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Source: *The Journal of Symbolic Logic*, Vol. 27, No. 1 (Mar., 1962), pp. 19-52

Published by: [Association for Symbolic Logic](#)

Stable URL: <http://www.jstor.org/stable/2963676>

Accessed: 31/01/2011 22:06

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## THE PURE CALCULUS OF ENTAILMENT<sup>1</sup>

ALAN ROSS ANDERSON and NUEL D. BELNAP, JR.

The "implicational paradoxes" are treated by most contemporary logicians somewhat as follows:

"The two-valued propositional calculus sanctions as valid many of the obvious and satisfactory inferences which we recognize intuitively as valid, such as

$$(A \rightarrow .B \rightarrow C) \rightarrow .A \rightarrow B \rightarrow .A \rightarrow C,$$
<sup>2</sup>

and

$$A \rightarrow B \rightarrow .B \rightarrow C \rightarrow .A \rightarrow C;$$

it consequently suggests itself as a candidate for a formal analysis of implication. To be sure, there are certain odd theorems such as

$$A \rightarrow .B \rightarrow A$$

and

$$A \rightarrow .B \rightarrow B$$

which might offend the naive, and indeed these have been referred to in the literature as 'paradoxes of implication.' But this terminology reflects a misunderstanding. 'If A, then if B then A' really means no more than 'Either not-A, or else not-B or A,' and the latter is clearly a logical truth; hence so is the former. Properly understood, there are no 'paradoxes' of implication.

"Of course this is a rather weak sense of 'implication,' and one may for certain purposes be interested in a stronger sense of the word. We find a

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Received May 28, 1960.

<sup>1</sup> This research was supported in part by the Office of Naval Research, Group Psychology Branch, Contract No. SAR/Nonr-609(16). Permission is granted for reproduction, translation, publication, and disposal in whole or in part by or for the U. S. Government.

<sup>2</sup> Throughout we use the following notation. The sole primitive constants are  $\rightarrow$ ,  $(, )$ , and we assume that propositional variables are specified.  $A, B, C, D, \dots A_4 \dots$  are metalinguistic variables ranging over wffs, which are specified in the usual way. Parentheses are omitted under the conventions of Church, *Introduction to mathematical logic*: outermost parentheses are omitted; a dot may replace a left-hand parenthesis, the mate of which is to be restored at the end of the parenthetical part in which the dot occurs (otherwise at the end of the formula); otherwise parentheses are to be restored by association to the left. Example:  $(A \rightarrow .B \rightarrow C) \rightarrow .A \rightarrow B \rightarrow .A \rightarrow C$  abbreviates  $((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)))$ . We use axiom schemata, throughout, instead of a rule of substitution (even when discussing systems, e.g. Church 1951, which are formulated with a substitution rule); and we translate the notation of cited authors into our own.

formalization of a stronger sense in modal logics, where we consider strict implication, taking 'if A then B' to mean 'It is impossible that (A and not-B).' And, *mutatis mutandis*, some rather odd formulas are provable here too. But again nothing 'paradoxical' is going on; the matter just needs to be understood properly — that's all.

"And the weak sense of 'if ... then —' can be given formal clothing, after Tarski-Bernays as in Łukasiewicz 1929<sup>3</sup>, as follows.

$$A \rightarrow .B \rightarrow A$$

$$A \rightarrow B \rightarrow .B \rightarrow C \rightarrow .A \rightarrow C$$

$$A \rightarrow B \rightarrow A \rightarrow A$$

with a rule of *modus ponens*."

The position just outlined will be found stated in many places and by many people; we shall refer to it as the Official view. We agree with the Official view that there are no paradoxes of implication, but for reasons which are quite different from those ordinarily given. To be sure there is a misunderstanding involved, but it does not consist in the fact that the strict and material "implication" relations are "odd kinds" of implication, but rather in the claim that material and strict "implication" are "kinds" of implication at all. We will defend the view that material "implication" is *not* an implication relation in detail in the course of this paper, but it might help at the outset to give an example which will indicate the sort of criticism we plan to lodge.

Let us imagine a logician who offers the following formalization as an explication or reconstruction of implication in formal terms. In addition to the rule of *modus ponens* he takes as primitive the following three axioms:

$$A \rightarrow A$$

$$A \rightarrow B \rightarrow .B \rightarrow C \rightarrow .A \rightarrow C$$

$$A \rightarrow B \rightarrow .B \rightarrow A$$

One might find those who would object that "if ... then —" doesn't seem to be symmetrical, and that the third axiom is objectionable. But our logician has an answer to that. "There is nothing paradoxical about the third axiom; it is just a matter of understanding the formalism properly. 'If A then B' means simply 'Either A and B are both true, or else they are both false,' and if we understand the arrow in that way, then our rule will never allow us to infer a false proposition from a true one, and moreover all the axioms are evidently logical truths. The implication relations of this system may not *exactly* coincide with the intuitions of naive, untutored

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<sup>3</sup> See the list of references at the end of this paper.

folk, but it is quite adequate for my needs, and for the rest of us who are reasonably sophisticated. And it has the important property, common to all kinds of implication, of *never leading from truth to falsehood.*"

There are of course some differences between the situation just sketched and the Official view outlined above, but in point of perversity, muddle-headedness, and downright error, they seem to us entirely on a par. Of course proponents of the view that material and strict "implication" have something to do with implication have frequently apologized by saying that the name "material implication" is "somewhat misleading," since it suggests a closer relation with implication than actually obtains. But we can think of lots of no more "misleading" names for the relation: "material conjunction," for example, or "material disjunction," or "immaterial negation." "Material implication" is *not* a "kind" of implication, or so we hold; it is no more a kind of implication than a blunderbuss is a kind of buss.

This brief polemical blast will serve to set the tone for our subsequent investigations, which will concern the matter of trying to give a formal analysis of the notion of logical implication,<sup>4</sup> variously referred to also as "entailment", or "the converse of deducibility" (Moore 1920), expressed in such logical locutions as "if . . . then —," "implies," "entails," etc., and answering to such conclusion-signalling logical phrases as "therefore," "it follows that," "hence," "consequently," and the like. There have been numerous previous attempts to capture the notion, but we do not take it as our task here to survey the literature.<sup>5</sup> The exposition will rather be analytic, and we draw only on such material as is directly relevant to a point under discussion.

We shall proceed as follows. In section I we consider the implicational part  $H_I$  of the intuitionist propositional calculus, especially from the point of view of Fitch 1952. In sections II and III we argue that  $H_I$  is unsatisfactory as a formalization of entailment on two distinct counts, and in section IV we offer a system  $E_I$  (derived from Ackermann 1956) intended to capture the notion of implication or deducibility. Section V is devoted to showing that  $E_I$  is free of features which embarrass the other formal theories considered.

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<sup>4</sup> We should add, in a soberer tone, that though we would prefer to follow Kneale 1946 in substituting "Philonian junction" for "material implication," we are nevertheless prepared to award the word "implication" to the modern logical tradition. Indeed we are not in a position to do much else; the literature has a claim on the word stemming from Squatter's Rights. But for the purposes of this study we use it as a synonym for what has more recently been called "entailment," pointing out in justification that some authors *have* urged that "material implication" and "strict implication" have a close connection with the notion of a logically valid inference.

<sup>5</sup> Such surveys are essayed in Belnap 1960 and Bennett 1954.

## I

Since we wish to interpret " $A \rightarrow B$ "<sup>6</sup> as "A entails B," or "B is deducible from A," we clearly want to be able to *assert*  $A \rightarrow B$  whenever there exists a deduction of B from A; i.e., we will want a rule of "Entailment Introduction" (hereafter " $\rightarrow I$ ") having the property that if

$\begin{array}{c} \text{---} A \\ \vdots \\ B \end{array}$	hypothesis (hereafter "hyp")
$B$	[conclusion]

is a valid deduction of B from A, then  $A \rightarrow B$  shall follow from that deduction.

Moreover, the fact that such a deduction exists, or correspondingly that an entailment  $A \rightarrow B$  holds, warrants the *inference* of B from A. That is, we expect also that an Elimination Rule (henceforth " $\rightarrow E$ ") will obtain for  $\rightarrow$ , in the sense that whenever  $A \rightarrow B$  is asserted, we shall be entitled to infer B from A.

So much is simple and obvious, and presumably not open to question. Problems arise, however, when we ask what constitutes a "valid deduction" of B from A. How may we fill in the dots in the proof scheme above?

At least one rule seems as simple and obvious as the foregoing. Certainly the supposition A warrants the (trivial) inference that A; and if B has been deduced from A, we are entitled to infer B on the supposition A. That is, we may *repeat* ourselves:

1	$\begin{array}{c} \text{---} A \\ \vdots \end{array}$	hyp
<i>i</i>	$B$	?
<i>j</i>	$B$	<i>i</i> , repetition (henceforth "rep")
	$\vdots$	

This rule leads immediately to the following theorem, the *law of identity*:<sup>7</sup>

<sup>6</sup> We are taking entailment to be a relation between propositions, rather than sentences or statements, and with this understanding we will in the future observe a distinction between use and mention only where it seems essential.

<sup>7</sup>  $A \rightarrow A$  represents the archetypal form of inference, the trivial foundation of all reasoning, in spite of those who would call it "merely a case of stuttering." Strawson (1952, p. 15) says that

a man who repeats himself does not reason. But it is inconsistent to assert and deny the same thing. So a logician will say that a statement has to itself the relationship [entailment] he is interested in.

Strawson has got the cart before the horse: we hold that the reason A and  $\bar{A}$  are inconsistent is precisely because A follows from itself, rather than conversely. Note that in the system E of Anderson and Belnap 1958, we have  $A \rightarrow A \rightarrow \overline{A \wedge \bar{A}}$  but not  $\overline{A \wedge \bar{A}} \rightarrow A \rightarrow A$ , just as we have  $A \rightarrow B \rightarrow \overline{A \wedge \bar{B}}$  but not  $\overline{A \wedge \bar{B}} \rightarrow A \rightarrow B$ .

1	— A	hyp
2	— A	1, rep
3	A → A	1-2, →I

But obviously more is required if a theory of entailment is to be developed, and we therefore consider initially a device contained in the variant of natural deduction due to Fitch 1952, which allows us to construct *within* proofs of entailment, further proofs of entailment called “subordinate proofs,” or “subproofs.” In the course of a deduction, under the supposition that A (say), we may begin a new deduction, with a new hypothesis:

1	— A	hyp
	⋮	
<i>i</i>	— B	hyp
	⋮	

The new subproof is to be conceived of as an “item” of the proof of which A is the hypothesis, entirely on a par with A and any other propositions occurring in that proof. And the subproof of which B is hypothesis might itself have a consequence (by →I) occurring in the proof of which A is the hypothesis.

We next ask whether or not the hypothesis A holds also under the assumption B. In the system of Fitch 1952, the rules are so arranged that (for example) the hypothesis A may also be repeated under the assumption that B, such a repetition being called a “reiteration” to distinguish it from repetitions within the same proof or subproof:

1	— A	hyp
	⋮	
<i>i</i>	— B	hyp
	⋮	
<i>j</i>	A	1 reiteration (“reit”)
	⋮	

We designate as  $H_I^*$  the system defined by the five rules, →I, →E, hyp, rep, and reit. A proof is *categorical* if all hypotheses in the proof have been discharged by use of →I; and A is a *theorem* if A is the last step of a categorical proof. These rules lead naturally and easily to proofs of intuitively satisfactory theorems about entailment, such as the following *law of transitivity*<sup>8</sup>:

<sup>8</sup> Lewis (Lewis and Langford 1932, p. 496) doubts whether this proposition should be regarded as a valid principle of deduction: it would never lead to any inference  $A \rightarrow C$  which would be questionable when  $A \rightarrow B$  and  $B \rightarrow C$  are given premisses; but it gives the inference  $B \rightarrow C \rightarrow A \rightarrow C$  whenever  $A \rightarrow B$  is a premiss. Except as an elliptical statement for “ $(A \rightarrow B) \wedge (B \rightarrow C) \rightarrow A \rightarrow C$  and  $A \rightarrow B$  is true,” this inference is dubious.

