soc. A* 262 (1961), 455-475]. (Received Nov. 13, 1963.)

G. KREISEL. *The subformula property and reflection principles.*

Throughout we use arithmetisations which express the syntactic relations considered (i.e., as a minimal requirement satisfy the derivability conditions of HB II). Informally, an infinite proof tree for the rules and axioms \mathcal{R} is a well-founded tree of formulae at whose terminal nodes are axioms, and each F at a non-terminating node is an immediate consequence of its immediate descendants F_1, F_2, \dots by one of the given finitary or infinitary 'rules' (all in \mathcal{R}). Formally, if \mathcal{R} is applied to formulae in the notation S , and the formal system S_1 is an extension of S , let $<$ be a partial ordering for which transfinite induction can be proved in S_1 (since formal systems are considered, $<$ denotes of course a formula or its Gödel number, not a set theoretic object; the same convention applies below.) *Definition.* Let $O(p; a, b)$ be a formula for which it can be proved in S_1 that, with free variable p , (i) the field of $\{\langle a, b \rangle : O(p; a, b)\}$, say $<_p$, is a set of Gödel numbers of formulae which are related to their immediate predecessors by \mathcal{R} , (ii) $a(p)$ is the top element A_p of $<_p$, obtained recursively from p and (iii) φ_p is an order preserving mapping of $<_p$ into $<$. Then p is called an \mathcal{R} -proof of A_p relative to S_1 . *Definition.* \mathcal{R} has the subformula property if for each formula A there is a formula $A_1(n)$ with free variable n , such that, for some \bar{n} , $\vdash [A \leftrightarrow A_1(O(\bar{n}))]$ and for each rule in \mathcal{R} , where $\nu(n, \ulcorner F \urcorner) = \ulcorner F_n \urcorner$ (with free variables n and $\ulcorner F \urcorner$), $\vdash [\ulcorner \lnot F \leftrightarrow A_1(O(\bar{n})) \urcorner] \rightarrow \bigwedge n \forall p \vdash [\ulcorner F_n \leftrightarrow A_1(O(\bar{p})) \urcorner]$ where \vdash means provability in S ; thus only formulae enumerated by $A_1(n)$ can occur in an \mathcal{R} -proof of A

up to equivalence w.r.t.S. *Definition.* \mathcal{R} is S-valid for subformulae of A, if, with free variables $\ulcorner F \urcorner$ and m , for each rule in \mathcal{R} : $\vdash \wedge n \forall p \{A_1(p) \wedge [\ulcorner F_n \leftrightarrow A_1(0^{(p)}) \urcorner] \wedge \ulcorner F \leftrightarrow A_1(0^{(m)}) \urcorner\} \rightarrow A_1(m)$. *Theorem 1.* Suppose \mathcal{R} consists of a finite number of possibly infinitary rules having the subformula property, and suppose \mathcal{R} is S_1 -valid. Then, for each A containing the free variable m (but not p) the reflection principle $\vdash_1 \{[\text{Prov}(p) \wedge \ulcorner A_p \urcorner = \ulcorner A(0^{(m)}) \urcorner] \rightarrow A(m)\}$ holds. Proof by transfinite induction applied to $<_p$ and A_1 . The theorem can be extended to infinite sets \mathcal{R} , but the statement is too long for this abstract. *Corollary.* Under the conditions of the theorem, if $<_p$ is recursive and well-founded and the set of terminal elements is recursive, and if \mathcal{R} is S-valid, then A can be proved in S by adding transfinite induction on a recursive well ordering.

Schütte's infinitary cut free formulations for arithmetic and ramified analysis satisfy the subformula property. Takeuti's cut free system of analysis does not. *Theorem 2.* The Π_1^1 comprehension axiom of analysis, i.e. $\forall S \lambda x [x \in S \leftrightarrow A(x)]$ where A contains at most one set quantifier, is not provable on a recursive proof tree for rules \mathcal{R} which satisfy the subformula property and are valid for elementary analysis S. (Proof by use of the corollary and simple model theory.) — Theorem 2 constitutes a limitation of 'Gentzen's method' for the proof theory of analysis. Evidently every true Π_1^1 formula has a recursive proof tree satisfying the conditions of Theorem 2 by use of unsecured sequences and the Brouwer rule (of the next abstract). (Received Dec. 11, 1963.)

