TWENTY-EIGHTH ANNUAL MEETING OF THE ASSOCIATION FOR SYMBOLIC LOGIC

The twenty-eighth annual meeting of the Association for Symbolic Logic was held in conjunction with the annual meeting of the Eastern Division of the American Philosophical Association at the Statler-Hilton Hotel, Washington, D.C., on Friday and Saturday, December 27–28, 1963.

Professors Alan Ross Anderson and Hugues Leblanc presided over the Friday morning session for contributed papers, Professor Leon Henkin over the Friday afternoon session which included Martin Davis’ invited address, *First Order, Second-Order, and Higher Order Logic*, followed by contributed papers. The Council of the Association met at lunch.

The Saturday morning joint session with APA featured a symposium on Frege with Rulon Wells, commentary by James Bartlett, and with James F. Thomson in the chair. Professor S. C. Kleene presided over the Saturday afternoon session for contributed papers. The first seventeen papers below were presented in person, the remaining four by title.


Limiting ourselves to sequents of the kind \(A_1, A_2, \ldots, A_n \rightarrow B\), where \(A_1, A_2, \ldots, A_n\) (\(n \geq 0\)), and \(B\) are wffs of the propositional calculus, we reexamine — in the wake of R. E. Vesley's "On strengthening intuitionistic logic" (*Notre Dame Journal of Formal Logic*, vol. IV, no. 1) — the problem of converting so-called structural and intelim rules for \(PC_I\), Gentzen's intuitionistic calculus of sequents, into rules for \(PC_C\), Gentzen's classical calculus of sequents. It has been known since 1935, 1960, and 1962, respectively, that one can pass from \(PC_I\) to \(PC_C\) by strengthening the intelim rules for \(\neg\), or those for \(\forall\), or those for \(\equiv\). Whether, however, one can do so by strengthening the intelim rules for \(\forall\) or those for \&' has so far remained an open question. Sharpening here the concept of an intelim rule for \(\forall\) or \&', we establish that: (1) any intelim rule for \(\forall\) or \&' which is provable in \(PC_C\) is also provable in \(PC_I\), and (2) any intelim rule for \(\forall\) or \&' which is closed under substitution and admissible in \(PC_C\) is also admissible in \(PC_I\). The answer to the above question is thus negative. (Received Oct. 10, 1963.)

William T. Parry. *Categorical forms in many-sorted logic with identity.*

Smiley (JSL XXVII 58) applies many-sorted logic to the traditional syllogism. He briefly indicates how a system MSI, adding identity to basic many-sorted logic, expresses 8 doubly quantified forms, corresponding to "quantification of the predicate." These forms merit further consideration.

Since the quantifiers have non-empty ranges, four forms express Aristotelian categoricals: \(\Pi_a \Sigma b(a = b)\) is \(A\); Every \(a\) is (some) \(b\); \(\Pi_a \Pi b(a \neq b)\), E; \(\Sigma a \Sigma b(a = b)\), I; \(\Sigma a \Pi b(a \neq b)\). O. *Distribution* now has a simple definition: distributed terms are those having the universal quantifier. The scholastic doctrine of distribution as logical descent follows. Non-scholastic "explanations" seldom explain.

The other four forms express non-standard categoricals discussed since Aristotle, such as "Every \(\{No, Some, Not\ every\} a\) is every \(b\," which we assign Greek vowels. \(\Pi a \Pi b(a = b)\) is *alpha*; \(\Pi a \Sigma b(a \neq b)\), *eta*; \(\Sigma a \Pi b(a = b)\), *iota*; \(\Sigma a \Sigma b(a \neq b)\), *omega*. They mean: Every \(\{No, Some, Not\ everywhere\} a\) is the only \(b\). The *eight forms* directly express De Morgan’s "exemplar" forms, 'Any (some) one \(a\) is (is not) any (some) one \(b\." Ockham was apparently first to understand that alpha is doubly singular.
He and Pseudo-Scotus understood that iota's predicate is singular. But De Morgan misinterpreted $sp$ as $pA$s (its subaltern), and $spb$ as $pO$s (its superaltern). A sign for unit classes permits Venn-type diagrams.

Other points: (1) $a(b$ iff $(ab(bu))$. (2) With iota and eta, rules of distribution and quantity change. (3) Reichenbach's rule for use of 'any' is amended. (4) Interpretation of alpha divides 'traditions' on quantification of the predicate into three: Aristotelian, alpha always false; Ockham-De Morgan, alpha doubly singular; Hamiltonian (collective), terms co-extensive. MSI gives a symbolic version of the second. (Received July 28, 1963.)

DON D. ROBERTS. Peirce's existential graphs and natural deduction.

Late in 1896 Charles S. Peirce invented a system of logic diagrams which he called the "existential graphs" (EG). He meant for this system to be a deductive treatment of both non-relative and relative logic. The first part of the system, part Alpha, treats of the logic of propositions; the second part, Beta, treats of the logic of quantification.

Peirce's usual method of providing his system with rules of inference but no axioms is suggestive of the method characteristic of natural deduction systems. And the purpose of this paper is to make the following three points:

1. The idea basic to natural deduction systems — that of using the rule of conditionalization as a primitive rule of inference — did in fact occur to Peirce. In 1900 he stated this as a primitive rule for an algebraic system of notation which he modeled after EG. Although Peirce gives no explicit development of this idea, he nevertheless anticipates Gentzen and Jaśkowski by 34 years.

2. The Alpha part of EG, when augmented by the rule of conditionalization as a primitive rule of inference, is adequate to the entire logic of propositions. This is proved by establishing, as theorems in EG, the graphical equivalents of the axioms and rule of inference of an algebraic formulation of the propositional calculus which is known to be adequate. The strategy for a consistency proof for Alpha is outlined.

3. The Beta part of EG can be made adequate to the logic of quantification by adding to the Alpha rules (the propositional basis) appropriate formulations of the rules of universal instantiation, universal generalization, existential instantiation, and existential generalization.

The ease of operation characteristic of this system, and illustrated in the proofs for (2), suggests that certain heuristic values are associated with EG. (Received Oct. 14, 1963.)

GEORGE GOE. First-order logic without existential import.

The provability in first-order logical calculi of wffs not valid in the empty domain has often been taken exception to. In fact, assertions of the existence of objects cannot be regarded as logical truths. Though a non-empty domain is presupposed in all axiomatic systems outside pure logic, it is clearly desirable for the latter sharply to separate from all others, as theorems of its own, those wffs which are valid in every domain, while those valid in every non-empty domain should be derivable within the pure logic system from a special assumption formula.

The presence of the controversial theorems is in most traditional first-order calculi related to the use therein of free variables in the wffs that occur as lines in derivations, which has also been independently criticized as a duplicate way of expressing universality; but some systems which dispense with that use of free variables none the less are complete with respect to validity in every non-empty domain. Insights are gained in syntactically and semantically tracing the origin of the controversial theorems in both types of systems (note that such wffs as '($x$)Fx $\supset$ Fy' or 'Fx $\supset$ (3y)Fy' are valid in the empty domain if they are regarded as semantically equivalent to their closures).
A system of first-order logic wherein only closed wffs occur as lines in derivations, and wherein all and only wffs valid in every domain are theorems, can be formulated with five axiom schemata, and modus ponens as sole rule of derivation. Every wff valid in every non-empty domain can be derived in this system from \((\exists x) (Fx \lor \neg Fx)\), or from any prenex form opening with an existential quantifier, taken as an assumption formula. Thus the addition of a single axiom (not an infinity, as by a schema) transforms the system into one equivalent to traditional first-order logical systems. (Received Oct. 15, 1963.)