1	— A → B		hyp
2	— B → C		hyp
3	— A → B		1 reit
4	— A		hyp
5	— A → B		3 reit
6	B		4 5 →E
7	B → C		2 reit
8	C		6 7 →E
9	A → C		4-8 →I
10	B → C → A → C		2-9 →I
11	A → B → B → C → A → C		1-10 →I

The proof method also has the advantage, in common with other systems of natural deduction, of motivating proofs: in order to prove  $A \rightarrow B$ , (perhaps under some hypothesis or hypotheses) we follow the simple and obvious strategy of playing both ends against the middle: breaking up the conclusion to be proved, and setting up subproofs by hyp until we find one with a variable as last step. Only then do we begin applying reit, rep, and  $\rightarrow E$ .

Our description of  $H_I^*$  has been somewhat informal, and for the purpose of checking proofs it would be desirable to have a more rigorous formulation. We think of the following as designed primarily for *testing* rather than *constructing* proofs (though of course the distinction is heuristic rather than logical), and we therefore call this the *Test Formulation* of Fitch's system.

In order to motivate the Test Formulation, we notice that with each  $i$ -th step  $A_i$  in the proofs above there is associated first a number of vertical lines to the left of  $A_i$  (which we shall call the *rank* of  $A_i$ ), secondly a class of formulas (including  $A_i$ ) which are candidates for application of the rule of repetition to yield a next step for the proof, thirdly a class of candidates for application of the rule of reiteration, to yield the next step, and fourthly a hypothesis (if the proof has one) which may together with the final step of the deduction furnish an entailment as next step, as a consequence of the deduction. Accordingly we define a *proof* as consisting of a sequence  $A_1, \dots, A_n$  of wffs, not necessarily distinct, for each  $A_i$  of which is defined a *rank*  $R(A_i)$ , a class of *repeatable wffs*  $Rep(A_i)$ , a class of *reiteratable wffs*  $Reit(A_i)$ , and (if  $R(A_i) > 0$ ) an *immediate hypothesis*  $H(A_i)$ . These are all defined by simultaneous induction as follows:

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On the contrary, Ackermann 1956 is surely right that "unter der Voraussetzung  $A \rightarrow B$  ist der Schluss von  $B \rightarrow C$  auf  $A \rightarrow C$  logisch zwingend." The mathematician is involved in no ellipsis in arguing that "if the lemma is deducible from the axioms, then this entails that the deducibility of the theorem from the axioms is entailed by the deducibility of the theorem from the lemma."

- Basis: (a)  $A_1$  is any wff.  
 (b)  $R(A_1) = 1$ .  
 (c)  $\text{Rep}(A_1) = \{A_1\}$ .  
 (d)  $\text{Reit}(A_1) = \wedge$ .  
 (e)  $H(A_1) = A_1$ .

Now suppose we have  $A_j$ ,  $R(A_j)$ ,  $\text{Rep}(A_j)$ ,  $\text{Reit}(A_j)$ , and  $H(A_j)$ , for every  $j < k$ . Then  $A_1, \dots, A_k$  is a proof provided  $A_1, \dots, A_{k-1}$  is a proof and  $A_k$  satisfies one of the following five conditions:

- (1) (hyp) (a)  $A_k$  is any wff.  
 (b)  $R(A_k) = R(A_{k-1}) + 1$   
 (c)  $\text{Rep}(A_k) = \{A_k\}$   
 (d)  $\text{Reit}(A_k) = \text{Reit}(A_{k-1})$   
 (e)  $H(A_k) = A_k$
- (2) (rep) If  $A_j$  is in  $\text{Rep}(A_{k-1})$ , then  
 (a)  $A_k = A_j$   
 (b)  $R(A_k) = R(A_{k-1})$   
 (c)  $\text{Rep}(A_k) = \text{Rep}(A_{k-1}) \cup \{A_k\}$   
 (d)  $\text{Reit}(A_k) = \text{Reit}(A_{k-1})$   
 (e)  $H(A_k) = H(A_{k-1})$  (if  $H(A_{k-1})$  is defined; otherwise  $H(A_k)$  is undefined)
- (3) (reit) If  $A_j$  is in  $\text{Reit}(A_{k-1})$ , then  
 (a)  $A_k = A_j$   
 (b)–(e) as in (2)
- (4) ( $\rightarrow E$ ) If  $A_j$  is in  $\text{Rep}(A_{k-1})$  and  $A_j \rightarrow B (= A_i)$  is in  $\text{Rep}(A_{k-1})$ , then  
 (a)  $A_k = B$   
 (b)–(e) as in (2)
- (5) ( $\rightarrow I$ ) (a)  $A_k = H(A_{k-1}) \rightarrow A_{k-1}$   
 (b)  $R(A_k) = R(A_{k-1}) - 1$   
 (c)  $\text{Rep}(A_k) = \text{Reit}(A_{k-1}) \cup \{A_k\}$   
 (d) If  $R(A_k) > 0$ , then  $\text{Reit}(A_k) = \text{Reit}(A_j)$ , where  $H(A_{k-1}) = A_{j+1}$ ; and if  $R(A_k) = 0$ ,  $\text{Reit}(A_k) = \wedge$ .  
 (e) If  $R(A_k) > 0$ , then  $H(A_k) = H(A_j)$ , where  $H(A_{k-1}) = A_{j+1}$ ; and if  $R(A_k) = 0$ ,  $H(A_k)$  is undefined.

Then  $A$  is a *theorem* if there is a proof in which  $A$  has rank zero.

The basis clause and rule (1) enable us to begin new deductions, and (2)–(5) correspond to rep, reit,  $\rightarrow E$  and  $\rightarrow I$  respectively. If we use a sequence

of  $n$  vertical strokes to represent a rank of  $n$ , we may arrange proofs as follows:

	$R(A_i)$	Step	Rule	$Rep(A_i)$	$Reit(A_i)$	$H(A_i)$
$A_1$ :		$A \rightarrow B$	hyp	$\{A_1\}$	$\wedge$	$A_1$
$A_2$ :		$B \rightarrow C$	hyp	$\{A_2\}$	$\{A_1\}$	$A_2$
$A_3$ :		$A \rightarrow B$	$A_1$ reit	$\{A_{2-3}\}$	$\{A_1\}$	$A_2$
$A_4$ :		$A$	hyp	$\{A_4\}$	$\{A_{2-3}\}$	$A_4$
$A_5$ :		$A \rightarrow B$	$A_3$ reit	$\{A_{4-5}\}$	$\{A_{2-3}\}$	$A_4$
$A_6$ :		$B$	$A_{4-5} \rightarrow E$	$\{A_{4-6}\}$	$\{A_{2-3}\}$	$A_4$
$A_7$ :		$B \rightarrow C$	$A_2$ reit	$\{A_{4-7}\}$	$\{A_{2-3}\}$	$A_4$
$A_8$ :		$C$	$A_{6-7} \rightarrow E$	$\{A_{4-8}\}$	$\{A_{2-3}\}$	$A_4$
$A_9$ :		$A \rightarrow C$	$A_{4-8} \rightarrow I$	$\{A_{2-3}, A_9\}$	$\{A_1\}$	$A_2$
$A_{10}$ :		$B \rightarrow C \rightarrow .A \rightarrow C$	$A_{2-9} \rightarrow I$	$\{A_1, A_{10}\}$	$\wedge$	$A_1$
$A_{11}$ :		$A \rightarrow B \rightarrow .B \rightarrow C \rightarrow .A \rightarrow C$	$A_{1-10} \rightarrow I$	$\{A_{11}\}$	$\wedge$	—

Then if we connect the lines indicating rank, we have a format that looks much like the proof of the law of transitivity given earlier.

The foregoing formulation makes explicit the techniques involved in constructing subproofs, and similar formulations may be found for other systems to be considered subsequently. But making the procedure explicit leads to some loss as regards intuitive naturalness and obviousness, and we shall in the sequel continue to use the rather less formal approach with which we began, relying on the reader to see that the whole matter could be discussed more rigorously, along the lines just indicated.

As a further simplification we allow reiterations directly into subsubproofs, etc., with the understanding that a complete proof requires that reiterations be performed always from one proof into another proof immediately subordinate to it. As an example (step 6 below), we prove the *self-distributive law* ( $H_{I2}$ , below):

1		$A \rightarrow .B \rightarrow C$	hyp
2		— $A \rightarrow B$	hyp
3		— $A$	hyp
4		— $A \rightarrow B$	2 reit
5		$B$	3 4 $\rightarrow E$
6		$A \rightarrow .B \rightarrow C$	1 reit
7		$B \rightarrow C$	3 6 $\rightarrow E$
8		$C$	5 7 $\rightarrow E$
9		$A \rightarrow C$	3-8 $\rightarrow I$
10		$A \rightarrow B \rightarrow .A \rightarrow C$	2-9 $\rightarrow I$
11		$(A \rightarrow .B \rightarrow C) \rightarrow .A \rightarrow B \rightarrow .A \rightarrow C$	1-10 $\rightarrow I$

Fitch 1952 shows (essentially) that the set of theorems of  $H_I^*$  stemming from these rules is identical with the pure implicational fragment  $H_I$  of the intuitionist propositional calculus of Heyting 1930, which consists of the following two axioms, with  $\rightarrow E$  as the sole rule:

$$H_{I1}. \quad A \rightarrow .B \rightarrow A$$

$$H_{I2}. \quad (A \rightarrow .B \rightarrow C) \rightarrow .A \rightarrow B \rightarrow .A \rightarrow C$$

In order to introduce terminology and to exemplify a pattern of argument which we shall have further occasion to use, we shall reproduce Fitch's proof that the two formulations are equivalent.

To see that the subproof formulation  $H_I^*$  contains the axiomatic formulation  $H_I$ , we deduce the axioms of  $H_I$  in  $H_I^*$  ( $H_{I2}$  was just proved and  $H_{I1}$  is proved on p. 28 below) and then observe that the only rule of  $H_I$  is also a rule of  $H_I^*$ . It follows that  $H_I^*$  contains  $H_I$ .

To see that the axiomatic system  $H_I$  contains the subproof formulation  $H_I^*$ , we first introduce the notion of a *quasi-proof* in  $H_I^*$ ; a quasi-proof differs from a proof only in that we may introduce axioms or theorems of  $H_I$  as steps (and of course use these, and steps derived from them, as premisses for rep, reit, or  $\rightarrow E$ ). Clearly this does not increase the stock of theorems of  $H_I^*$ , since we may think of a step  $A$ , inserted under this rule, as coming by reiteration from a previous proof of  $A$  in  $H_I^*$  (which we know exists since  $H_I^*$  contains  $H_I$ ).

Our object then is to show how subproofs in a quasi-proof in  $H_I^*$  may be systematically eliminated in favor of theorems of  $H_I$  and uses of  $\rightarrow E$ , in such a way that we are ultimately left with a sequence of formulas all of which are theorems of  $H_I$ . This reduction procedure always begins with an *innermost subproof*, by which we mean a subproof  $Q$  which has no proofs subordinate to it. Let  $Q$  be an innermost subproof of a quasi-proof  $P$  of  $H_I^*$ , where the steps of  $Q$  are  $A_1, \dots, A_n$ , let  $Q'$  be the sequence  $A_1 \rightarrow A_1, A_1 \rightarrow A_2, \dots, A_1 \rightarrow A_n$ , and let  $P'$  be the result of replacing the subproof  $Q$  of  $P$  by the sequence  $Q'$  of wffs. Our task is now to show that  $P'$  is a quasi-proof, by showing how to insert theorems of  $H_I$  among the wffs of  $Q'$ , in such a way that each step of  $Q'$  may be justified by reit, rep, or  $\rightarrow E$ . First we need a theorem of  $H_I$ :

- |  |                               |
|--|-------------------------------|
| 1. $(A \rightarrow .B \rightarrow A \rightarrow A) \rightarrow .(A \rightarrow .B \rightarrow A) \rightarrow .A \rightarrow A$ | $H_{I2}$                      |
| 2. $A \rightarrow .B \rightarrow A \rightarrow A$  | $H_{I1}$                      |
| 3. $A \rightarrow .B \rightarrow A$  | $H_{I1}$                      |
| 4. $A \rightarrow A$   | 1 2 3 $\rightarrow E$ (twice) |

Then an inductive argument shows that we may justify steps in  $Q'$  as follows:

$A_1 \rightarrow A_1$  is justified by the theorem above.