G. KREISEL. *Reflection principle for Heyting's arithmetic.*

HA denotes intuitionistic first order arithmetic extended by the addition of (free) function symbols and the corresponding extension of the classes of terms, formulae, axiom schemata. The well orderings mentioned below are *standard*, i.e. are uniquely determined up to isomorphism w.r.t. mappings in a class C by requiring that basic ordinal functions on them are in C, C being closed w.r.t. elementary operations; for ε_0 one needs the ordinal functions of addition, multiplication, exponentiation to base 2, and their inverses, cf. axioms A1–A12, pp. 202–203 of Schütte's *Beweis-theorie*. Let $HA_{\varepsilon_0}^n$ be obtained from HA by (i) adding the principle of transfinite induction on ε_0 applied to formulae with $\leq n$ logical constants, (ii) restricting proofs to those containing formulae with $\leq n$ logical constants. Note that by enumeration of formulae with n logical constants $HA_{\varepsilon_0}^n$ is equivalent to a finite set of axioms. $HA_{\varepsilon_0}^0$ is quantifier free. Informally let HA^+ be based on cut free intuitionistic predicate logic and two infinitary rules: $\lambda x A(x)$ is inferred from $A(0), A(0'), \dots$ (ω -rule) and $A(x)$ is inferred from $\alpha(0^{(p)}) = 0 \rightarrow A(x), \alpha(0^{(p)}) = 0' \rightarrow A(x), \dots$ (Brouwer's F rule which is equivalent to the ω -rule and the inference of $A(x)$ from $\lambda x [\alpha(0^{(p)}) = x \rightarrow A(x)]$.) Formally, in the notation of the preceding abstract, let S_1 be $HA_{\varepsilon_0}^n$, $<$ a standard ε_0 ordering, $<_p$ a formula defining, say, primitive recursive (p.r.) proof trees for HA^+ , with p.r. $a(p)$ and φ_p (where one simply builds into the definition of $<_p$ that the p be HA^+ -proofs relative to S_1). *Lemma 1.* (syntactic). For each A of HA , $\vdash_{\varepsilon_0}^0 \wedge n \{ \text{Prov}_{HA}(n, \ulcorner A \urcorner) \rightarrow (\text{Prov}^+[\pi(n)] \wedge \ulcorner A_{\pi(n)} \urcorner = \ulcorner A \urcorner) \}$ with a constant π of $HA_{\varepsilon_0}^0$. (Standard proof by cut elimination which is easily modified to give p.r. trees as in embedding recursive in p.r. well orderings.) *Lemma 2.* For A containing $\leq n$ logical constants, $\vdash_{\varepsilon_0}^{n+1} \{ [\text{Prov}^+(p) \wedge \ulcorner A_p \urcorner = \ulcorner A \urcorner] \rightarrow A \}$ (by use of a formula A_1 with $n+1$ logical constants which enumerates all subformulae of A). *Theorem.* For each A, $\vdash_{\varepsilon_0} [\text{Prov}_{HA}(p, \ulcorner A \urcorner) \rightarrow A]$, combining Lemma 1 and 2. [HA^+ is merely an auxiliary.] *Corollary 1.* ε_0 induction is not finitely axiomatisable over HA. *Corollary 2.* (Formal analogue to Brouwer's bar theorem.) If $\vdash_{HA} \forall x A(\alpha, x), \forall x A(\alpha, x)$ can be proved by cut free rules (Brouwer's 'Elementarschlüsse', footnote 8 *Math. Ann.* 97 (1927) 60–75) together with his basic rules F and ζ ('kanonisierte Form'). More specifically, there is a primitive recursive P_A for which $\vdash_{HA} \forall x P_A[\bar{\alpha}(x)]$ and