ALBERT A. MULLIN. Studies in arithmetic.

Circa 300 B.C. Euclid of Alexandria (Elements, Book IX, Proposition 14) stated the spirit, and if one interprets thirdness properly, the essence, of the Fundamental Theorem of Arithmetic (FTA), which is so useful for, e.g., (i) Gödel numbering wffs of axiomatics (or if the reader wishes, for approaching Leibniz' Characteristica Universalis), (ii) serving as the paradigm par excellence for no small part of mathematics (e.g., algebraic number theory à la Kummer-Kronecker-Hilbert); analytic number theory; combinatorial topology, especially graph theory, and less directly through the Fundamental Theorem of finitely generated abelian groups), and (iii) more applied matters of value, such as circuit analysis.

This monograph, whose syllabus is available from the author, introduces a new formulation of FTA, which for some good reason no doubt, has escaped the notice of the community, and then proceeds to classify all effective (multiplicative) models for FTA. The models for FTA are collected into three forms (classes of models), viz., (i) the primordial Euclidean form (the product of the first power of primes alone), (ii) the mixed form (in which a model is determined, in general, by both primes and composites as with Gauss' model, circa A.D. 1800), and (iii) the pure non-Euclidean form (in which a model is determined by primes alone).

The point of departure for the new method is to use an induction with the Gaussian model for FTA upon itself, i.e., upon the exponents of the Gaussian model for FTA. To this extent the method is analogous to Euclid's Algorithm for the GCD of two natural numbers in that it applies a given rule to a portion of the object obtained from a prior application of the given rule until the process terminates in a finite number of steps.

In this mathematically logical investigation it is shown that modest portions of mathematics (notably algebra, analysis, number theory, geometry, and topology) and mathematical logic (notably recursive function theory, the theory of mathematical induction, and intuitionistic analysis) can be related by a common method, among diverse methods. (Received June 23, 1963.)

BRUCE LERCHER. Strong reduction and normal form in combinatorial logic.

Weak reduction, \(U \geq V\), is generated by axiom schemes for the atomic combinators \(I, K, S: IX \geq X; KXY \geq X; SXYZ \geq XZ(YZ)\). The left-hand sides are weak redexes. For strong reduction, \(U \longrightarrow V\), change the "\(\geq\)" to "\(\rightarrow\)" and add the rule: If \(X \longrightarrow Y\), then \([x]X \longrightarrow [x]Y\). The abstraction \([x]X\) is defined by algorithm (abcf) of Curry and Feys Combinatory Logic (where strong reduction was first defined). \(X\) is in normal form iff (i) \(X\) is an indeterminate; (ii) \(X = xX_1 \ldots X_n\), where \(X_1, \ldots, X_n\) are in normal form; (iii) \(X = [x]Y\), where \(Y\) is in normal form. (This definition is due to L. E. Sanchis in his Penn State Ph.D. thesis.)

Theorem 1. If \(X\) is strongly irreducible, then it is in normal form.

This is the converse of the main theorem about strong reduction in Combinatory Logic. Examples of preliminary results follow. In each statement, \(x, x_1, \ldots, x_k\) are indeterminates not occurring in \(X\).

Lemma 1. Let \(X = [x]Y\). Then \(X\) is weakly irreducible iff every component of \(Y\) which is a weak redex contains an occurrence of \(x\).
Lemma 2. If $X$ is weakly irreducible, there is at most one weakly irreducible $Y$ such that $X = \{x\} Y$.

Theorem 2. If $X$ is strongly irreducible, there is a weakly irreducible $Y$ such that $X \equiv \{x_1, \ldots, x_k\} Y$.

The proof here is by induction on the number of occurrences of atomic combiners in X. The induction step is handled by a secondary induction on $k$, where in that induction step, we assume $X = \{x_1, \ldots, x_{k-1}\} Z$ and consider cases for the possible forms of $Z$. All are trivial except $Z = \text{SUV}$. We show there are $Y_1$ and $Y_2$ which are weakly irreducible such that $U = \{x_k\} Y_1$, $V = \{x_k\} Y_2$, and $Z = \{x_k\} Y_1 Y_2$. The assumption that $Y_1 Y_2$ is weakly reducible then is shown to lead to contradiction.

Theorem 3. If $X$ is weakly irreducible and there is a $k \geq 0$ and indeterminates $y_1, \ldots, y_k$ such that $[y_1, \ldots, y_k] X$ is strongly irreducible, then $X$ is in normal form.

John Corcoran. Generative structure of logics.

In his Two-valued Iterative Systems of Mathematical Logic Post gives characterizations of all two-valued logics in such a way as to answer the question: given a set $S$ of truth functions what is the logic which results using $S$ as a set of “primitive functions”? We continue his work. We characterize all sets of truth functions in such a way as to answer the question: given a truth-function logic $L$ what is the class of all sets $S$ such that $S$ is a primitive set for $L$? In order to answer this question, we discover which two-valued iterative logics can be based on a single function. We call equivalence classes of such functions Sheffer sets.

In order to get a more detailed knowledge of the interrelations among two-valued iterative logics and Sheffer sets, we define a measure which assigns to certain sets of truth-functions numbers which can be thought of as the fractions of the set of all truth functions included in those sets. We show that all two-valued logics and all Sheffer sets are measurable, and we compute their measures. We find that a Sheffer set has non-zero measure iff the corresponding two-valued logic does as well. Moreover, we find that the Sheffer set of the complete logic has measure one-fourth — intuitively, that any one of a quarter of all truth functions can be used as a primitive on which to base a functionally complete logic.

Finally, we study the structure relations between two-valued iterative logics and can conclude that the only non-trivial isomorphism between two-valued iterative logics is duality — the function that maps each truth-function into its dual. (Received Oct. 4, 1963.)

Leon Henkin. A class of non-normal models for classical sentential logic.