If  $A_i$  was by rep in  $Q$ , then  $A_1 \rightarrow A_i$  is by rep in  $Q'$ .

If  $A_i$  was by reit in  $Q$ , then in  $Q'$  insert  $A_i \rightarrow .A_1 \rightarrow A_i$  ( $H_{I1}$ ) and use  $\rightarrow E$  to get  $A_1 \rightarrow A_i$  (the minor premiss being an item of the quasi-proof in  $P$  to which  $Q$  is subordinate, hence also preceding  $Q'$  in  $P$ ).

If  $A_i$  was by  $\rightarrow E$  in  $Q$ , with premisses  $A_j$  and  $A_j \rightarrow A_i$ , then in  $Q'$  we have  $A_1 \rightarrow A_j$  and  $A_1 \rightarrow .A_j \rightarrow A_i$ . Then insert  $H_{I2}$  and use  $\rightarrow E$  twice to get  $A_1 \rightarrow A_i$  as required.<sup>9</sup>

Finally, the step  $A_1 \rightarrow A_n$ , treated as a consequence of  $Q$  in  $P$ , may be treated in  $Q'$  as a consequence by rep of the immediately preceding step  $A_1 \rightarrow A_n$ .

Repeated application of this reduction then converts any proof in  $H_I^*$  into a sequence of formulas all of which are theorems of  $H_I$ ; hence the latter system contains the former, and the two are equivalent. Notice incidentally that the choice of axioms for  $H_I$  may be thought of as motivated by a wish to prove  $H_I$  and  $H_I^*$  equivalent: they are *exactly* what is required to carry out the inductive argument above.

(We retain the terminology *quasi-proof* and *innermost subproof* for use in later arguments which are closely similar to the foregoing.)

The axioms of  $H_I$  also enable us to prove a slightly different form of the result above. We consider proofs with no subproofs, but with multiple hypotheses, and we define a *proof of B from hypotheses  $A_1, \dots, A_n$*  (in the Official way) as a sequence  $S_1, \dots, S_m, B$  of wffs, each of which is either an axiom, or one of the hypotheses  $A_i$ , or a consequence of predecessors by  $\rightarrow E$ . Then we arrive by very similar methods at the Official form of the *Deduction Theorem*: if there exists a proof of  $B$  from the hypotheses  $A_1, \dots, A_n$ , then there exists a proof of  $A_n \rightarrow B$  on the hypotheses  $A_1, \dots, A_{n-1}$ ; and conversely.

We return now to consideration of  $H_{I1}$ , which is proved in  $H_I^*$  as follows:

1	— A	hyp
2	— B	hyp
3	— A	1 reit
4	B $\rightarrow$ A	2-3 $\rightarrow I$
5	A $\rightarrow$ .B $\rightarrow$ A	1-4 $\rightarrow I$

Thus far the theorems proved by the subordinate proof method have all seemed natural and obvious truths about our intuitive idea of *entailment*. But here we come upon a theorem which shocks our intuitions (at least our untutored intuitions), for the theorem seems to say that anything whatever has  $A$  as a logical consequence, provided only that  $A$  is true.

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<sup>9</sup> And of course at some stage of the reduction  $A_i$  may be a theorem of  $H_I$  inserted at a previous stage of the reduction, in which case we simply regard it as an insertion in  $P'$ . Here and elsewhere we assume this step tacitly.

To be sure, there is an interpretation<sup>10</sup> we could place on the arrow which would make  $A \rightarrow B \rightarrow A$  true, but if the formal machinery is offered as an analysis or reconstruction of the notion of entailment, or formal deducibility, the principle seems outrageous, — such at least is almost certain to be the initial reaction to the theorem, as anyone who has taught elementary logic very well knows. Such theorems as  $A \rightarrow B \rightarrow A$  and  $A \rightarrow B \rightarrow B$  are of course familiar, and much discussed under the heading of “implicational paradoxes.” The commonest attitude among logicians (as was remarked at the outset) is simply to accept the paradoxes as in some sense true, and at the same time to recognize that the formalism doesn’t *quite* capture our intuitive use of “if . . . then —” (such in fact being our own attitude until the late fall of 1958). In what follows we will try to analyze the sources of the discomfort occasioned by such “implicational” paradoxes, and in the course of the argument offer reasons for believing that  $H_I$  simply doesn’t capture a notion of implication or entailment at all, or at least no more than (say) an equivalence relation captures the notion of “if . . . then —.” True, equivalence relations and entailment are both transitive and reflexive, but this similarity is not sufficient to enable us to *identify* entailment with any equivalence relation, since entailment is not symmetrical, and to say so is to make a false statement. Likewise, entailment and the connective of  $H_I$  are both transitive and reflexive, but again this similarity is not sufficient to enable us to identify implication or entailment with the arrow of  $H_I$ , since in  $H_I$  one can prove  $A \rightarrow B \rightarrow A$ , which is false of entailment and its converse, formal deducibility. Such a view is of course heterodox in logical circles, and will be rejected by many. Curry 1959, for example, has argued (in four closely-reasoned pages) that  $A \rightarrow B \rightarrow A$ , “far from being paradoxical,” is,

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<sup>10</sup> Myhill 1953 has shown that in  $H_I$   $A \rightarrow B$  is interdeducible with  $(\exists v)(v \wedge ((v \wedge A) \rightarrow B))$ , under plausible assumptions about strict implication. In a forthcoming paper, Belnap proves that in the system E of Anderson and Belnap 1958, with propositional quantifiers added, intuitionistic implication  $A \supset B$  may be defined as  $(\exists v)(v \wedge ((v \wedge A) \rightarrow B))$ , and intuitionistic negation  $\neg A$  as  $A \supset (p)p$ . Some may therefore wish to regard intuitionistic implication as a kind of enthymematic implication, as when we say that it “follows” from the assertion that Socrates is a man, that he is also mortal. But this is *not* a relation of deducibility or entailment; the latter holds only when all required premisses are stated explicitly in the antecedent. We agree (in spirit) with the classical position regarding enthymemes, as stated e.g. by Joseph 1925:

An enthymeme indeed is not a particular form of argument, but a particular way of stating an argument. The name is given to a syllogism with one premiss — or, it may be, the conclusion — suppressed. Nearly all syllogisms are, as a matter of fact, stated as enthymemes, except in the examples of a logical treatise, or the conduct of a formal disputation. It must not be supposed, however, that we are the less arguing in syllogism, because we use one member of the argument without its being explicitly stated.

We note in passing that the principle of exportation has the effect of confusing valid arguments with enthymemes.

for any *proper* implication, “a platitude.” A “proper” implication is defined by Curry as any implication which has the following properties: there is a proof of  $B$  from the hypotheses  $A_1, \dots, A_{n-1}, A_n$  (in the Official sense of “proof from hypotheses”) if and only if there is a proof of  $A_n \rightarrow B$  from the hypotheses  $A_1, \dots, A_{n-1}$ . On these grounds  $A \rightarrow B \rightarrow A$  is indeed a platitude: there is surely a proof of  $A$  from the hypotheses  $A, B$ ; and hence for any “proper” implication, a proof of  $B \rightarrow A$  from the hypothesis  $A$ ; and hence a proof without hypotheses of  $A \rightarrow B \rightarrow A$ .<sup>11</sup>

Curry goes on to dub the implicational relation of  $H_I$  “absolute implication” on the grounds that  $H_I$  is the minimal system having this property. But we notice at once that  $H_I$  is “absolute” only relatively, i.e., relatively to the Official definition of “proof from hypotheses.” From this point of view, our remarks to follow may be construed as arguing the impropriety of accepting the Official definition of “proof from hypotheses,” as a basis for defining a “proper implication”; as we shall claim, the Official view captures neither “proof” (a matter involving logical necessity), nor “from” (a matter requiring relevance). But even those with intuitions so sophisticated that  $A \rightarrow B \rightarrow A$  seems tolerable might still find some interest in an attempt to analyze our initial feelings of repugnance in its presence.

Why does  $A \rightarrow B \rightarrow A$  seem so queer? We believe that its oddness is due to two isolable features of the principle, which we consider forthwith.

## II

**Necessity.** For more than two millenia logicians have taught that logic is a *formal* matter, and that the validity of an inference depends not on material considerations, but on formal considerations alone. The companion view that the validity of a valid inference is no accident of nature, but rather a property a valid inference has necessarily, has had an equally illustrious history. But both of these conditions are violated if we take the arrow of  $H_I$  to express implication. For if  $A$  is contingent, then  $A \rightarrow B \rightarrow A$  says that an entailment  $B \rightarrow A$  follows from or is deducible from a contingent proposition — in defiance of the condition that formal considerations *alone* validate valid inferences. And if  $A$  should be a true contingent proposition, then  $B \rightarrow A$  is also contingently true, and an entailment is established as holding because of an accident of nature.

It has been said in defence of  $A \rightarrow B \rightarrow A$  as an entailment that at least it is “safe,” in the sense that if  $A$  is true, then it is always safe to infer  $A$

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<sup>11</sup> Curry calls this a proof of  $A \rightarrow B \rightarrow A$  “from nothing.” We remark that this expression invites the interpretation “there is nothing from which  $A \rightarrow B \rightarrow A$  is deducible,” in which case we would seem to have done little toward showing that it is true. But of course Curry is not confused on this point; he means that  $A \rightarrow B \rightarrow A$  is deducible “from” the null set of premisses — in the reason-shattering, Official sense of “from.”

from an arbitrary B, since we run no risk of uttering a falsehood in doing so; this thought ("Safety First") seems to be behind attempts, in a number of elementary logic texts, to justify the claim that  $A \rightarrow .B \rightarrow A$  has something to do with implication. In reply we of course admit that if A is true then it is "safe" to say so (i.e.,  $A \rightarrow A$ ). But saying that A is true on the irrelevant assumption that B, is *not* to deduce A from B, nor to establish that B implies A, in *any* sensible sense of "implies." Of course we *can* say "Assume that snow is puce. Seven is a prime number." But if we say "Assume that snow is puce. *It follows that* (or *consequently*, or *therefore*, or *it may validly be inferred that*) seven is a prime number," then we have simply spoken falsely. A man who assumes the continuum hypothesis, and then remarks that it is a nice day, is not inferring the latter from the former, — even if he keeps his supposition fixed firmly in mind while noting the weather. And since a (true) A does not follow from an (arbitrary) B, we reject  $A \rightarrow .B \rightarrow A$  as expressing a truth of entailment or implication, a rejection which is in line with the view (often expressed even by those who hold that  $A \rightarrow .B \rightarrow A$  expresses a fact about implication or entailment or "if ... then —") that entailments, if true at all, are necessarily true.

How can we modify the formulation of  $H_I$  in such a way as to guarantee that the logical truths expressible in it shall be necessary, rather than contingent? As a start, we might reflect that in our usual mathematical or logical proofs, we demand that all the conditions required for the conclusion be stated in the hypothesis of a theorem. After the word "*Proof:*" in a mathematical treatise, mathematical writers seem to feel that no more hypotheses may be introduced — and it is regarded as a criticism of a proof if not all the required hypotheses are stated explicitly at the outset. Of course additional machinery *may* be invoked in the proof, but this must be of a logical character, i.e., in addition to the hypotheses, we may use only logically necessary propositions in the argument. These considerations suggest that we should be allowed to import into a deduction (i.e., into a subproof by *reit*) only propositions which, if true at all, are necessarily true: i.e., we should reiterate only entailments. And indeed such a restriction on reiteration would immediately rule out  $A \rightarrow .B \rightarrow A$  as a theorem, while countenancing all the other theorems we have proved thus far. We call the system with reiteration so restricted  $S_{4I}^*$ , and proceed to prove it equivalent to the following axiomatic formulation, which we call  $S_{4I}$ , since it is the pure strict "implicational" fragment of Lewis's  $S_4$ .<sup>12</sup>

<sup>12</sup> This was conjectured by Lemmon *et al.* 1956, and is verifiable by a *sequenzenkalkül* formulation of  $S_4$  got by adding standard Gentzen rules for negation and disjunction to the formulation of  $S_{4I}$  of Kripke 1960. Formulations of this kind are also available for the system  $E_I$  discussed below; see Belnap 1960 and Kripke 1960. Kripke's formulation is simpler, and alone leads to a decision procedure for  $E_I$ , while Belnap's formulation allows a more direct interpretation with respect to  $E_I$ .