$\vdash_{\text{HA}} \{P_A[\bar{a}(x)] \rightarrow A(\alpha, x)\}$. Consequently, the set of unsecured sequences of P_A can be mapped recursively into the standard ε_0 -ordering, and the schema of transfinite induction w.r.t. these sequences can be proved in HA. *Corollary 3.* The results of Kleene's [JSL 27 (1962) 11–18] for *closed* formulae of HA extend to all formulae. (In fact, Lemma 1 affords a uniform method of extending metamathematical results for predicate logic to HA.) *Corollary 4.* If Prov_Z is the proof predicate for classical arithmetic Z , $\vdash'_{\varepsilon_0} [\text{Prov}_Z(p, \ulcorner A \urcorner) \rightarrow A]$ where \vdash'_{ε_0} denotes provability by ε_0 -induction and classical logic. (Use Gödel's translation of A into negative form A^- ; $\text{Prov}_Z(p, \ulcorner A \urcorner) \rightarrow \text{Prov}_{\text{HA}}[\ulcorner p \urcorner, \ulcorner A^- \urcorner]$, hence $\vdash^{\omega+1}_{\varepsilon_0} \rightarrow A^-$, $\vdash_Z (A \leftrightarrow A^-)$; in fact, generally, metamathematical results for HA go over immediately to the classical system, but not conversely: HA is the better system for *proof* theory of first order arithmetic). *Corollary 5.* The schema $[\text{Prov}_{\text{HA}}(p, \ulcorner A \urcorner) \rightarrow A]$ (for arbitrary formulae of HA including free variables) is equivalent over HA to the schema of transfinite induction on ε_0 . [By the *Theorem*, the schema follows. Conversely, by Gentzen, *Math. Ann.* 119 (1943) 140–161, for each B, there is a p.r. $\varrho(n)$, $\vdash_{\text{HA}} \text{Prov}[\varrho(n), \ulcorner A(0^{(n)}) \urcorner]$, where $A(n)$ expressed transfinite induction applied to B up to ω_n when $\omega_0 = \omega$, $\omega_{p+1} = \omega^{\omega^p}$; by the reflection principle, $\Lambda n A(n)$, i.e. ε_0 -induction applied to B.] Hence the limit of the autonomous progression based on HA and extended by the reflection principle is the first critical ε -number [cf. Feferman, *Notices A.M.S.* 8 (1961), 495]. (Received Dec. 11, 1963.)

G. KREISEL. *Reflection principles and ω -consistency.*

For uniform treatment of classical and intuitionistic systems the proper formulation of ω -consistency of a system (S) is this: with free variables n (over proofs) and $\ulcorner A \urcorner$ (over formulae): (*) $\text{Prov}[n, \ulcorner \neg \Lambda x A(x) \urcorner] \rightarrow \neg \Lambda x \forall p \text{Prov}[p, \ulcorner A(0^{(x)}) \urcorner]$. (Then the ω -consistency of HA yields immediately the ω -consistency of Z ; $\text{Prov}_{\text{HA}}[n, \ulcorner \forall x A(x, m) \urcorner] \rightarrow \forall x \forall p \text{Prov}_{\text{HA}}[p, \ulcorner A(0^{(x)}, 0^{(m)}) \urcorner]$ is weaker.)