A system $\mathcal{L} = \{\text{L}, +, \cdot, \rightarrow\}$ is an $A$-lattice if $\mathcal{L}_1 = \{\text{L}, +, \cdot\}$ is a distributive lattice and is an anti-automorphism of $\mathcal{L}_1$ (i.e., a one-one mapping of $L$ onto itself such that $-(x + y) = -x \cdot -y$ for all $x, y \in \text{L}$). Let $T$ be the filter of $\mathcal{L}_1$ generated by all elements $x + -q_x$, where $q = 1, 3, 5, \ldots$ and $x \in \text{L}$. Theorem: If $T \neq \text{L}$ then $(\mathcal{L}, T)$ is a characteristic model for classical sentential logic with primitives $\vee, \wedge, \neg$. (I.e., among sentential formulas the classical tautologies, and only these, assume values lying entirely in $T$ for arbitrary assignments of $L$-elements to variables.) This generalizes the corresponding theorem for De Morgan lattices (i.e., $A$-lattices in which $- x = x$ for all $x \in \text{L}$), due to Monteiro (Academia Brasileira de Ciências, 1960, pp. 1–7). A representation theorem for $A$-lattices is obtained, generalizing the corresponding result for De Morgan lattices due to Bialynicki-Birula and Rasiowa (Bulletin de l’Académie Polonaise des Sciences, 1957, pp. 259–261). Finite $A$-lattices are constructed which are not homomorphic to any De Morgan lattice, and are hence non-normal in the sense of Church (Boletín de la Sociedad Matemática Mexicana, 1953, pp. 41–52). (Received Oct. 25, 1963.)
William Craig. *An implicit definition of satisfaction in n-th order theory.*

Let $2 \leq n < \omega$, $0 \leq p < \omega$, and let $R_0, \ldots, R_{p-1}$ be variables of order $\leq n$. For each type $\tau' = (\tau'_0, \ldots, \tau'_{r-1})$ let $\tau'_0, \ldots, \tau'_{r-1}$ be the immediate predecessors of $\tau'$. Let $Q = \{\tau_0, \ldots, \tau_{r-1}\}$ consist of the predecessors of the types of $R_0, \ldots, R_{p-1}$, the $n$ monadic types of order $\leq n$ and the two types corresponding to order 1 and rank 2 or 3 respectively. Let $L$ consist of those formulas of [arithmeticized] pure simple type theory in which only variables of order $\leq n$ ($< n$) occur free (bound), and $L'$ ($L''$) of those $\psi$ in $L$ (in $L'$) such that if a variable occurs free (bound) in $\psi$ then either its type is in $Q$ or it is one of $R_0, \ldots, R_{p-1}$ (its type is in $Q$). Elaboration of Vaught's argument in XXXIII 306, 307 shows:

**Lemma 1.** There are $\theta_i$ in $L'$ whose free variables are $N, X, V, Z, u, W_i$ of appropriate type, $i < q$, which yield, for any $h$ mapping an infinite $A$ onto $A^{(\omega)}$ and any $N, X, V, f$ satisfying (iv) and (v) below, a mapping $h^*$ as follows: (i) the domain of $h^*$ consists of the monadic $\langle n-1\rangle$-th order sets $Z$ on $A$; (ii) each $h^*(Z)$ is a sequence whose $(f(u)q + i)$-th term is an object $W_i$ of type $\tau_i$ on $A$, $u \in N$; and (iii) every such sequence having eventually period $q$ is in the range of $h^*$. **Lemma 2.** There is a recursive $g$ which for any $\psi$ in $L'$ yields a $g(\psi)$ in $L''$ having the same infinite models. **Theorem 1.** Some $\varphi \in L$ defines implicitly, for any infinite $A$ of the appropriate kind, the satisfaction relation between sequences and formulas $\psi \in L'$. [Indeed, the definition is by induction on the length of $g(\psi)$ and hence "predicative"] More precisely, we construct a $\varphi \in L$ such that: (1) The only variables free in $\varphi$ are $R_0, \ldots, R_{p-1}$ and $S$; and (2) $\langle A, R_0, \ldots, R_{p-1}, S \rangle$ is a [standard] model of $\varphi$ if and only if $S$ consists of exactly those $\langle N, X, V, Z, y \rangle$ of the appropriate kind such that for some $h$ of $A$ onto $A^{(\omega)}$ and some $f$: (iv) $f$ maps $\langle N, X \rangle$ isomorphically onto $\langle \omega, Sc \rangle$ where $Sc$ is the successor relation; (v) $V$ is a function such that $V(z, u) = (h(z))(f(u))$ for each $z \in A$ and $u \in N$; (vi) $f(y) \in L'$; and (vii) $h^*(Z)$ satisfies $g(f(y))$ in $\langle A, R_0, \ldots, R_{p-1} \rangle$. **Theorem 2.** Consider any infinite $A = \langle A, R_0, \ldots, R_{p-1} \rangle$ of the appropriate kind and the unique $S$ such that $\varphi$ is true for $(A, S)$; then for each $\chi \in L'$ which contains free only $R_0, \ldots, R_{p-1}$ and variables $N, X, V, Z, y$ appropriate as arguments for $S, \forall N \ldots \forall y[S(N, \ldots, y) \equiv \chi]$ is false for $(A, S)$. For proof, let $T$ be the theory whose set $W$ of valid sentences consists of those $\psi$ in $L$ which are true in $\mathbb{A}$, and as on p. 47 of *Undecidable Theories* prove: **Lemma 3.** Let $\pi_0, \ldots, R_{p-1}, N, X, V, u$ define a set $P \subseteq \omega$ in $T$ [relative to formulas $\Delta_\omega(N, X, u)$ and $\lambda(N, X, V)$] if and only if, for each $n \in \omega$, $\forall N \forall X \forall V[\lambda \rightarrow \forall u[\Delta_n \rightarrow \pi]]$ or $\forall N \forall X \forall V[\lambda \rightarrow \forall u[\Delta_n \rightarrow \pi]]$ is valid in $T$ depending on whether $n \in P$ or $n \notin P$; for functions modify the notion of UT of definability in $T$ similarly; let $D$ be a function from $\omega$ into $\omega$ such that, if $n = \forall N \forall X \forall V[\lambda \rightarrow \psi]$, then $D(n) = \forall N \forall X \forall V[\lambda \rightarrow \forall u[\Delta_n \rightarrow \psi]]$; assume that $T$ is consistent and that $\exists N \exists X \exists \lambda$ and all $\forall N \forall X \forall V[\lambda \rightarrow \exists u[\Delta_n \rightarrow \psi]]$ is valid in $T$; then $D$ and $W$ are not both definable in $T$. **Corollary.** Beth's (and hence Craig's) theorem for first order theories (languages) has no [literal] analogue for $T$ ($L'$ or $T$). (Received Oct. 16, 1963.)

Robert McNaughton. *A solvability algorithm for Büchi's sequential calculus.*

Using the theory of finite automata, J. R. Büchi has settled the basic questions about the sequential calculus (SC), which is a semantically interpreted system of second-order logic (without axioms and rules of inference) having quantifiable variables ranging over natural numbers and over monadic predicates of natural numbers. ("On a decision method in restricted second-order arithmetic," *Logic, methodology and philosophy of science: proceedings of the 1960 international congress*, Stanford University Press, 1962.) That is to say, he has shown the precise limits of definability in SC, and has presented a decision procedure for it.

One of the open questions that are listed at the end of Büchi's paper concerns the possibility of a solvability algorithm for SC: namely, is there an effective manner
of determining whether or not there exists a finite automaton satisfying a given condition expressed in SC? The present paper answers this question in the affirmative, using Büchi's theorem 1 (the main theorem of his paper, from which the above results follow), to the effect that every formula of SC without free individual variables is effectively equivalent to a formula having a certain form, and thus saying something rather complicated about the infinite history of a finite automaton. My algorithm uses the concept of a game in which two players play in turn infinitely often, and where the winning and losing depends on whether or not a certain event has occurred infinitely often. If there is a finite automaton satisfying a condition expressed in SC, then the algorithm continues beyond the affirmative answer to the synthesis of it. If there is none, the algorithm produces, in addition to the negative answer, a further algorithm that can generate, for a given finite automaton, an infinite input history that falsifies the original condition. (Received Oct. 1, 1963.)