S4<sub>I</sub>1.  $A \rightarrow A$

S4<sub>I</sub>2.  $(A \rightarrow .B \rightarrow C) \rightarrow .A \rightarrow B \rightarrow .A \rightarrow C$

S4<sub>I</sub>3.  $A \rightarrow B \rightarrow .C \rightarrow .A \rightarrow B$

It is a trivial matter to prove the axioms of S4<sub>I</sub> in S4<sub>I</sub>\*, and the only rule of S4<sub>I</sub> ( $\rightarrow E$ ) is also a rule of S4<sub>I</sub>\*; hence S4<sub>I</sub>\* contains S4<sub>I</sub>. To establish the converse, we show how to convert any quasi-proof of a theorem  $A$  in S4<sub>I</sub>\* into a proof of  $A$  in S4<sub>I</sub>.

**THEOREM.** Let  $A_1, \dots, A_n$  be the propositional items of an innermost subproof  $Q$  of a quasi-proof  $P$ , and let  $Q'$  be the sequence  $A_1 \rightarrow A_1, \dots, A_1 \rightarrow A_n$ , and finally let  $P'$  be the result of replacing the subproof  $Q$  in  $P$  by the sequence of propositions  $Q'$ . Then  $P'$  is a quasi-proof.

*Proof.* By induction on  $n$ . For  $n = 1$  we note that  $A_1 \rightarrow A_1$  is an instance of S4<sub>I</sub>1. Then assuming the theorem for all  $i < n$ , consider  $A_1 \rightarrow A_n$ .

Case 1.  $A_n$  is by repetition in  $Q$  of  $A_i$ . Then treat  $A_1 \rightarrow A_n$  in  $Q'$  as a repetition of  $A_1 \rightarrow A_i$ .

Case 2.  $A_n$  is a reiteration in  $Q$  of  $B$ . Then  $B$  has the form  $C \rightarrow D$ , by the restriction on reiteration. Then insert  $C \rightarrow D \rightarrow .A_1 \rightarrow .C \rightarrow D$  in  $Q'$  by S4<sub>I</sub>3, and treat  $A_1 \rightarrow .C \rightarrow D$  (i.e.,  $A_1 \rightarrow A_n$ ) as a consequence of  $C \rightarrow D$  (i.e.,  $B$ ), and S4<sub>I</sub>3 by  $\rightarrow E$ .

Case 3.  $A_n$  follows in  $Q$  from  $A_i$  and  $A_i \rightarrow A_n$  by  $\rightarrow E$ . Then by the inductive hypothesis we have  $A_1 \rightarrow A_i$  and  $A_1 \rightarrow .A_i \rightarrow A_n$  in  $Q'$ . Then  $A_1 \rightarrow A_n$  is a consequence of the latter and S4<sub>I</sub>2, with two uses of  $\rightarrow E$ .

Finally, the step  $A_1 \rightarrow A_n$ , regarded as a consequence of  $Q$  in  $P$ , may now be regarded as a repetition of the final step  $A_1 \rightarrow A_n$  of  $Q'$  in  $P'$ .

Hence  $P'$  is a quasi-proof. And repeated application of this technique to  $P'$  eventually leads to a sequence  $P''$  of wffs, each of which is a theorem of S4<sub>I</sub>. Hence S4<sub>I</sub> includes S4<sub>I</sub>\*, and the two are equivalent.

A deduction theorem of the more usual sort is provable also for S4<sub>I</sub>:

**THEOREM.** If there is a proof of  $B$  on hypotheses  $A_1, \dots, A_n$ , (in the Official sense), where each  $A_i$ ,  $1 \leq i \leq n$ , has the form  $C \rightarrow D$ , then there is a proof of  $A_n \rightarrow B$  on hypotheses  $A_1, \dots, A_{n-1}$ . (Barcan Marcus 1946; see also Kripke 1960.)

(We remark that a Test Formulation of S4<sub>I</sub> arises if we add to the "if" clause of (3) the requirement that  $A_j$  be of the form  $B \rightarrow C$ . Notice again that the choice of S4<sub>I</sub>1–3 may be thought of as motivated exactly by the wish to prove an appropriate deduction theorem.)

The restriction on reiteration then suffices to remove one objectionable feature of H<sub>I</sub>, since it is now no longer possible to establish an entailment  $B \rightarrow A$  on the (perhaps contingent) ground that  $A$  is simply true. But of course it is well known that the "implication" relation of S4 is also paradoxical, since we can easily establish that an arbitrary irrelevant proposition  $B$  "implies"  $A$ , provided  $A$  is a necessary truth.  $A \rightarrow A$  is necessarily true,

and from it and S4<sub>13</sub> follows  $B \rightarrow A \rightarrow A$ , where  $B$  may be totally irrelevant to  $A \rightarrow A$ . This defect leads us to consider an alternative restriction on  $H_1$ , designed to exclude such fallacies of relevance.

### III

**Relevance.** For more than two millenia logicians have taught that a necessary condition for the validity of an inference from  $A$  to  $B$  is that  $A$  be relevant to  $B$ . Virtually every logic book up to the present century has a chapter on fallacies of relevance, and many contemporary elementary texts have followed the same plan. (Notice that contemporary writers, in the later and more formal chapters of their books, seem explicitly to contradict the earlier chapters, when they try desperately to con the students into accepting material or strict "implication" as a "kind" of implication relation, in spite of the fact that these relations countenance fallacies of relevance.) But the denial that relevance is essential to a valid argument (a denial which is implicit in the view that strict "implication" is an implication relation) seems to us flatly in error.

Imagine, if you can, a situation as follows. A mathematician writes a paper on Banach spaces, and after proving a couple of theorems he concludes with a conjecture. As a footnote to the conjecture, he writes: "In addition to its intrinsic interest, this conjecture has connections with other parts of mathematics which might not immediately occur to the reader. For example, if the conjecture is true, then the first order functional calculus is complete; whereas if it is false, then it implies that Fermat's last conjecture is correct." The editor replies that the paper is obviously acceptable, but he finds the final footnote perplexing; he can see no connection whatever between the conjecture and the "other parts of mathematics," and none is indicated in the footnote. So the mathematician replies, "Well, I was using 'if . . . then —' and 'implies' in the way that logicians have claimed I was: the first order functional calculus *is* complete, and necessarily so, so anything implies that fact — and if the conjecture is false it is presumably impossible, and hence implies anything. And if you object to this usage, it is simply because you have not understood the technical sense of 'if . . . then —' worked out so nicely for us by logicians." And to this the editor counters: "I *understand* the technical bit all right, but it is simply not correct. In spite of what most logicians say about us, the standards maintained by this journal require that the antecedent of an 'if . . . then —' statement must be *relevant* to the conclusion drawn. And you have given no evidence that your conjecture about Banach spaces is relevant either to the completeness theorem or to Fermat's conjecture."

The editor's point is of course that though the technical meaning is clear, it is simply not the same as the meaning ascribed to "if . . . then —"

in the pages of his journal (nor, we suspect, in the pages of this JOURNAL). Furthermore, he has put his finger precisely on the difficulty: to argue from the necessary truth of  $A$  to *if B then A* is simply to commit a fallacy of relevance. The fancy that relevance is irrelevant to validity strikes us as ludicrous, and we therefore make an attempt to explicate the notion of relevance of  $A$  to  $B$ .

For this we return to the notion of proof from hypotheses, the leading idea being that we want to infer  $A \rightarrow B$  from "a proof of  $B$  from the hypothesis  $A$ ." As we pointed out before, in the usual axiomatic formulations of propositional calculi the matter is handled as follows. We say that  $A_1, \dots, A_n$  is a proof of  $B$  from the hypothesis  $A$ , if  $A = A_1$ ,  $B = A_n$ , and each  $A_i$  ( $i > 1$ ) is either an axiom or else a consequence of predecessors among  $A_1, \dots, A_n$  by one of the rules. But in the presence of a deduction theorem of the form: from a proof of  $B$  on the hypothesis  $A$ , to infer  $A \rightarrow B$ , this definition leads immediately to fallacies of relevance; for if  $B$  is a theorem independently of  $A$ , then we have  $A \rightarrow B$  where  $A$  may be irrelevant to  $B$ . For example, in a system with  $A \rightarrow A$  as an axiom, we have

1	B	hyp
2	A $\rightarrow$ A	axiom
3	B $\rightarrow$ .A $\rightarrow$ A	1-2, $\rightarrow$ I

In this example we indeed proved  $A \rightarrow A$ , but it is crashingly obvious that we did not prove it *from* the hypothesis  $B$ : the defect lies in the definition, which fails to take seriously the word "from" in "proof *from* hypotheses." And this fact suggests a solution to the problem: we should devise a technique for keeping track of the steps used, and then allow application of the introduction rule only when  $A$  is relevant to  $B$  in the sense that  $A$  is *used* in arriving at  $B$ .

As a start in this direction, we suggest prefixing a star (say) to the hypothesis of a deduction, and also to the conclusion of an application of  $\rightarrow$ E just in case at least one premiss has a star, steps introduced as axioms being unstarred. Restriction of  $\rightarrow$ I to cases where in accordance with these rules both  $A$  and  $B$  are starred would then exclude theorems of the form  $A \rightarrow B$ , where  $B$  is proved independently of  $A$ .

In other words, what is wanted is a system for which there is provable a deduction theorem, to the effect that there exists a proof of  $B$  *from* the hypothesis  $A$  if and only if  $A \rightarrow B$  is provable. And we now consider the question of choosing axioms in such a way as to guarantee this result. In view of the rule  $\rightarrow$ E, the implication in one direction is trivial; we consider the converse.

Suppose we have a proof

*	$A_1$	hyp
	$\vdots$	
	$A_t$	
	$\vdots$	
*	$A_n$	

of  $A_n$  from the hypothesis  $A_1$ , and we wish to convert this into an axiomatic proof of  $A_1 \rightarrow A_n$ . A natural and obvious suggestion would be to consider replacing each starred  $A_t$  by  $A_1 \rightarrow A_t$  (since the starred steps are the ones to which  $A_1$  is relevant), and try to show that the result is a proof without hypotheses. What axioms would be required to carry the induction through?

For the basis case we obviously require as an axiom  $A \rightarrow A$ . And in the inductive step, where we consider steps  $A_t$  and  $A_t \rightarrow A_j$  of the original proof, four cases may arise.

(1) Neither premiss is starred. Then in the axiomatic proof,  $A_t$ ,  $A_t \rightarrow A_j$ , and  $A_j$  all remain unaltered, so  $\rightarrow E$  may be used as before.

(2) The minor premiss is starred and the major one is not. Then in the axiomatic proof we have  $A_1 \rightarrow A_t$  and  $A_t \rightarrow A_j$ ; so we need to be able to infer  $A_1 \rightarrow A_j$  from these (since the star on  $A_t$  guarantees a star on  $A_j$  in the original proof).

(3) The major premiss is starred and the minor one is not. Then in the axiomatic proof we have  $A_1 \rightarrow A_t \rightarrow A_j$  and  $A_t$ , so we need to be able to infer  $A_1 \rightarrow A_j$  from these.

(4) And finally both may be starred, in which case we have  $A_1 \rightarrow A_t \rightarrow A_j$  and  $A_1 \rightarrow A_t$  in the axiomatic proof, from which again we need to infer  $A_1 \rightarrow A_j$ .

Summarizing: the proof of an appropriate deduction theorem where relevance is demanded would require the axiom  $A \rightarrow A$  together with the validity of the following inferences:

From  $A \rightarrow B$  and  $B \rightarrow C$  to infer  $A \rightarrow C$ ;

From  $A \rightarrow B \rightarrow C$  and  $B$  to infer  $A \rightarrow C$ ;

From  $A \rightarrow B \rightarrow C$  and  $A \rightarrow B$  to infer  $A \rightarrow C$ .