Theorem 1. If the reflection principle for S can be proved in S_1 for the formulae $\neg \Lambda x A(x)$ and $A(p)$ with free variable p , then (*) can be proved in S_1 . Furthermore a proof of (*) is obtained primitive recursively from given proofs of the reflection principles. *Theorem 2.* Suppose (i) for a certain $\pi(\ulcorner A \urcorner)[\mu(\ulcorner A \urcorner), \sigma(\ulcorner A \urcorner, x)]$ being the canonical representations of $\ulcorner \neg \Lambda x A(x) \urcorner$, $\ulcorner A(0^{(x)}) \urcorner$ respectively as functions of x and $\ulcorner A \urcorner$, $\vdash_2 \text{Prov}_1[\pi(a), \ulcorner \Lambda n \{ \text{Prov}[n, \mu(0^{(a)})] \rightarrow \neg \Lambda x \forall p \text{Prov}[p, \sigma(0^{(a)}, x)] \} \urcorner]$ with free variable a , (ii) in S_2 the reflection principle for S_1 can be proved for the formula: $\Lambda n \{ \text{Prov}[n, \mu(a)] \rightarrow \neg \Lambda x \forall p \text{Prov}[p, \sigma(a, x)] \}$. Then the ω -consistency of S can be proved in S_2 . (Proof by Theorem 1.) *Corollary 1.* The ω -consistency of HA can be proved by ε_1 induction (first established by Parsons, *Archiv f. math. Logik u. Grundlagenforschung*, 6 (1962) 30–34). Take for S_1 the system $\cup \text{HA}_{\varepsilon_0}^n$, and for S_2 the system $\text{HA}_{\varepsilon_1}^3$, proving the reflection principle for S_1 as in the preceding abstract for HA. Since the instructions for proving the reflection principle are quite explicit, condition (i) of Theorem 1 is satisfied not only in S_2 , but in primitive recursive arithmetic. — A proof that the corollary is optimal is given in the review of Parson's paper in *Math. Revs.* 25 (1963) 746–747; it uses $\vdash_{\text{HA}} [(\omega\text{-Con HA}) \rightarrow (\omega\text{-Con HA}_{\varepsilon_0}^n)]$ for each n , which depends on the general result of Shoenfield:

$$\vdash_{\text{HA}} [\Lambda x \forall y \text{Prov}_S(y, \ulcorner A(0^{(x)}) \urcorner) \wedge \omega\text{-Con S}] \rightarrow \omega\text{-Con}(S \cup \{ \Lambda x A(x) \}).$$

Corollary 2. For quantifier-free $A(x)$, $\vdash'_{\varepsilon_0} \text{Prov}_{\text{HA}}[p, \ulcorner \neg \Lambda x A(x, m) \urcorner] \rightarrow \neg A[\pi(p, m), m]$, where $\text{HA}_{\varepsilon_0}'$ is $\text{HA}_{\varepsilon_0}^0$ extended by definition of functions by ε_0 -recursion. First, by the reflection principle, in $\text{HA}_{\varepsilon_0}' \rightarrow \neg \Lambda x \forall p \text{Prov}_{\text{HA}}[p, \ulcorner A(0^{(x)}) \urcorner]$. But by verifiability of quantifierfree $A(x)$, $\vdash_{\text{HA}} A(x) \rightarrow \forall p \text{Prov}_{\text{HA}}[p, \ulcorner A(0^{(x)}) \urcorner]$, and so $\vdash'_{\varepsilon_0} \text{Prov}_{\text{HA}}[p, \ulcorner \neg \Lambda x A(x, m) \urcorner] \rightarrow \neg \Lambda x A(x, m)$. In intuitionistic predicate logic, for quantifier-free $P(x)$, if $\vdash \neg \Lambda x P(x)$ then also $\vdash \forall x \neg P(x)$ (by use of cut free proofs). Extending this to infinite cut free proofs we have $\vdash'_{\varepsilon_0} \forall x \{ \text{Prov}_{\text{HA}}[p, \ulcorner \neg \Lambda x A(x, m) \urcorner] \rightarrow \neg A(x, m) \}$. *Corollary 3.* (Quantifier