A formulation of first-order logic was made as the basis for an efficient proof procedure to be realized in a program for a digital computer. The formalism is quantifier-free, and contains symbols for no truth-functional connectives except negation. Atomic formulae are formed from predicate symbols, function symbols, individual variables, and individual constants in the usual way; these, and negations of these, are literals. A finite set of literals (including the empty set) is a clause; the semantics of the system construes clauses as disjunctions. Finite sets of clauses (including the empty set) are construed as conjunctions of their elements. The empty clause is always false, the empty conjunction always true. Individual variables are construed as universally quantified with scope the entire clause in which they occur. Existential quantification is achieved via function symbols.

Only one principle of inference, the Resolution Principle, is used; there are no axioms or axiom schemata. The Resolution Principle is a two-premise rule, with clauses as the premises. The system is proved to be complete, and it is shown that the problem of instantiation, hitherto a barrier to efficient proof-procedures, is completely solved; no choices are "creatively" made of terms with which to instantiate free variables — the Resolution Principle makes the choice automatically.

The system is compared, via examples, with earlier systems, and we show that in our system proofs are shorter and are discovered more quickly.

The Resolution Principle, as a price of its power, is complex and machine-oriented. The human cannot in general determine in one gestalt that the principle is applicable to a pair of given clauses, nor what the result of applying it would be. This must be determined by the application of certain algorithms to the two clauses as inputs. The Resolution Principle is in fact defined by these algorithms. (Received Oct. 14, 1963.)


Many problems of philosophy arise in the context of attempts to maintain that some particular linguistic expressions are univocal. Solutions which reject these expressions as not proper or which reject the original intuition of univocity, and which go no further, fail to take into account certain unusual relationships between the instances of use of the expressions. For example it may be difficult to draw less than arbitrary distinctions separating pairs of usages which, if identified, lead to contradiction. The claim is made that more may be said about linguistic expressions which have various incompatible usages than is said in merely noting these usages. The properties of these family resemblances are made precise in a number of ways by means of the imposition of various mathematical structures. The geometric-like invariants assigned to representations of linguistic phenomena provide potentially useful tools for classification and analysis. These structures are applicable to a number of variants of well-known formal languages. In von Neumann-Robinson class
theory the addition of an axiom is suggested which, although it arises from philosophical considerations, resembles the crucial axiom of the theory. A "spiral" type theory (of function variables) is exhibited in which the collection of the various types is not disjoint or nested but has more complicated intersection and order properties. Some light is shed, belatedly, on the puzzles of modal logic by pointing out that the intuition of univocity about possibility may have caused them, by noticing that several features of Hintikka's solutions are amenable to this analysis, and by suggesting that a further generalization of Hintikka's results to systems of modal systems and so on should be undertaken. Some methods of approximation for the empirical determination of these invariants provide a possible but as yet impractical method for the analysis of rational discourse in ordinary language. (Received Oct. 14, 1963.)

R. Solovay. 2\(\mathbb{N}\)0 can be anything it ought to be.
The methods of Cohen are used to study the possible values of the function 2\(\mathbb{N}\) in various models for Z - F set theory. (In all models considered, the axiom of choice is valid.) For example, the following theorem is proved:

If Z - F is consistent, it remains consistent if the axiom of choice and the assertions

2\(\mathbb{N}\)0 = \(\mathbb{N}\)9, 2\(\mathbb{N}\)1 = \(\mathbb{N}\)5 and 2\(\mathbb{N}\)2 = \(\mathbb{N}\)10

are added as axioms. (Received Oct. 14, 1963.) (At the meeting this paper was introduced by A. Church.)

W. A. Howard. Bar induction, bar recursion and the \(\Sigma^1\) axiom of choice.
Let \(\mathbb{Z}\)2 denote classical second order arithmetic with no comprehension axiom but with primitive recursion on type level 1 and certain trivial functional of type level 2. Let \(\mathbb{H}\)2 be the corresponding system based on intuitionistic predicate logic. Let (BI) denote the schema of bar induction of type 0 and let (BR) denote the schema of bar recursion of type 0 [3]. For any formula \(\mathcal{A}\) of \(\mathbb{Z}\)2 let \(\mathcal{A}\)' be G"odel's translation of \(\mathcal{A}\) into the system \(\mathbb{K}\) of primitive recursive functionals of finite type given in [1].

Theorem 1. For any formula \(\mathcal{A}\), if \(\mathcal{A}\) is provable in \(\mathbb{H}\)2 + (BI) then \(\mathcal{A}\)' is provable in \(\mathbb{Z}\) + (BR).

Let (SP-BI) denote (BI) in which \(\mathcal{P}(c)\) in the hypothesis \(\alpha\forall\chi \mathcal{P}(\alpha(x))\) of (BI) is quantifier free. For any formula \(\mathcal{A}\), let \(\mathcal{A}\) denote the "negative" version of \(\mathcal{A}\) obtained by prefixing every disjunction and existential quantifier by double negation.

Theorem 2. If \(\mathcal{A}\) is provable in \(\mathbb{Z}\)2 + (SP-BI) then \(\neg(\mathcal{A} \rightarrow \mathcal{A})\) is provable in \(\mathbb{H}\)2 + (BR).

Theorem 3. The \(\Sigma^1\) axiom of choice (see [2] and [3]) is provable in \(\mathbb{Z}\)2 + (SP-BI).

Theorem 4. The comparability of well-orderings is provable in \(\mathbb{Z}\)2 + (SP-BI).

Theorems 2 and 3 provide a constructive justification, which is simpler than that provided by [3], of the use of the \(\Sigma^1\) axiom of choice in \(\mathbb{Z}\)2. Indeed, we have justified the stronger system \(\mathbb{Z}\)2 + (SP-BI).

Finally, we show that (BI) of type 1 is reducible to (BI) of type 0 and that (BR) of type 1 is reducible to (BR) of type 0.


(Rceived Dec. 2, 1963.)

Raymond M. Smullyan. Superinductive classes and ordinal numbers.
We think of the ordinal numbers as just those sets which can be obtained, starting with the empty set 0, and by taking the successor \(x^+\) (i.e. \(x \cup \{x\}\)) of any set \(x\) already obtained, and by taking the union \(U\) of any set \(y\) of sets already obtained. This idea is best formalized in a class-set theory such as VNB. Define a class \(\mathcal{A}\) to be super-
inductive iff: (i) $0 \in A$; (ii) for every set $x$, $x \in A \Rightarrow x^+ \in A$; (iii) for every set $y$, $y \subseteq A \Rightarrow \bigcup y \in A$. Now we define $\text{On}(x)$ [read ‘$x$ is an ordinal number’] iff $x$ belongs to every superinductive class. To verify that a given property $P$ (defined by a normal formula) holds for every ordinal number, it obviously suffices to show that the class of all $x$’s having the property is superinductive. By this method, one easily establishes all the usual properties of the ordinal numbers.