It then seems plausible to consider the following axiomatic system as capturing the notion of relevance:

$A \rightarrow A$	<i>(identity)</i>
$A \rightarrow B \rightarrow C \rightarrow A \rightarrow C$	<i>(transitivity)</i>
$(A \rightarrow B \rightarrow C) \rightarrow B \rightarrow A \rightarrow C$	<i>(permutation)</i>
$(A \rightarrow B \rightarrow C) \rightarrow A \rightarrow B \rightarrow A \rightarrow C$	<i>(self-distribution)</i>

And without further proof we state that for this system  $W_1'$  we have the following

THEOREM.  $A \rightarrow B$  is a theorem of  $W_I'$  just in case there is a proof of  $B$  from the hypothesis  $A$  (in the starred sense).

Equivalent systems have been investigated by Moh Shaw-Kwei 1950 and Church 1951. (See also Kripke 1960.) Church calls his system the "weak positive implicational propositional calculus," ( $W_I$ ), and uses the following axioms:

- |            |   |                       |
|------------|---|-----------------------|
| $W_{I1}$ . | $A \rightarrow A$   | <i>(identity)</i>     |
| $W_{I2}$ . | $A \rightarrow B \rightarrow C \rightarrow A \rightarrow C \rightarrow B$   | <i>(transitivity)</i> |
| $W_{I3}$ . | $(A \rightarrow B \rightarrow C) \rightarrow B \rightarrow A \rightarrow C$ | <i>(permutation)</i>  |
| $W_{I4}$ . | $(A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$               | <i>(contraction)</i>  |

The proof that  $W_I'$  and  $W_I$  are equivalent is left to the reader.

A generalization of the deduction theorem above was proved by both Moh Shaw-Kwei and Church; modified to suit present purposes, it may be stated as follows:

THEOREM. If there exists a proof of  $B$  on the hypotheses  $A_1, \dots, A_n$ , in which all of  $A_1, \dots, A_n$  are used in arriving at  $B$ , then there is a proof of  $A_n \rightarrow B$  from  $A_1, \dots, A_{n-1}$  satisfying the same condition.

So put, the result acquires a rather peculiar appearance: it seems odd that we should have to use *all* the hypotheses. One would have thought that for a group of hypotheses to be relevant to a conclusion, it would suffice if *some* of the hypotheses were used + at least if we think of the hypotheses as taken conjointly. (Cf. the "entailment" theorem of Belnap 1960.) The peculiarity arises because of a tendency (thus far not commented on) to confound  $(A_1 \wedge \dots \wedge A_n) \rightarrow B$  with  $A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_n \rightarrow B$ .<sup>13</sup> But in the former case, we would not expect to require that *all* the  $A_i$  be relevant to  $B$ . We shall give reasons presently, deriving from another formulation of  $W_I$ , for thinking it sensible to require each of the  $A_i$  in the nested implication to be relevant to  $B$ ; a feature of the situation which will lead us to make a sharp distinction between  $(A_1 \wedge \dots \wedge A_n) \rightarrow B$  and  $A_1 \rightarrow \dots \rightarrow A_n \rightarrow B$ . (It is presumably the failure to make this distinction which leads Curry 1959 to say of the relation considered in Church's theorem that it is one "which is not ordinarily considered in deductive methodology at all.")

We feel that the star formulation of the deduction theorem makes clearer what is at stake in Church's calculus. On the other hand Church's own deduction theorem has the merit of allowing for proof of nested entailments in a more perspicuous way than is available in the star formulation. Our next task therefore is to try to combine these approaches so as to obtain the advantages of both.

Returning now to a consideration of subordinate proofs, it seems natural

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<sup>13</sup> The tendency is not universal; see for example Lewis and Langford 1932, p. 165, and Barcan Marcus 1953.

to try to extend the star treatment, using some other symbol for deductions carried out in a subproof, but retaining the same rules for carrying this symbol along. We might consider a proof of contraction in which the inner hypothesis is distinguished by a dagger rather than a star:

1		A → .A → B	*	hyp	
2			A	†	hyp

(the different relevance marks reflecting the initial assumption that the two formulas, as hypotheses, are irrelevant to each other). Then generalizing the starring rules, we might require that in applications of  $\rightarrow E$ , the conclusion B must carry all the relevance marks of both premisses A and  $A \rightarrow B$ , thus:

1		A → .A → B	*	hyp	
2			A	†	hyp
3			A → .A → B	*	1 reit
4			A → B	* †	2 3 $\rightarrow E$
5			B	* †	2 4 $\rightarrow E$

To motivate the restriction on  $\rightarrow I$ , we recall that in proofs involving only stars, it was required that both A and B have stars, and that the star was discharged on  $A \rightarrow B$  in the conclusion of a deduction. This suggests the following generalization: that in drawing the conclusion  $A \rightarrow B$  by  $\rightarrow I$ , we require that the relevance symbol on A also be present among those of B, and that in the conclusion  $A \rightarrow B$  the relevance symbol of A (like the hypothesis A itself) be discharged. Two applications of this rule then lead from the proof above to

6		A → B	*	2-5 $\rightarrow I$
7		(A → .A → B) → .A → B		1 6 $\rightarrow I$

But of course the easiest way of handling the matter is to use classes of numerals to mark the relevance conditions, since then we may have as many nested subproofs as we wish, each with a distinct numeral (which we shall write as a subscript) for its hypothesis. More precisely we allow that: (1) one may introduce a new hypothesis  $A_{\{k\}}$ , where  $k$  should be different from all subscripts on hypotheses of proofs to which the new proof is subordinate — it suffices to take  $k$  as the rank of A (see Test Formulations); (2) from  $A_a$  and  $A \rightarrow B_b$  we may infer  $B_{a \cup b}$ ; (3) from a proof of  $B_a$  from the hypothesis  $A_{\{k\}}$ , where  $k$  is in  $a$ , we may infer  $A \rightarrow B_{a - \{k\}}$ ; and reit and rep retain subscripts (where  $a, b, c$ , range over classes of numerals).

As an example we prove the *law of assertion*.

1	— A <sub>{1}</sub>	hyp
2	— A → B <sub>{2}</sub>	hyp
3	— A <sub>{1}</sub>	1 reit
4	B <sub>{1,2}</sub>	2 3 →E
5	A → B → B <sub>{1}</sub>	2-4 →I
6	A → . A → B → B	1-5 →I

To see that this generalization  $W_I^*$  is also equivalent to  $W_I$ , observe first that the axioms of  $W_I$  are easily proved in  $W_I^*$ ; hence  $W_I^*$  contains  $W_I$ . The proof of the converse involves little more than repeated application, beginning with innermost subproofs, of the techniques used in proving the deduction theorem for  $W_I$ ; it will be left to the reader.

(We remark that a Test Formulation of  $W_I$  arises if to the formulation for  $H_I$  we add the following clauses defining a set of "relevance indices"  $\text{Rel}(A_i)$  for each  $i$ -th step.

- Basis: (f)  $\text{Rel}(A_1) = \{1\}$
- (1) (f)  $\text{Rel}(A_k) = \{R(A_k)\}$
- (2) (f)  $\text{Rel}(A_k) = \text{Rel}(A_j)$
- (3) (f)  $\text{Rel}(A_k) = \text{Rel}(A_j)$
- (4) (f)  $\text{Rel}(A_k) = \text{Rel}(A_j) \cup \text{Rel}(A_i)$
- (5) If  $\text{Rel}(H(A_{k-1})) \subset \text{Rel}(A_{k-1})$ , then
- (a)–(e) as before, and
- (f)  $\text{Rel}(A_k) = \text{Rel}(A_{k-1}) - \text{Rel}(H(A_{k-1}))$ .

If the subscripting device is taken as an explication of relevance, then it is seen that Church's  $W_I$  does secure relevance, since  $A \rightarrow B$  is provable in  $W_I$  only if  $A$  is relevant to  $B$ .

But if  $W_I$  is taken as an explication of entailment, then the requirement of necessity for a valid inference is lost. Consider the following special case of the law of assertion, just proved:

$$A \rightarrow . A \rightarrow A \rightarrow A.$$

This says that if  $A$  is true, then it follows from  $A \rightarrow A$ . But it seems reasonable to suppose that any logical consequence of  $A \rightarrow A$  should be necessarily true. (Note that in the familiar systems of modal logic, consequences of necessary truths are necessary.) We certainly do in practice recognize that there are truths which do not follow from a law of logic — but  $W_I$  obliterates this distinction. It seems evident, therefore, that a satisfactory theory of implication will require both relevance (like  $W_I$ ) and necessity (like  $S4_I$ ).

## IV

**Necessity and relevance.** We therefore consider the system  $E_I^{14}$  which arises when we recognize that valid inferences require both necessity and relevance. Since the restrictions are most transparent as applied to the subproof format, we begin by considering the system  $E_I^*$  which results from imposing the restriction on reiteration (of  $S4_I^*$ ) together with the subscript requirements (of  $W_I^*$ ).<sup>15</sup> We summarize the rules of  $E_I^*$  as follows:

- (1) Hyp. A step may be introduced as the hypothesis of a new subproof, and each new hypothesis receives a unit class  $\{k\}$  of numerical subscripts, where  $k$  is the rank of A.
- (2) Rep.  $A_a$  may be repeated, retaining the relevance indices  $a$ .
- (3) Reit.  $(A \rightarrow B)_a$  may be reiterated, retaining  $a$ .
- (4)  $\rightarrow E$ . From  $A_a$  and  $(A \rightarrow B)_b$  to infer  $B_{a \cup b}$ .
- (5)  $\rightarrow I$ . From a proof of  $B_a$  on hypothesis  $A_{\{k\}}$  to infer  $(A \rightarrow B)_{a - \{k\}}$ , provided  $k$  is in  $a$ .

(For a Test Formulation of  $E_I^*$  we simply combine the restriction on reiteration for  $S4_I^*$  with the subscript requirements for  $W_I^*$ .)

It develops that an axiomatic counterpart of  $E_I^*$  has also been considered in the literature,  $E_I^*$  in fact being equivalent to the pure implicational fragment  $E_I'$  of Ackermann 1956,<sup>16</sup> defined by the following axioms and rules:

- $E_{I1}$ .  $A \rightarrow A$
- $E_{I2}$ .  $A \rightarrow B \rightarrow . B \rightarrow C \rightarrow . A \rightarrow C$
- $E_{I3}$ .  $A \rightarrow B \rightarrow . C \rightarrow A \rightarrow . C \rightarrow B$
- $E_{I4}$ .  $(A \rightarrow . A \rightarrow B) \rightarrow . A \rightarrow B$

Rules:  $\rightarrow E$

- ( $\delta$ ) From  $A$  and  $B \rightarrow . A \rightarrow C$  to infer  $B \rightarrow C$ .

( $\delta$ ) is essentially a rule of permutation, giving the *effect* of the formula  $(B \rightarrow . A \rightarrow C) \rightarrow . A \rightarrow . B \rightarrow C$ . The latter, however, is not provable in Ackermann's calculus, and addition of such a law of permutation would reduce

<sup>14</sup> We reserve this designation for the formulation of Anderson, Belnap, and Wallace, 1960. Axioms:  $A \rightarrow A \rightarrow B \rightarrow B$ ,  $A \rightarrow B \rightarrow . B \rightarrow C \rightarrow . A \rightarrow C$ , and  $(A \rightarrow . A \rightarrow B) \rightarrow . A \rightarrow B$ ; rule:  $\rightarrow E$ .

<sup>15</sup>  $E_I^*$  is not, however, the intersection of  $S4_I^*$  and  $W_I^*$ . We are indebted to Saul Kripke for a counter-example:  $A \rightarrow . (A \rightarrow . A \rightarrow B) \rightarrow . A \rightarrow B$  is provable in both the latter systems, but not in  $E_I^*$ .

<sup>16</sup> The present order of exposition was adopted for reasons connected with the philosophical points to be made, but it should not be allowed to obscure the extent of our considerable indebtedness to this paper of Ackermann.