fier-free formulation of ω -consistency.) If $\vdash_{\text{HA}} \neg \lambda x A(x)$, there is a functional $F(\alpha)$ defined by transfinite recursion on an ordering $< \varepsilon_1$, such that $\alpha[F(\alpha)]$ is not a proof in HA of $A[F(\alpha)]$. By Theorem 1, for some n , $\vdash_{\varepsilon_0}^n \neg \lambda x \forall p \text{Prov}[p, \ulcorner A(0^{(x)}) \urcorner]$, hence $\vdash_{\varepsilon_0}^n \neg \lambda x \text{Prov}[\alpha(x), \ulcorner A(0^{(x)}) \urcorner]$. Apply Corollary 2. — Among other relations one may note (i) ω -consistency (for HA and similar systems) is equivalent to the reflection principle for 2 quantifier prenex formulae and their negations, but not for more (and the same applies to Z, because the negation of a prenex formula is provable in Z if and only if it is provable in HA), (ii) if for such a formula A, $\vdash_{\varepsilon_0}^n A$, then $\vdash_{\text{HA}} (\omega\text{-Con HA}) \rightarrow A$ because, by Shoenfield's theorem $\vdash_{\text{HA}} [(\omega\text{-Con HA}) \rightarrow (\omega\text{-Con HA}_{\varepsilon_0}^n)]$ and now apply the reflection principle of $\text{HA}_{\varepsilon_0}^n$, (iii) $\vdash_{\text{HA}} (\omega\text{-Con HA}) \rightarrow \text{Con}(\bigcup \text{HA}_{\varepsilon_1}^n)$ and, in fact, the reflection principle for one quantifier formulae and their negations for $\bigcup \text{HA}_{\varepsilon_0}^n$. (Received Dec. 11, 1963.)

J. N. CROSSLEY and R. J. PARIKH. *On isomorphisms of recursive well-orderings.*

If \mathfrak{A} is a well-ordering and a is in the field of \mathfrak{A} , we write $|a|$ for the ordinal a represents. Let Γ be an ordinal and let \mathfrak{A} be a well-ordering of the integers of type Γ ; then a function φ from Γ^n to Γ is said to be *recursively representable* on \mathfrak{A} if there is a partial recursive function f such that

$$|f(a_1, \dots, a_n)| = \varphi(|a_1|, \dots, |a_n|).$$

(If \mathfrak{A} is a recursive well-ordering, then f can be taken to be general recursive.)

Theorem. Let $\mathfrak{A}, \mathfrak{B}$ be recursive well-orderings of (the same) ordinal Γ . Then \mathfrak{A} is recursively isomorphic to \mathfrak{B} if any of the following holds:

- $1 + \Theta$ is recursively representable on \mathfrak{A} and \mathfrak{B} and $\Gamma < \omega \cdot 2$,
- addition is recursively representable on \mathfrak{A} and \mathfrak{B} and $\Gamma < \omega^2$,
- multiplication and addition are recursively representable on \mathfrak{A} and \mathfrak{B} and $\Gamma < \omega^{\omega\omega}$,
- exponentiation and successor are recursively representable on \mathfrak{A} and \mathfrak{B} and $\Gamma < \varepsilon_{\omega}$,

e) (Kreisel) every ordinal $< \Gamma$ is representable by a term obtained from ordinal functions $\varphi_1, \dots, \varphi_n$ and a finite number of ordinals $\Gamma_1 \dots \Gamma_m < \Gamma$ and the functions $\varphi_1, \dots, \varphi_n$ are recursively representable on \mathfrak{A} and \mathfrak{B} .

In a)–d), the bounds are the best possible for the corresponding functions.

Using definitions analogous to the above and calling a co-ordinal *recursive* if it contains a recursive well-ordering, (see J. N. Crossley, *Constructive Order Types*, I–IV, abstracts, this JOURNAL), we have:

Corollary: The co-ordinals on which the functions in the theorem are recursively representable, are recursive and unique up to the bounds given in the theorem (and no further).

f) Let $f_{\mu}(\alpha)$ be Schütte's functions (abstract, meeting of the ASL, Oxford, 1963). Then the co-ordinals on which the function $f(\mu, \alpha) = f_{\mu}(\alpha)$ is recursively representable are recursive and unique for any ordinal κ_n where κ_n is the n -th strongly critical number. (Received Oct. 14, 1963.)