Since the formula $\text{On}(x)$ is itself not normal, we do not immediately have the class $\text{On}$ of all ordinal numbers. [We would have this class at once, if we were working in impredicative class-set theory rather than VNB]. To establish in VNB the existence of $\text{On}$, one proves the equivalence of $\text{On}(x)$ with any of the usual definitions (by normal formulas), or one can use either Th. 1 or Th. 2 below.

The following modification is applicable to ZF as well as VNB. For any set $A$ and any subset $B$ of $A$, define $B$ to be superinductive relative to $A$ iff (i) $0 \in A \Rightarrow 0 \in B$; (ii) for every $x \in B$, $x^+ \in A \Rightarrow x^+ \in B$; (iii) for every subset $x$ of $B$, $\bigcup x \in A \Rightarrow \bigcup x \in B$. Let $A^*$ be the intersection of all subsets of $A$ which are superinductive relative to $A$.

**Theorem 1.** $x$ is an ordinal number iff there exists a set $A$ such that $x \in A^*$. A more elegant characterization is given by: **Theorem 2.** $x$ is an ordinal number iff $x = x^*$.

(Rceived Dec. 27, 1963.)

G. Kreisel and G. E. Sacks. **Metarecursive sets I.**

In preparation for abstract recursion theory we analyse classical recursion theory in terms of the following notions. A universe $F$ (of formal objects) including $\omega$, a collection $\mathfrak{F}$ of sequences of elements from $F$ (computations), including the set $\mathfrak{I}$ of finite sequences of some distinguished object in $F$, say $1$; a collection $\mathfrak{E}$ of functions: $\mathfrak{F} \rightarrow \mathfrak{F}$, and pairing functions $\mathfrak{F}^2 \rightarrow \mathfrak{F}$ which induce $\mathfrak{R}^n$: $\mathfrak{F}^n \rightarrow \mathfrak{F}$. Derived notions are the collections $\mathfrak{F}^*$ of (unordered) sets in $\mathfrak{F}$, $\mathfrak{E}_0$ of subsets of $\mathfrak{F}$ whose characteristic functions are in $\mathfrak{E}$; $\mathfrak{R}^\ast$ of subsets of $\mathfrak{F}$ which are the ranges of functions in $\mathfrak{E}$; $\mathfrak{A}$ of the field of subsets of $\mathfrak{F}$ generated by $\mathfrak{R}^\ast$ and included in some set $\epsilon \mathfrak{F}^*$, finally, for $X \subseteq F$, $Y \subseteq F$ we mean by: $X$ is $\mathfrak{A}$ in $Y$ that for some $f \in \mathfrak{R}^3$ (a ranging over $F$, $a$ over $\mathfrak{F}$, $\epsilon$ over $\mathfrak{F}$), $\langle \langle \rangle \rangle$ denoting the empty sequence):

$a \epsilon X \leftrightarrow \forall a \epsilon_1 [(a^* \subseteq Y \land a_1^* \subseteq CY \land f(a, a, a_1) = 1],$

$a \epsilon X \leftrightarrow \forall a \epsilon_1 [(a^* \subseteq Y \land a_1^* \subseteq CY \land f(a, a, a_1) = \langle 1, 1 \rangle],$ and

$\forall a \epsilon_1 a_1 \epsilon_1 a \epsilon_1 a_1^* \epsilon_1 a_1^* \subseteq CY \land f(a, a, a_1^*) = f(a, a', a_1') \Rightarrow f(a, a', a_1') = f(a, a, a_1])$ where $a^*$ denotes the (unordered) set consisting of the elements of $a$. In classical recursion theory $F$ itself is $\omega$, $\mathfrak{F}$ are ordered (and thus well-ordered) finite sequences of natural numbers, $\mathfrak{E}$ recursive functions, $\mathfrak{F}$ unordered finite sets, $\mathfrak{E}_0$ recursive sets, $\mathfrak{A}$ recursively enumerable, $\mathfrak{B}$ bounded (and thus $\mathfrak{A} = \mathfrak{F}$), $X$ is $\mathfrak{B}$ in $Y$ if $X$ is recursive in $Y$. These may be construed as model theoretic invariants of the general models of certain large classes of formal systems of arithmetical or concatenation theory. The same invariants of $\omega$-models for analogous systems yield for $F$ the recursive ordinals. These may be considered either as transfinite sequences of $1$ of length $\omega_1$ (first non-recursive ordinal), or, more conventionally, as the initial well-ordered segment $W$ of length $\omega_1$ of some hyperarithmetic ordering of the natural numbers (cf. e.g. Gandy, *Bull. Acad. Pol. Sci.* 8 (1960) 571–575).

Any two such $W$, $W'$ are isomorphic by mappings which are $\Pi_1 \land \Sigma_1$ on $W$ and $W'$. $\mathfrak{F}$ are the hyperarithmetic subsequences of $W$, which we call *metafinite*, $\mathfrak{E}$ the functions which are $\Pi_1 \land \Sigma_1$ on $\mathfrak{F}$ with values in $\mathfrak{F}$ which we call *metarecursive*, $\mathfrak{F}$ hyperarithmetic subsets of $W$ i.e. absolutely $\Pi_1 \land \Sigma_1$. $\mathfrak{E}_0$ are sets that are $\Pi_1 \land \Sigma_1$ on $\mathfrak{F}$. $\mathfrak{A}$ are $\Pi_1$ subsets of $\mathfrak{F}$ which we call *metarecursively enumerable*. $\mathfrak{B} \subseteq \mathfrak{F}$. If $X$ is metarecursive in $Y$ then $X$ is hyperarithmetic in $Y$, but, in general, not conversely; e.g. the sets of natural numbers which are metarecursive in $O$ can be enumerated by a function hyperarithmetic in $O$ (and not metarecursive in $O$).
\(\mathcal{A} \cap \mathcal{B}\) consists of exactly those \(\mathcal{B}\) sets whose members contain only ordinals less than some \(\alpha < \omega_1\). (Received Nov. II, 1963.)

G. Kreisel and G. E. Sacks. Metarecursive Sets II.

For notation see preceding abstract. For \(\mathcal{A} \subseteq \mathcal{B}(X), X \subseteq F\), we call \(M_0\) maximal simple for \(\langle \mathcal{A}, X \rangle\) if (i) \(X - M_0\) is infinite, (ii) \((\mathcal{A} \cup \mathcal{B}) [M \supseteq M_0 \rightarrow (M - M_0)\) is finite or \(X - M = (X - M_0)\) is finite]]; metamaximal simple if (i') \(X - M_0\) is unbounded, (ii') \((\mathcal{A} \cup \mathcal{B}) [M \supseteq M_0 \rightarrow (M - M_0) bounded or \(X - M\) is bounded]; and strongly (meta)maximal simple if (i') is unchanged, but 'bounded' is replaced by 'meta' finite'.