Ackermann's system to Church's. Moreover, rule  $(\delta)$  requires the restriction (as Ackermann 1956 points out) that the minor premiss  $A$  be a theorem, else one could pass from a contingently true  $A$ , by way of  $A \rightarrow A \rightarrow .A \rightarrow A$ , to  $A \rightarrow A \rightarrow A$ , thus again obliterating the distinction between necessary and contingent truth.  $(\delta)$  is therefore non-normal, in the sense that no corresponding entailment is provable.

This situation may be remedied, however, if we replace  $(\delta)$  by the following axiom

$E_{I5}$ .  $A \rightarrow A \rightarrow B \rightarrow B$ ,

which yields an equivalent set of theorems, as we now see. Let  $E_I'$  be  $E_{I1-4}$ ,  $\rightarrow E$  and  $(\delta)$ , and let  $E_I''$  be  $E_{I1-5}$  and  $\rightarrow E$ .  $E_{I5}$  is immediate in  $E_I'$ , from  $E_{I1}$  and  $A \rightarrow A \rightarrow B \rightarrow .A \rightarrow A \rightarrow B$  by  $(\delta)$ ; hence  $E_I'$  contains  $E_I''$ . To prove  $(\delta)$  in  $E_I''$  we assume the first two steps below obtained, and proceed as indicated (after noting that in the pure implicational calculus the minor premiss of  $(\delta)$  must be an entailment):

- |   |                          |
|---|--------------------------|
| 1. $A \rightarrow B$  |                          |
| 2. $C \rightarrow .A \rightarrow B \rightarrow D$                             |                          |
| 3. $B \rightarrow B \rightarrow .A \rightarrow B$                             | $1 E_{I2} \rightarrow E$ |
| 4. $A \rightarrow B \rightarrow D \rightarrow .B \rightarrow B \rightarrow D$ | $3 E_{I2} \rightarrow E$ |
| 5. $C \rightarrow .B \rightarrow B \rightarrow D$                             | $2\ 4$ transitivity      |
| 6. $C \rightarrow D$  | $5 E_{I5}$ transitivity  |

Hence  $E_I'$  and  $E_I''$  are equivalent.

Proof of equivalence of  $E_I^*$  with  $E_I'$  proceeds most easily by way of  $E_I''$ . That  $E_I^*$  contains  $E_I''$  is trivial; the construction of proofs of the axioms by the subproof technique is left to the reader. To reduce proofs in  $E_I^*$  to proofs in  $E_I''$ , we require the following

LEMMA. If  $A_a$  is a step of a subproof  $Q$  (of a quasi-proof in  $E_I^*$ ) with hypothesis  $H_{\{k\}}$ , where  $k$  is not in  $a$ , then we may also obtain  $A \rightarrow B \rightarrow B_a$  in  $Q$  (for arbitrary  $B$ ), with the help of theorems of  $E_I''$  and  $\rightarrow E$ .

*Proof.* We first require a theorem of  $E_I''$ .

- $E_{I6}$ .  $A \rightarrow B \rightarrow .A \rightarrow B \rightarrow C \rightarrow C$
- |   |                                |
|---|--------------------------------|
| 1. $A \rightarrow B \rightarrow .A \rightarrow A \rightarrow .A \rightarrow B$  | $E_{I3}$                       |
| 2. $(A \rightarrow A \rightarrow .A \rightarrow B) \rightarrow .A \rightarrow B \rightarrow C \rightarrow .A \rightarrow A \rightarrow C$ | $E_{I2}$                       |
| 3. $A \rightarrow B \rightarrow .A \rightarrow B \rightarrow C \rightarrow .A \rightarrow A \rightarrow C$                                | $1\ 2$ transitivity            |
| 4. $(A \rightarrow B \rightarrow C \rightarrow .A \rightarrow A \rightarrow C) \rightarrow .A \rightarrow B \rightarrow C \rightarrow C$  | $E_{I5}\ E_{I3} \rightarrow E$ |
| 5. $A \rightarrow B \rightarrow .A \rightarrow B \rightarrow C \rightarrow C$   | $3\ 4$ transitivity            |

The proof of the lemma is by induction. The *first*  $A_a$  in the proof, with  $k$

not in  $a$ , must be obtained by reiteration, hence has the form  $C \rightarrow D$ . Then insert  $C \rightarrow D \rightarrow .C \rightarrow D \rightarrow B \rightarrow B$  ( $E_16$ ) and use  $\rightarrow E$  to obtain  $C \rightarrow D \rightarrow B \rightarrow B_a$ , i.e.,  $A \rightarrow B \rightarrow B_a$ . Any other  $A_a$  with  $k$  not in  $a$ , must either be obtained from reit (in which case we argue as before) or else from  $C_c$  and  $C \rightarrow A_b$  by  $\rightarrow E$ , where  $a = b \cup c$  and  $k$  is in neither  $b$  nor  $c$ . Then by the inductive hypotheses, we may obtain  $C \rightarrow A \rightarrow A_c$  and  $C \rightarrow A \rightarrow B \rightarrow B_b$ . Then insert

$$C \rightarrow A \rightarrow A \rightarrow .A \rightarrow B \rightarrow .C \rightarrow A \rightarrow B \quad (E_{12})$$

and use  $\rightarrow E$  to get  $A \rightarrow B \rightarrow .C \rightarrow A \rightarrow B_c$ ; and insertion of

$$(A \rightarrow B \rightarrow .C \rightarrow A \rightarrow B) \rightarrow .C \rightarrow A \rightarrow B \rightarrow B \rightarrow .A \rightarrow B \rightarrow B \quad (E_{12})$$

and two uses of  $\rightarrow E$  then leads to the desired result  $A \rightarrow B \rightarrow B_{b \cup c}$ , i.e.,  $A \rightarrow B \rightarrow B_a$ . Notice that only theorems of  $E_1''$  and  $\rightarrow E$  were used in obtaining the lemma.

To prove that  $E_1^*$  is contained in  $E_1''$ , we again consider an innermost subproof  $Q$  of a proof  $P$  in  $E_1^*$ , with steps  $A_{1\{k\}}, \dots, A_{ia_i}, \dots, A_{na_n}$  and we let  $A_{ia'_i}$  be  $(A_1 \rightarrow A_i)_{a_i - \{k\}}$  or  $A_{ia_i}$  according as  $k$  is or is not in  $a_i$ . Then we replace  $Q$  by the sequence  $Q'$  of  $A_{ia'_i}$ , obtaining  $P'$ , and the theorem may be stated as follows:

**THEOREM.** Under these conditions theorems of  $E_1''$  may be inserted in  $Q'$  in such a way as to make  $P'$  a quasi-proof. (Anderson 1959.)

*Proof.* Basis.  $A_{1\{k\}}$  is  $A_1 \rightarrow A_1$ , and may be treated as an insertion of  $E_{11}$ .

If any  $A_a$  in  $Q$  is a consequence of a preceding step by rep or reit, then  $A_{a'}$  may be treated in  $Q'$  as a repetition.

If any  $A_a$  is a consequence of  $B_b$  and  $(B \rightarrow A)_c$  in  $Q$ , where  $a = b \cup c$ , then we distinguish four cases:

(i)  $k$  is in neither  $b$  nor  $c$ . Then  $B_{b'}$  is  $B_b$ ,  $(B \rightarrow A)_{c'}$  is  $(B \rightarrow A)_c$ , and  $A_{b \cup c}$ , i.e.,  $A_{a'}$ , follows by  $\rightarrow E$  in  $Q'$ .

(ii)  $k$  is in  $b$  but not  $c$ . Then  $B_{b'}$  is  $(A_1 \rightarrow B)_{b - \{k\}}$ , and  $(B \rightarrow A)_{c'}$  is  $(B \rightarrow A)_c$ . Then insert  $E_{12}$ , and use  $\rightarrow E$  twice to get  $(A_1 \rightarrow A)_{(b \cup c) - \{k\}}$ , i.e.,  $A_{a'}$ .

(iii)  $k$  is in  $c$  but not  $b$ . Then  $B_{b'}$  is  $B_b$ , and  $(B \rightarrow A)_{c'}$  is  $(A_1 \rightarrow .B \rightarrow A)_{c - \{k\}}$ . Since  $k$  is not in  $b$ , we have by the lemma  $(B \rightarrow A \rightarrow A)_b$ . Then insertion of

$$(A_1 \rightarrow .B \rightarrow A) \rightarrow .B \rightarrow A \rightarrow A \rightarrow .A_1 \rightarrow A \quad (E_{12})$$

yields  $(A_1 \rightarrow A)_{(b \cup c) - \{k\}}$ , i.e.,  $A_{a'}$ , by two uses of  $\rightarrow E$ .

(iv)  $k$  is in both  $c$  and  $b$ . Then we have  $(A_1 \rightarrow B)_{b - \{k\}}$  and  $(A_1 \rightarrow .B \rightarrow A)_{c - \{k\}}$ . Inserting  $E_{12}$  we have  $A_1 \rightarrow B \rightarrow .B \rightarrow A \rightarrow .A_1 \rightarrow A$ , whence  $(B \rightarrow A \rightarrow .A_1 \rightarrow A)_{b - \{k\}}$  by  $\rightarrow E$ , and  $(A_1 \rightarrow .A_1 \rightarrow A)_{(b \cup c) - \{k\}}$  by transitivity. Then insert  $E_{14}$  and use  $\rightarrow E$  to get  $(A_1 \rightarrow A)_{a - \{k\}}$ , i.e.,  $A_{a'}$ , as required.

To complete the proof we observe that the restriction on the introduction rule guarantees that  $k$  is in  $a_n$ , hence the final step  $A_{na'_n}$  of  $Q'$  is  $(A_1 \rightarrow A_n)_{a_n - \{k\}}$ . But then the conclusion  $(A_1 \rightarrow A_n)_{a_n - \{k\}}$  from  $Q$  by  $\rightarrow I$  in  $P$  can be regarded in  $P'$  as a repetition of the last wff of  $Q'$ .

Hence  $P'$  is a quasi-proof.

Repeated application of these techniques then leads ultimately to a sequence of wffs each of which is either an axiom of  $E_I''$ , or else a consequence of predecessors by  $\rightarrow E$ , and hence a theorem of  $E_I''$ . And since  $E_I''$  is equivalent with  $E_I'$ , this establishes the equivalence of  $E_I^*$  with Ackermann's implicational calculus.

The equivalence of  $E_I^*$  with  $E_I'$  gives us an easy proof technique for  $E_I'$ , and we now state a summary list of laws of entailment, proofs of which will be left to the reader.

*Identity*:  $A \rightarrow A$

*Transitivity*: (Suffixing)  $A \rightarrow B \rightarrow B \rightarrow C \rightarrow A \rightarrow C$

(Prefixing)  $A \rightarrow B \rightarrow C \rightarrow A \rightarrow C \rightarrow B$

*Contraction*:  $(A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$

*Self-distribution*:  $(A \rightarrow B \rightarrow C) \rightarrow A \rightarrow B \rightarrow A \rightarrow C$

*Restricted permutation*:  $(A \rightarrow B \rightarrow C \rightarrow D) \rightarrow B \rightarrow C \rightarrow A \rightarrow D$

*Restricted conditioned modus ponens*:  $B \rightarrow C \rightarrow (A \rightarrow B \rightarrow C \rightarrow D) \rightarrow A \rightarrow D$

*Restricted assertion*:  $A \rightarrow B \rightarrow A \rightarrow B \rightarrow C \rightarrow C$

*Specialized assertion*:  $A \rightarrow A \rightarrow B \rightarrow B$

*Replacement of the middle*:  $D \rightarrow B \rightarrow (A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow D \rightarrow C)$

*Replacement of the third*:  $C \rightarrow D \rightarrow (A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B \rightarrow D)$

*Prefixing in the consequent*:  $(A \rightarrow B \rightarrow C) \rightarrow A \rightarrow D \rightarrow B \rightarrow D \rightarrow C$

*Suffixing in the consequent*:  $(A \rightarrow B \rightarrow C) \rightarrow A \rightarrow C \rightarrow D \rightarrow B \rightarrow D$

The foregoing seems to us to be a strong and natural list of valid entailments, all of which are *necessarily* true, and in each of which the antecedent is *relevant* to the consequent. And of course each is provable in all of the systems  $H_I$ ,  $S4_I$ , and  $W_I$  considered previously. But the latter all contain in addition "paradoxical" assertions, which we now discuss under two headings; fallacies of modality, and fallacies of relevance.

## V

**Fallacies of modality.** We have in earlier discussions remarked that entailments, if true at all, are necessarily true, and that if  $A$  follows from a law of logic, then  $A$  is necessarily true. We begin consideration of modal fallacies by showing that a theory of necessity is contained in  $E_I^*$ , and that it has the expected properties. We define "it is necessary that  $A$ " as follows (Anderson and Belnap 1959):

$$NA =_{df} A \rightarrow A \rightarrow A.^{17}$$

<sup>17</sup> Of course we do not mean to suggest that in saying that  $A$  is necessary we *mean* that  $A$  follows from  $A \rightarrow A$ ; the definition indicates rather that if one *did* mean this by "A is necessary," or "NA", then it would turn out that  $N$  had all the right properties as we see below. We are indebted to A. N. Prior for pointing out to us in correspondence

The motivation for this choice of a definition of  $NA$  lies in the belief that  $A$  is necessary if and only if  $A$  follows from a logical truth. The choice of  $A \rightarrow A$  as *the* logical truth from which a necessary  $A$  must follow is justified by the fact that if  $A$  follows from any true entailment  $B \rightarrow C$ , then it follows from  $A \rightarrow A$ , a fact expressed in the following theorem.

1	—	$B \rightarrow C_{(1)}$		hyp	
2		—	$B \rightarrow C \rightarrow A_{(2)}$	hyp	
3			—	$A \rightarrow A_{(3)}$	hyp
4				$B \rightarrow C_{(1)}$	1 reit
5				$B \rightarrow C \rightarrow A_{(2)}$	2 reit
6				$A_{(1,2)}$	4 5 $\rightarrow E$
7				$A_{(1,2,3)}$	3 6 $\rightarrow E$
8				$A \rightarrow A \rightarrow A_{(1,2)}$	3-7 $\rightarrow I$
9				$B \rightarrow C \rightarrow A \rightarrow NA_{(1)}$	2-8 $\rightarrow I$
10				$B \rightarrow C \rightarrow .B \rightarrow C \rightarrow A \rightarrow NA$	1-9 $\rightarrow I$

We find support for this proposal in the fact that in such systems as  $M$  (Feys 1937; von Wright 1951),  $S4$ , and  $S5$ ,  $NA$  (either taken as primitive, or defined as not-possible-not  $A$ ) is strictly equivalent to (hence inter-substitutable with)  $A \rightarrow A \rightarrow A$ . (This observation for  $S4$  and  $S5$  was made by Lemmon *et. al.* 1956.)

We give as examples some further proofs of theorems involving necessity.

1	—	$A \rightarrow A \rightarrow A_{(1)}$		hyp	
2		—	$A_{(2)}$	hyp	
3			—	$A_{(2)}$	2 rep
4				$A \rightarrow A$	2-3 $\rightarrow I$
5				$A_{(1)}$	1 4 $\rightarrow E$
6				$NA \rightarrow A$	1-5 $\rightarrow I$

1	—	$A \rightarrow A \rightarrow B_{(1)}$		hyp	
2		—	$B \rightarrow B_{(2)}$	hyp	
3			—	$A_{(3)}$	hyp
4				$A_{(3)}$	3 rep
5				$A \rightarrow A$	3-4 $\rightarrow I$
6				$A \rightarrow A \rightarrow B_{(1)}$	1 reit
7				$B_{(1)}$	5 6 $\rightarrow E$
8				$B_{(1,2)}$	2 7 $\rightarrow E$
9				$B \rightarrow B \rightarrow B_{(1)}$	2-8 $\rightarrow I$
10				$A \rightarrow A \rightarrow B \rightarrow NB$	1-9 $\rightarrow I$

that, conversely, with plausible assumptions ( $NA \rightarrow A$ ,  $A \rightarrow B \rightarrow N(A \rightarrow B)$ , and  $NA \rightarrow .A \rightarrow B \rightarrow NB$ ) about  $N$  (taken as primitive),  $NA$  and  $A \rightarrow A \rightarrow A$  entail each other.

The first says that necessity implies truth (a special case of the *specialized law of assertion*), and the second that if B follows from the *law of identity*, then B is necessarily true (which accords with previous informal observations). We add some other easily provable theorems.

$$\begin{aligned} A \rightarrow B &\rightarrow .NA \rightarrow NB \\ A \rightarrow B &\rightarrow N(A \rightarrow B) \\ A \rightarrow B \rightarrow C &\rightarrow .A \rightarrow B \rightarrow NC \\ NB &\rightarrow .(A \rightarrow .B \rightarrow C) \rightarrow .A \rightarrow C \\ NA &\rightarrow NNA \end{aligned}$$

The first expresses distributivity of necessity over entailment. The second (a special case of *restricted assertion*) says that entailments, if true at all, are necessarily true. The third that if C follows from an entailment, then the necessity of C also follows from that entailment (which we may also express by saying that if C follows from an entailment, then if the entailment is true, then C is necessary). The next corresponds to Ackermann's rule ( $\delta$ ), expressing his requirement that, for application of ( $\delta$ ), B must be a *logische Identität*: where B is necessary, we may infer  $A \rightarrow C$  from  $A \rightarrow .B \rightarrow C$ . And the last says that necessity implies necessary necessity (so to speak): a corollary of  $A \rightarrow B \rightarrow N(A \rightarrow B)$  together with the fact that necessity is defined in terms of entailment.

Modal fallacies arise when it is claimed that entailments follow from, or are entailed by, *contingent* propositions. In the pure theory of entailment, contingency can be carried only by propositional variables (in view of  $A \rightarrow B \rightarrow N(A \rightarrow B)$ ), and the following guarantees that  $E_I^*$  is free of modal fallacies.

**THEOREM.** If A is a propositional variable, then for no B and C is  $A \rightarrow .B \rightarrow C$  provable in  $E_I^*$ . (Ackermann 1956.)

*Proof.* Consider the following matrix (adapted from Ackermann):

$$\begin{array}{ccc} & 0 & 1 & 2 \\ \begin{array}{c} 0 \\ 1 \\ *2 \end{array} & \left| \begin{array}{ccc} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{array} \right. & & \end{array}$$

The axioms of  $E_I^*$  all take the value 2 for all assignments of values under this matrix. Moreover,  $\rightarrow E$  preserves this property. But for any  $A \rightarrow .B \rightarrow C$ , where A is a propositional variable, we can assign A the value 1, giving  $A \rightarrow .B \rightarrow C$  the value 0 regardless of the value of B and C. Hence no such  $A \rightarrow .B \rightarrow C$  is provable.

The theorem serves to rule out such formulas as  $A \rightarrow .A \rightarrow A$  and  $A \rightarrow .B \rightarrow B$ , which are standard "implicational paradoxes" embodying modal fallacies, and this fact clearly accords with (untutored) intuitions. Consider  $A \rightarrow .A \rightarrow A$ .

Though “snow is white” and “that snow is white entails that snow is white” are both true — the latter necessarily so — it seems implausible that “snow is white” should *entail* that it entails itself. It does entail itself, of course, but the color of snow seems irrelevant to that fact of logic. We would think someone arguing rather badly if he tried to convince us that snow is white entails itself by showing us some snow.

Also ruled out as involving fallacies of modality are such formulas as the unrestricted “law” of assertion:

$$A \rightarrow .A \rightarrow B \rightarrow B,$$

and the “law” of permutation

$$(A \rightarrow .B \rightarrow C) \rightarrow .B \rightarrow .A \rightarrow C,$$

which would lead from  $A \rightarrow B \rightarrow .A \rightarrow B$  to the “law” of assertion.

$A \rightarrow .B \rightarrow A$  also involves a modal fallacy, as we have argued before. Of the systems considered previously  $A \rightarrow .B \rightarrow A$  is provable only in  $H_I$ , but corresponding fallacies are to be found in both  $S4_I$ :

$$A \rightarrow B \rightarrow .C \rightarrow .A \rightarrow B, \text{ and}$$

$$B \rightarrow .A \rightarrow A,$$

and in  $W_I$ :

$$A \rightarrow .A \rightarrow A \rightarrow A, \text{ (i.e., } A \rightarrow NA).$$

The latter is of special interest in connection with our proposed definition of necessity. There are six fairly well known pure “implicational” calculi in the literature, three of which make modal distinctions ( $E_I$ ,  $S4_I$ , and  $S5_I$ ), and three of which do not ( $W_I$ ,  $H_I$ , and  $P_I$ , the “implicational” fragment of the two-valued calculus). Consider the following set of schemata.

- (1)  $A \rightarrow A \rightarrow B \rightarrow B$
- (2)  $A \rightarrow B \rightarrow .B \rightarrow C \rightarrow .A \rightarrow C$
- (3)  $(A \rightarrow .A \rightarrow B) \rightarrow .A \rightarrow B$
- (4)  $A \rightarrow B \rightarrow .C \rightarrow .A \rightarrow B$
- (5)  $(A \rightarrow B \rightarrow C \rightarrow .A \rightarrow B) \rightarrow .A \rightarrow B$
- (6)  $A \rightarrow .A \rightarrow A \rightarrow A$

Then for the systems maintaining modal distinctions we have

$$E_I = \{(1), (2), (3)\},$$

$$S4_I = \{(1), (2), (3), (4)\}, \text{ and}$$

$$S5_I = \{(1), (2), (3), (4), (5)\}.$$

And each of the significant systems which collapse modal distinctions can

be got by adding (6) to the appropriate modal system:

$$E_I + (6) = W_I,$$

$$S4_I + (6) = H_I, \text{ and}$$

$$S5_I + (6) = P_I.^{18}$$

So (6), which may be written  $A \rightarrow NA$ , is precisely the thesis which destroys modality. We take this as indirect evidence for the appropriateness of the definition  $NA =_{df} A \rightarrow A \rightarrow A$ . And if we accept this definition as appropriate, then these results provide strong grounds for rejecting  $W_I$ ,  $H_I$ , and  $P_I$ , as analyses of implication or entailment.

*Fallacies of relevance.* Many of the foregoing modal fallacies also embody fallacies of relevance, sanctioning the inference from A to B even though A and B may be totally disparate in meaning. The archetype of fallacies of relevance is  $A \rightarrow B \rightarrow A$ , which would enable us to infer that Bach wrote the Coffee Cantata from the premiss that the Van Allen belt is doughnut-shaped — or indeed from any premiss you like.

In arguing that  $E_I$  satisfies a principle of relevance, we venture, somewhat gingerly, on new ground. We offer two conditions, the first as necessary and sufficient, the second as necessary only.

1. The subscripting technique as applied in  $W_I$  and  $E_I$  may be construed as a formal analysis of the intuitive idea that for A to be relevant to B it must be possible to *use* A in a deduction of B from A. It need not be *necessary* to use A in the deduction of B from A — and indeed this is a familiar situation in mathematics and logic. It not infrequently happens that the hypotheses of a theorem, though all relevant to a conclusion, are subsequently found to be unnecessarily strong. An example is provided by Gödel's original incompleteness theorem, which required the assumption of  $\omega$ -consistency. Rosser subsequently showed that this condition was not required for the proof of incompleteness — but surely no one would hold that  $\omega$ -consistency was *irrelevant* to Gödel's result. Similarly in the following example (due to Smiley 1959), effort is wasted, since the antecedent is used in the proof of the consequent, though it need not be.

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<sup>18</sup> These six postulates are not independent as they stand. Independence can be secured, though not elegantly, by substituting (4')  $A \rightarrow B \rightarrow C \rightarrow C \rightarrow A \rightarrow B$  for (4), and (5')  $(A \rightarrow B \rightarrow A \rightarrow B \rightarrow C \rightarrow A \rightarrow B) \rightarrow (A \rightarrow B \rightarrow C \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$  for (5). We sketch the proof of independence: for (1), with elements 0, 1, and 2 (2 designated), let  $0 \rightarrow A = A \rightarrow 2 = 2$ , and otherwise  $A \rightarrow B = 0$ ; by a result of Jaskowski 1948, (2) is independent in the system got by replacing  $A \rightarrow B$  by D throughout (4') and (5'), and hence independent in  $P_I$ ; (3) alone is not satisfied when the arrow is interpreted as material equivalence; for (4'), with elements 0, 1, and 2 (2 designated), let  $0 \rightarrow A = A \rightarrow A = 2$ ,  $2 \rightarrow 1 = 1$ , and otherwise  $A \rightarrow B = 0$ ; (5') is not in  $H_I$ ; (6) is not in  $S5_I$ .