Theorem 1 (Generalisation of Friedberg XXV 165). For \(X = \omega, \mathcal{A} = \mathcal{B}\) there is a maximal simple set. [The proof is uniform for both interpretations. Besides some elementary closure conditions on \(\mathcal{B}\) such as \(\exists \in \mathcal{B}\), it uses essentially the facts: the collection \(\mathcal{B}\) is metarecursively enumerable over \(\omega\), i.e. there is \(f^* \in \mathcal{B}\) such that for each \(X \in \mathcal{B}\), there is an \(n \in \exists X : X = \{a \in \mathcal{V}_b f^*(n, a, b) = 0\}\); and a finite collection of (meta) finite sets is (meta) finite.] Corollary to Theorem. For \(X = F, \mathcal{A} = \mathcal{B}\) there is a maximal simple set. By a different proof, again uniformly, we get Theorem 2. There is a strongly maximal simple set. For the analogue of Post's problem one needs additional closure conditions on \(F\), \(\exists\) and \(\mathcal{B}\), in particular, that (a) a metafinite collection of metafinite sets is metafinite, (b) \(F\) is metafinite, (c) \(F\) is well-ordered by a metarecursive relation, (d) \(\mathcal{B}\) closed under definition by recursion on that well-ordering, and (e) there is a metarecursive isomorphism between \(F\) and \(\exists\).

Theorem 3 (Generalisation of Friedberg's: XXIII, 225). There are two (meta) recursively enumerable subsets of \(F\) which are (meta) recursively incomparable. Analysis of the proof gives: Theorem 4: There are two unbounded metarecursively enumerable subsets of \(F\) which are metarecursively incomparable. A different proof yields such subsets of \(\omega\) (instead of \(F\)) in the metarecursive case (e.g. by (b)). By Spector [XXI, 412] any two bounded \(\mathcal{B}\) subsets of \(F\) are hyperarithmetically comparable. For the sake of easy comparison with the literature we formulate here our results for subsets of \(F\) instead of subsets of \(\exists\); but in an abstract theory it is best to consider an alphabet \(F\) containing at least two letters, and for \(\exists\) (meta)finite sequences [cf. Smullyan, Theory of formal systems, Princeton 1961, esp. pp. 88–89, and also G. Higman, Proc. Roy. Soc. A 262 (1961), 455–475]. (Received Nov. 13, 1963.)

G. Kreisel. The subformula property and reflection principles.

Throughout we use arithmetisations which express the syntactic relations considered (i.e., as a minimal requirement satisfy the derivability conditions of HB II). Informally, an infinite proof tree for the rules and axioms \(\mathcal{B}\) is a well-founded tree of formulae at whose terminal nodes are axioms, and each \(F\) at a non-terminating node is an immediate consequence of its immediate descendants \(F_1, F_2, \ldots\) by one of the given finitary or infinitary 'rules' (all in \(\mathcal{B}\)). Formally, if \(\mathcal{B}\) is applied to formulae in the notation \(S\), and the formal system \(S_1\) is an extension of \(S\), let < be a partial ordering for which transfinite induction can be proved in \(S_1\) (since formal systems are considered, < denotes of course a formula or its Gödel number, not a set theoretic object; the same convention applies below.) Definition. Let \(\varphi(p; a, b)\) be a formula for which it can be proved in \(S_1\) that, with free variable \(p\), (i) the field of \(\{a, b\} : 0(p; a, b)\}, say <\(p\), is a set of Gödel numbers of formulae which are related to their immediate predecessors by \(\mathcal{B}\), (ii) \(a(p)\) is the top element \(A_p\) of <\(p\), obtained recursively from \(p\) and (iii) \(\varphi_p\) is an order preserving mapping of <\(p\) into <. Then \(p\) is called an \(\mathcal{B}\)-proof of \(A_p\) relative to \(S_1\). Definition. \(\mathcal{B}\) has the subformula property if for each formula \(A\) there is a formula \(A_1(n)\) with free variable \(n\), such that, for some \(\rho\), \(\vdash [A \leftrightarrow A_1(0^{(n)})]\) and for each rule in \(\mathcal{B}\), where \(\nu(\psi) = \Gamma^\nu\) (with free variables \(n\) and \(\Gamma^\nu\)), \(\vdash [\Gamma^\nu \rightarrow A_1(0^{(\nu)} \Rightarrow \lambda \mu \Phi \vdash [\Gamma^\nu \rightarrow A_1(0^(\mu))]]\) where \(\vdash\) means provability in \(S\); thus only formulae enumerated by \(A_1(n)\) can occur in an \(\mathcal{B}\)-proof of \(A\).
up to equivalence w.r.t. $S$. Definition. $R$ is $S$-valid for subformulae of $A$, if, with free variables $\{F^1\}$ and $m$, for each rule in $R$: $\vdash \forall n \forall p (A_1(p) \land \{F_n \leftrightarrow A_1(0(p))\} \land \{F \leftrightarrow A_1(0(m))\} \rightarrow A_1(m))$. Theorem 1. Suppose $R$ consists of a finite number of possibly infinitary rules having the subformula property, and suppose $R$ is $S_1$-valid. Then, for each $A$ containing the free variable $m$ (but not $p$) the reflection principle $\vdash (\text{Prov}(p) \land \forall A_p^m = \{A(0(m))\} \rightarrow A(m))$ holds. Proof by transfinite induction applied to $<_p$ and $A_1$. The theorem can be extended to infinite sets $R$, but the statement is too long for this abstract. Corollary. Under the conditions of the theorem, if $<_p$ is recursive and well-founded and the set of terminal elements is recursive, and if $R$ is $S$-valid, then $A$ can be proved in $S$ by adding transfinite induction on a recursive well ordering.

Schütte's infinitary cut free formulations for arithmetic and ramified analysis satisfy the subformula property. Takeuti's cut free system of analysis does not.

Theorem 2. The $\Pi^1_1$ comprehension axiom of analysis, i.e. $\forall x \in S \rightarrow A(x)$ where $A$ contains at most one set quantifier, is not provable on a recursive proof tree for rules $R$ which satisfy the subformula property and are valid for elementary analysis $S$. (Proof by use of the corollary and simple model theory.) — Theorem 2 constitutes a limitation of 'Gentzen's method' for the proof theory of analysis. Evidently every true $\Pi^1_1$ formula has a recursive proof tree satisfying the conditions of Theorem 2 by use of unsecured sequences and the Brouwer rule (of the next abstract). (Received Dec. 11, 1963.)