1	—	$A \rightarrow B_{(1)}$	hyp
2	—	$B \rightarrow A_{(2)}$	hyp
3	—	$B_{(3)}$	hyp
4	—	$B \rightarrow A_{(2)}$	2 reit
5		$A_{(2,3)}$	3 4 $\rightarrow E$
6		$A \rightarrow B_{(1)}$	1 reit
7		$B_{(1,2,3)}$	5 6 $\rightarrow E$
8		$A_{(1,2,3)}$	4 7 $\rightarrow E$
9		$B \rightarrow A_{(1,2)}$	3-8 $\rightarrow I$
10		$B \rightarrow A \rightarrow . B \rightarrow A_{(1)}$	2-9 $\rightarrow I$
11		$A \rightarrow B \rightarrow . B \rightarrow A \rightarrow . B \rightarrow A$	1-10 $\rightarrow I$

A similar proof yields

$$B \rightarrow A \rightarrow . A \rightarrow B \rightarrow . B \rightarrow A.$$

(The point in both cases is that the antecedent and the antecedent of the consequent can be made to “cycle,” producing one or the other as consequent of the consequent.)

The law  $A \rightarrow B \rightarrow . B \rightarrow A \rightarrow . B \rightarrow A$  requires special justification, because it violates a plausible condition due to Smiley 1959: he demands that every true entailment be a “substitution instance of a tautological implication whose components are neither contradictory nor tautological.” Thus Smiley’s criterion allows  $A \rightarrow A \rightarrow . A \rightarrow A$  (with the consequent necessary) on the grounds that it is a substitution instance of  $B \rightarrow B$  (with consequent possibly contingent). But it would exclude  $A \rightarrow B \rightarrow . B \rightarrow A \rightarrow . B \rightarrow A$  on the ground that the latter is *not* a substitution instance of a tautology with a possibly contingent consequent.

From the point of view we have been urging, Smiley’s criterion is misguided. The validity of an entailment has nothing to do with whether or not the components are true, false, necessary, or impossible; it has to do solely with whether or not there is a necessary connection between antecedent and consequent. Hence it is a mistake (we feel) to try to build a sieve which will “strain out” entailments from the set of material or strict “implications” present in some system of truth-functions, or of truth-functions with modality. Even so, however, it is of interest to note that if  $A$  and  $B$  are truth-functions, then the entailments  $A \rightarrow B$  provable in  $E$  (Anderson and Belnap 1958) are all substitution-instances of provable entailments with both  $A$  and  $B$  contingent. (This follows from Belnap 1959b; note that  $A \supset B \rightarrow . B \supset A \supset . B \supset A$ , where  $A \supset B = \bar{A} \vee B$ , is not a true entailment.) One might indeed think of Belnap 1959b as a modification of the criterion of von Wright 1957: “ $A$  entails  $B$ , if and only if, by means of logic, it is possible to come to know the truth of  $A \rightarrow B$  without coming to know the falsehood of  $A$  or the truth of  $B$ ,” (again of course where  $A$  and  $B$  are purely truth-functional).

But where A or B involves entailment, this sort of approach fails; and in particular, though of course  $A \rightarrow B$  should be a sufficient condition for the material "implication" from A to B, there seems no good reason to suppose that the result of substituting the horseshoe for the arrow throughout an arbitrary truth concerning entailments would be a truth-functional tautology. We seem to be unable to give criteria involving only extensional considerations for detecting the fundamentally intensional notion of entailment.

Though the entailment  $B \rightarrow A \rightarrow A \rightarrow B \rightarrow A \rightarrow B$  violates Smiley's condition, we can quote Smiley himself in support of it (and of its proof): "... inferences may be justified on more grounds than one, and the present theory requires not that there should be *no* cogent way of reaching the conclusion *without* using all the premisses, but only that there should be *some* cogent way of reaching it *with* all the premisses used."<sup>19</sup>

On the latter grounds we can justify  $A \rightarrow B \rightarrow B \rightarrow A \rightarrow B \rightarrow A$ ; but we notice that

$$A \rightarrow B \rightarrow A \rightarrow B \rightarrow A \rightarrow B,$$

is ruled out by a matrix which we discuss in connection with the second formal condition for relevance in the pure calculus of entailment.

2. Informal discussions of implication or entailment have frequently demanded "relevance" of A to B as a necessary condition for the truth of  $A \rightarrow B$ , where relevance is construed as involving some "meaning content" common to both A and B.<sup>20</sup> A formal condition for "common meaning content" becomes almost obvious once we note that commonality of meaning in propositional logic is carried by commonality of propositional variables. So we propose as a *necessary* (but not sufficient) condition for the relevance of A to B in the pure theory of entailment, that A and B must share a variable. If this property fails, then the variables in A and B may be assigned

<sup>19</sup> Actually, this condition won't do either, as is easily seen by examples. Proof of the uniqueness of the identity element in an Abelian group does not invoke the fact that the group operation is commutative; it nevertheless *follows* from the axioms for an Abelian group that the identity element is unique — or at any rate everyone says so. What is required is that there be *some* cogent way of reaching the conclusion with *some* of the (conjoined) premisses used. But this point will have to await discussion of truth-functions in connection with entailment.

<sup>20</sup> This call for common "meaning content" comes from a variety of quarters. Nelson (1930, p. 445) says that implication "is a necessary connection between meanings"; Duncan-Jones (1934, p. 71) that A implies B only when B "arises out of the meaning of" A; Baylis (1931, p. 397) that if A implies B then "the intensional meaning of B is identical with a part of the intensional meaning of A"; and Blanshard (1939, vol. 2, p. 390) that "what lies at the root of the common man's objection [to strict implication] is the stubborn feeling that implication has something to do with the *meaning* of propositions, and that any mode of connecting them which disregards this meaning and ties them together in despite of it is too artificial to satisfy the demand of thought."

propositional values, in such a way that the resulting  $A'$  and  $B'$  have no meaning content in common, and are totally irrelevant to each other.  $E_I$  avoids such fallacies of relevance, as is shown by the following

**THEOREM.** If  $A \rightarrow B$  is provable in  $E_I$ , then  $A$  and  $B$  share a variable. (Belnap 1959.)

*Proof.* Consider the following matrix (a finite adaptation of a matrix due to Sugihara 1955):

	0	1	2	3
0	3	3	3	3
1	0	2	2	3
*2	0	1	2	3
*3	0	0	0	3

The axioms of  $E$  take values 2 or 3 for all assignments of values to the variables, and the rule  $\rightarrow E$  preserves this property. But if  $A$  and  $B$  share no variables, then we may assign the value 3 to all the variables of  $A$  (yielding  $A = 3$ ), and 2 to all the variables of  $B$  (yielding  $B = 2$ ), and  $3 \rightarrow 2$  takes the undesignated value 0. Hence if  $A$  and  $B$  fail to share a variable,  $A \rightarrow B$  is unprovable.

We remark that the first of the two conditions has to do with entailment in its guise as the converse of deducibility, and in this sense is a purely syntactical completeness theorem:  $A$  is *relevant* to  $B$ , when  $A \rightarrow B$ , just if there exists a proof (satisfying certain conditions) of  $B$  *from* the hypothesis  $A$ . The second condition, however, concerns entailment conceived of as a relation of logical consequence, and is semantical in character, since it has to do with possible assignments of values to the propositional variables. The problem of finding suitable necessary and sufficient semantical conditions for relevance has proved refractory, and we leave it open.

The matrix given in the second condition above also satisfies  $(A \rightarrow .B \rightarrow C) \rightarrow .B \rightarrow .A \rightarrow C$ ; it follows that  $W_I$  is also free of fallacies of relevance (in the sense of satisfying the necessary condition of sharing variables), as we would expect from the previous discussion. Among fallacies of relevance which are *not* modal fallacies we mention

$$A \rightarrow A \rightarrow .B \rightarrow B.$$

It is interesting to note also that the same matrix can be used to rule out expressions involving fallacies of relevance even when antecedent and consequent *do* share a variable. We define *antecedent part of A* and *consequent part of A* inductively as follows:  $A$  is a consequent part of  $A$ , and if  $B \rightarrow C$  is a consequent {antecedent} part of  $A$ , then  $B$  is an antecedent {consequent} part of  $A$ , and  $C$  is a consequent {antecedent} part of  $A$ .

**THEOREM.** If  $A$  is a theorem of  $E_I$  (or, indeed, of  $W_I$ ), then every variable

occurring in A occurs at least once as an antecedent part and at least once as a consequent part of A. (Compare Smiley 1959, note 21.)

*Proof.* If a variable  $p$  occurs only as an antecedent {consequent} part of A, assign  $p$  the value 3 {0}; and assign all other variables in A the value 2. This assignment of values to the variables of A gives A the value 0. The proof is left to the reader.

The designation “antecedent” and “consequent” parts derives from a Gentzen formulation of  $E_I$  (Belnap 1959a). The ultimate premisses of a *sequenzen-kalkül* proof of B are primes  $p \vdash p$ , where, roughly speaking, the left  $p$  appears in B as an antecedent part, and the right  $p$  as a consequent part. For this reason, the theorem may be regarded as saying that the “tight” relevance condition satisfied by the primes is preserved in passing down the proof-tree to B.

As an immediate corollary, we see that no variable may occur just once in a theorem of  $E_I$ , which suggests that each variable-occurrence is essential to the theorem; theorems have no loose pieces, so to speak, which can be jiggled about while the rest of the theorem stays put. Hence

$$A \rightarrow B \rightarrow . B \rightarrow B,$$

$$A \rightarrow B \rightarrow . A \rightarrow A,$$

and Peirce’s “law”

$$A \rightarrow B \rightarrow A \rightarrow A,$$

all fail in  $E_I$ . The intuitive sense of Pierce’s formula is usually expressed in this way: one says “Well, if A follows simply from the fact that A entails something, then A must be true — since obviously A must entail something, itself for example.” But this reading, designed to make the formula sound plausible, seems also designed to pull the wool over our eyes, since an essential premiss is suppressed. It is of course true that if A follows from the fact that it entails *something it does entail* (an assumption hidden in the word “fact” above), *then* A must be true, and a theorem to this effect is provable in  $E_I$ :

$$A \rightarrow B \rightarrow . A \rightarrow B \rightarrow A \rightarrow A$$

(a special case of *restricted assertion*). But if A follows from the (quite possibly false) assumption that A entails B, this would hardly guarantee the truth of A.

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We have argued that valid inferences are necessarily valid, and that the antecedent in a valid inference must be relevant to the consequent. In view

of the long history of logic as a topic for investigation, and the near unanimity on these two points among logicians, it is surprising, indeed startling, that these issues should require re-arguing. That they do need arguing is a consequence of the almost equally unanimous contradictorily opposed feeling on the part of contemporary logicians that material and strict "implication" are implication relations, and that therefore necessity and relevance are not required for true implications. But, if we may be permitted to apply a result of the ingenious Bishop of Halberstadt,<sup>21</sup> if *both* of these views are correct, it follows that Man is a donkey.<sup>22</sup>

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<sup>21</sup> Ca. 1316–1390; see Boehner 1952, pp. 99–100.

<sup>22</sup> The results of this paper have been extended to entailment with truth-functions. With plausible axioms for negation, conjunction, and disjunction (due largely to Ackermann) we obtain a system E (Anderson and Belnap 1958) in which (1) all truth-functional tautologies are provable, (2) relevance, in the sense of variable-sharing, is retained (Belnap 1959), and (3) a syntactical completeness theorem, relatively to a subproof formulation, is provable (Anderson 1959). These results extend also to a pure functional calculus of first order.

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