G. Kreisel. Reflection principle for Heyting's arithmetic.

HA denotes intuitionistic first order arithmetic extended by the addition of (free) function symbols and the corresponding extension of the classes of terms, formulae, axiom schemata. The well orderings mentioned below are standard, i.e. are uniquely determined up to isomorphism w.r.t. mappings in a class $C$ by requiring that basic ordinal functions on them are in $C$, $C$ being closed w.r.t. elementary operations; for $\varepsilon_0$ one needs the ordinal functions of addition, multiplication, exponentiation to base 2, and their inverses, cf. axioms $A_1$–$A_{12}$, pp. 202–203 of Schütte's Beweistheorie. Let $HA^{\varepsilon_0}_{\varphi}$ be obtained from HA by (i) adding the principle of transfinite induction on $\varepsilon_0$ applied to formulae with $\leq n$ logical constants, (ii) restricting proofs to those containing formulae with $\leq n$ logical constants. Note that by enumeration of formulae with $n$ logical constants $HA^{\varepsilon_0}_{\varphi}$ is equivalent to a finite set of axioms. $HA^{\varepsilon_0}_{\varphi}$ is quantifier free. Informally let $HA^+$ be based on cut free intuitionistic predicate logic and two infinitary rules: $\forall x A(x)$ is inferred from $A(0), A(0'), \ldots$ ($\sigma$-rule) and $A(s)$ is inferred from $A(0(p)) = 0 \rightarrow A(x), A(0(p)) = 0' \rightarrow A(x), \ldots$ (Brouwer's $\sigma$ rule which is equivalent to the $\omega$-rule and the inference of $A(x)$ from $\forall x [A(0(p)) = x \rightarrow A(x)]$.) Formally, in the notation of the preceding abstract, let $S_1$ be $HA^{\varepsilon_0}_{\varphi}$, a standard $\varepsilon_0$ ordering, $<_p$ a formula defining, say, primitive recursive (p.r.) proof trees for $HA^+$, with p.r. $A(p)$ and $\varphi(p)$ (where one simply builds into the definition of $<_p$ that the $p$ be $HA^+$-proofs relative to $S_1$). Lemma 1. (syntactic). For each $A$ of HA, $\vdash_0 \forall n [\text{Prov}_{HA}(n, \forall A^m \rightarrow (\text{Prov}^+(\pi(n))) \land \forall A_n^m = \{A(0(m))\}]$ with a constant $\pi$ of $HA^{\varepsilon_0}_{\varphi}$. (Standard proof by cut elimination which is easily modified to give p.r. trees as in embedding recursive in p.r. well orderings.) Lemma 2. For $A$ containing $\leq n$ logical constants, $\vdash_0^{n+1} (\text{Prov}^+(\pi)) \land A^m = \{A\} \rightarrow A$ (by use of a formula $A_1$ with $n + 1$ logical constants which enumerates all subformulae of $A$). Theorem. For each $A$, $\vdash_0 \text{Prov}_{HA}(p, \forall A \rightarrow A)$, combining Lemma 1 and 2. $[HA^+]$ is merely an auxiliary. Corollary 1. $\varepsilon_0$ induction is not finitely axiomatisable over $HA$. Corollary 2. (Formal analogue to Brouwer’s bar theorem.) If $\vdash_0 HA \forall x A(x, s)$, $\forall x A(x, s)$ can be proved by cut free rules (Brouwer’s 'Elementarschlüsse', footnote 8 Math. Ann. 97 (1927) 60–75) together with his basic rules $\xi$ and $\zeta$ ('kanonisierte Form'). More specifically, there is a primitive recursive $P_\Lambda$ for which $\vdash_0 HA \forall x P_\Lambda(s(x))$ and
\[ \vdash_{HA} \{ P_A[\bar{s}(x)] \rightarrow A(x, x) \}. \]

Consequently, the set of unsecured sequences of \( P_A \) can be mapped recursively into the standard \( \varepsilon_0 \)-ordering, and the schema of transfinite induction w.r.t. these sequences can be proved in HA. **Corollary 3.** The results of Kleene's [JSL 27 (1962) 11-18] for closed formulae of HA extend to all formulae. (In fact, Lemma 1 affords a uniform method of extending metamathematical results for predicate logic to HA.) **Corollary 4.** If \( \vdash \) is the proof predicate for classical arithmetic \( Z \), \( \vdash_{\varepsilon_0} \{ \text{Prov}_Z(p, \{ A \}) \rightarrow A \} \) where \( \vdash_{\varepsilon_0} \) denotes provability by \( \varepsilon_0 \)-induction and classical logic. (Use G"odel's translation of A into negative form \( \neg A \); \( \vdash_{\varepsilon_0} \) \( \{ \text{Prov}_Z(p, \{ A \}) \rightarrow \text{Prov}_{HA}(\gamma(p), \{ A \}) \} \), hence \( \vdash_{\varepsilon_0} \{ x \rightarrow \neg A \} \); \( \vdash_{\varepsilon_0} \) \( \{ x \rightarrow A \} \); in fact, generally, metamathematical results for HA go over immediately to the classical system, but not conversely: HA is the better system for proof theory of first order arithmetic.) **Corollary 5.** The schema \( \{ \text{Prov}_{HA}(p, \{ A \}) \rightarrow A \} \) (for arbitrary formulae of HA including free variables) is equivalent over HA to the schema of transfinite induction on \( \varepsilon_0 \). (By the *Theorem*, the schema follows. Conversely, by Gentzen, *Math. Ann.* 119 (1943) 140-161, for each \( B \), there is a p.r. \( q(n) \), \( \vdash_{HA} \text{Prov}_Z[q(n), \{ A(0(n)) \}] \), where \( A(n) \) expressed transfinite induction applied to \( B \) up to \( \omega_n \) when \( \omega_0 = \omega \), \( \omega_{n+1} = \omega^\omega \); by the reflection principle, \( \exists A(n)(n) \), i.e. \( \varepsilon_0 \)-induction applied to \( B \).) Hence the limit of the autonomous progression based on HA and extended by the reflection principle is the first critical \( \varepsilon \)-number [cf. Feferman, *Notices A.M.S.* 6 (1961), 495]. (Received Dec. 31, 1963.)

G. KREISEL. **Reflection principles and \( \omega \)-consistency.**

For uniform treatment of classical and intuitionistic systems the proper formulation of \( \omega \)-consistency of a system (S) is this: with free variables \( n \) (over proofs) and \( \Gamma^\downarrow \) (over formulae): (*) \( \vdash \{ \text{Prov}_n, \{ \neg \Delta A(X) \} \rightarrow \neg \Delta x \forall \forall \text{Prov}_p, \{ A(0(x)) \} \} \). (Then the \( \omega \)-consistency of HA yields immediately the \( \omega \)-consistency of \( Z \); \( \vdash_{HA} \{ \forall x \forall \forall \forall \text{Prov}_n, \{ \forall x \forall \forall \forall \text{Prov}_p, \{ A(0(x)) \} \} \rightarrow \forall x \forall \forall \forall \text{Prov}_{HA}(p, \{ A(0(x)) \}) \} \) is weaker.)

**Theorem 1.** If the reflection principle for \( S \) can be proved in \( S_1 \) for the formulae \( \neg \Delta x A(x) \) (and \( A(p) \) with free variable \( p \), then (*) can be proved in \( S_1 \). Furthermore a proof of (*) is obtained primitive recursively from given proofs of the reflection principles. **Theorem 2.** Suppose (i) for a certain \( \pi(\Gamma^\downarrow) \subseteq \Gamma^\downarrow \), \( \sigma(\Gamma^\downarrow, x) \) being the canonical representations of \( \neg \Delta x A(x) \), \( \{ A(0(x)) \} \) respectively as functions of \( x \) and \( \Gamma^\downarrow \), \( \vdash_{HA} \{ \forall x \forall \forall \forall \text{Prov}_n, \{ A(n), \mu(0(n)) \} \rightarrow \neg \Delta x \forall \forall \forall \text{Prov}_p, \{ \sigma(0(n), x) \} \} \) with free variable \( a \), (ii) in \( S_2 \) the reflection principle for \( S_1 \) can be proved for the formula: \( \Lambda a \{ \text{Prov}_n, \{ a(n) \} \rightarrow \neg \Delta x \forall \forall \forall \text{Prov}_p, \{ p, a(n), x \} \} \). Then the \( \omega \)-consistency of \( S \) can be proved in \( S_2 \). (Proof by Theorem 1.) **Corollary 1.** The \( \omega \)-consistency of HA can be proved by \( \varepsilon_1 \) induction (first established by Parsons, *Archiv f. math. Logik u. Grundlagenforschung*, 6 (1962) 30-34). Take for \( S_1 \) the system \( U \cup HA_\omega \), and for \( S_2 \) the system \( HA_\omega \), proving the reflection principle for \( S_1 \) as in the preceding abstract for HA. Since the instructions for proving the reflection principle are quite explicit, condition (i) of Theorem 1 is satisfied not only in \( S_2 \), but in primitive recursive arithmetic. — A proof that the corollary is optimal is given in the review of Parson's paper in *Math. Revs.* 25 (1963) 746-747; it uses \( \vdash_{HA}(\omega-\text{Con} HA) \rightarrow (\omega-\text{Con} HA_\omega \rangle \) for each \( n \), which depends on the general result of Shoenfield:

\[ \vdash_{HA} \{ \Delta x \forall \forall \forall \text{Prov}_n(\{ y \}, \{ A(0(x)) \}) \} \wedge \omega-\text{Con} S \rightarrow \omega-\text{Con} (S \cup \{ \Delta x A(x) \}) \].

**Corollary 2.** For quantifier-free \( A(x) \), \( \vdash_{\varepsilon_0} \{ \text{Prov}_\omega[p, \{ \neg \Delta x A(x, m) \} \} \rightarrow \neg A[p, \{ m \}] \), where \( HA_\omega \) is \( HA_\omega \) extended by definition of functions by \( \varepsilon_0 \)-recursion. First, by the reflection principle, in \( HA_{\varepsilon_0} \rightarrow \neg \Delta x \forall \forall \forall \text{Prov}_\omega[p, \{ A(0(x)) \}] \). But by verifiability of quantifierfree \( A(x) \), \( \vdash_{HA} \{ \forall x \forall \forall \forall \text{Prov}_\omega[p, \{ A(0(x)) \}] \} \), and so \( \vdash_{\varepsilon_0} \{ \text{Prov}_\omega[p, \{ \neg \Delta x A(x, m) \} \} \rightarrow \neg \Delta x A(x, m) \}. \) In intuitionistic predicate logic, for quantifier-free \( P(x) \), if \( \vdash \neg \Delta x P(x) \) then also \( \forall x \neg P(x) \) (by use of cut free proofs). Extending this to infinite cut free proofs we have \( \vdash_{\varepsilon_0} \{ \forall x \forall \forall \forall \text{Prov}_\omega[p, \{ \neg \Delta x A(x, m) \} \} \rightarrow \neg A(x, m) \}. \) **Corollary 3.** (Quantifier-
fier-free formulation of $\omega$-consistency.} If $\vdash_{HA} \neg \Delta^0_1 A(x)$, there is a functional $F(x)$ defined by transfinite recursion on an ordering $< \varepsilon_1$, such that $\alpha[F(x)]$ is not a proof in HA of $A[F(x)]$. By Theorem 1, for some $n$, $\vdash_{HA} \neg \Delta^0_n \forall \varepsilon \exists \delta$ $\text{Prov}[\exists \varepsilon, \text{"}A(0^{(x)})\text{"}]$, hence $\vdash_{HA} \neg \Delta^0_n \text{Prov}[\varepsilon(x), \text{"}A(0^{(x)})\text{"}]$. Apply Corollary 2. — Among other relations one may note (i) $\omega$-consistency (for HA and similar systems) is equivalent to the reflection principle for 2 quantifier prenex formulae and their negations, but not for more (and the same applies to Z, because the negation of a prenex formula is provable in Z if and only if it is provable in HA), (ii) if for such a formula $A$, $\vdash_{HA} A$, then $\vdash_{HA} (\omega\text{-Con HA}) \rightarrow A$ because, by Shoenfield’s theorem $\vdash_{HA} [(\omega\text{-Con HA}) \rightarrow (\omega\text{-Con HA}^n)]$ and now apply the reflection principle of HA$_{\aleph_0}$, (iii) $\vdash_{HA} (\omega\text{-Con HA}) \rightarrow \text{Con}(U \text{ HA}^n)$ and, in fact, the reflection principle for one quantifier formulae and their negations for $U$ HA$_{\aleph_0}$.

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If $\mathfrak{A}$ is a well-ordering and $a$ is in the field of $\mathfrak{A}$, we write $|a|$ for the ordinal $a$ represents. Let $\Gamma$ be an ordinal and let $\mathfrak{A}$ be a well-ordering of the integers of type $\Gamma$; then a function $\varphi$ from $\Gamma^n$ to $\Gamma$ is said to be recursively representable on $\mathfrak{A}$ if there is a partial recursive function $f$ such that $|f(a_1, \ldots, a_n)| = \varphi(|a_1|, \ldots, |a_n|)$.

(If $\mathfrak{A}$ is a recursive well-ordering, then $f$ can be taken to be general recursive.)

**Theorem.** Let $\mathfrak{A}$, $\mathfrak{B}$ be recursive well-orderings of (the same) ordinal $\Gamma$. Then $\mathfrak{A}$ is recursively isomorphic to $\mathfrak{B}$ if any of the following holds:

a) $1 + \Theta$ is recursively representable on $\mathfrak{A}$ and $\mathfrak{B}$ and $\Gamma < \omega \cdot 2$,

b) addition is recursively representable on $\mathfrak{A}$ and $\mathfrak{B}$ and $\Gamma < \omega \omega$,

c) multiplication and addition are recursively representable on $\mathfrak{A}$ and $\mathfrak{B}$ and $\Gamma < \omega \omega$,

d) exponentiation and successor are recursively representable on $\mathfrak{A}$ and $\mathfrak{B}$ and $\Gamma < \varepsilon_{\omega_2}$,

e) (Kreisel) every ordinal $< \Gamma$ is representable by a term obtained from ordinal functions $\varphi_1, \ldots, \varphi_n$ and a finite number of ordinals $\Gamma_1, \ldots, \Gamma_m < \Gamma$ and the functions $\varphi_1, \ldots, \varphi_n$ are recursively representable on $\mathfrak{A}$ and $\mathfrak{B}$.

In a)—d), the bounds are the best possible for the corresponding functions.

Using definitions analogous to the above and calling a co-ordinal recursive if it contains a recursive well-ordering, (see J. N. Crossley, Constructive Order Types, I—IV, abstracts, this Journal), we have:

**Corollary:** The co-ordinals on which the functions in the theorem are recursively representable, are recursive and unique up to the bounds given in the theorem (and no further).

f) Let $f_\mu(x)$ be Schütte’s functions (abstract, meeting of the ASL, Oxford, 1963). Then the co-ordinals on which the function $f(\mu, x) = f_\mu(x)$ is recursively representable are recursive and unique for any ordinal $\kappa_n$ where $\kappa_\mu$ is the $n$-th strongly critical number. (Received Oct. 14, 1963.)