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First Degree Entailments

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Preface

This paper generalizes some results of Belnap 1959b (presented more elaborately in Anderson and Belnap 1962), to a non-classical system of quantification theory, based on an intensional relation of entailment.

The fundamental idea behind the present quantificational treatment is most easily made clear by reference to the propositional calculus E (for which see Anderson 1959, together with works cited above). Let $A \rightarrow B$ be an entailment in which only truth-functions occur in A and B. In the system E we can rewrite A in disjunctive normal form, and B in conjunctive normal form, obtaining an equivalent formula (in the sense of the double arrow $\leftrightarrow$):

$$A_1 \vee \cdots \vee A_m \rightarrow B_1 \,\&\, \cdots \,\&\, B_n\,.$$

In the system E, as classically, the validity of the foregoing formula reduces to the validity of

$$A_i \rightarrow B_j$$

for each i and j.

Now classically there are three conditions under which we regard $A_i \rightarrow B_j$ as valid. (1) Contradiction on the left, (2) excluded middle on the right, (3) sharing of an atom (i.e., a variable or negated variable). In the system E, however, only condition (3) assures validity: a necessary and sufficient condition for provability of $A \rightarrow B$ in E is that each $A_i \rightarrow B_j$ share an atom. The arrow of E thus secures *revelance* of antecedent to consequent.

The proof techniques for the generalization are essentially due to Kanger 1957 (building on Gentzen 1934). Such interest as the paper may have lies in the application of strong and interesting classical techniques to a non-classical system of logic.

I. The System LEQ_1

A syntactical completeness theorem for the system EQ of entailment with quantification may be found in ANDERSON 1959, and a semantical completeness theorem for a certain fragment of E has been proved by BELNAP 1959b. But the general problem of providing EQ with an adequate semantical theory has posed difficulties. As a partial solution to the problem, we here undertake a discussion of *first-degree entailments* (i.e., entailments in which no arrow occurs within the scope of an arrow).

Notation for the system LEQ_1. We will assume that propositional variables are specified, and that well-formed formulas ("wffs") are as usual in treatments of first order functional calculi (with negation, disjunction, and existential quantification as primitive). x, y, z, range over individual variables; F, G, H, range over function variables; and A, B, C, D, range over wffs (all these with or without subscripts). $\overline{A}$ is the negation of A, $A \vee B$ is the disjunction of A and B, and $\exists x$ is an existential quantifier. Ax is a formula in which x may occur free, and Ay is the result of replacing every free occurrence of x in Ax by y, in such a way that y is free in Ay where ever x is free in Ax.

The system LEQ_1 of *first degree entailments* consists of sequents (in the manner of GENTZEN 1934) of the form $J \to K$ (where J, K, L, M, N, and P, range over sequences of wffs).

Axioms have the form

$$J, A, K \to L, A, M\ ,$$

where A is an atom (i.e., A is either a propositional variable, or the denial of such, or else has the form $F(x_1, \ldots, x_n)$ or $\overline{F(x_1, \ldots, x_n)}$); and we have the following ten rules of inference.

(R1)
$$\frac{J, A, K \to L}{J, \overline{\overline{A}}, K \to L}$$

(R2)
$$\frac{J \to K, A, L}{J \to K, \overline{\overline{A}}, L}$$

(R3)
$$\frac{J, A, K \to L \quad J, B, K \to L}{J, A \vee B, K \to L}$$

(R4)
$$\frac{J \to K, A, B, L}{J \to K, A \vee B, L}$$

(R5)
$$\frac{J, \overline{A}, \overline{B}, K \to L}{J, \overline{A \vee B}, K \to L}$$

(R6)
$$\frac{J \to K, \overline{A}, L \quad J \to K, \overline{B}, L}{J \to K, \overline{A \vee B}, L}$$

(R7)
$$\frac{J, Ay, K \to L}{J, \exists x Ax, K \to L}$$

(R8)
$$\frac{J \to K, Ay, L, \exists x Ax}{J \to K, \exists x Ax, L}$$

(R9)
$$\frac{J, \overline{Ay}, K, \overline{\exists x Ax} \to L}{J, \overline{\exists x Ax}, K \to L}$$

(R10)
$$\frac{J \to K, \overline{Ay}, L}{J \to K, \overline{\exists x Ax}, L}$$

In (R 7) and (R 10) it is required that y not occur free in the conclusion; and we also require a rule for alphabetic change of bound variables.

II. Semantics

Our problem in this section is to provide appropriate semantics for LEQ_1, relatively to which LEQ_1 may be proved complete. To this end we begin by postulating the existence of propositions $\underline{p}$, $\underline{q}$, etc., and of classes $\underline{X}$, $\underline{Y}$, and $\underline{Z}$, of such propositions. (We conceive of propositions as intensional entities.) Associated with each proposition $\underline{p}$ we postulate the existence of a unique proposition $\underline{q}$ which is the denial (negation) of $\underline{p}$, where $\underline{q}$ is such that the truth of $\underline{p}$ entails the denial of $\underline{q}$, and the falsehood of $\underline{p}$ entails the truth of $\underline{q}$. We write $\underline{p}'$ for the denial of $\underline{p}$, and where $\underline{X}$ is a class of propositions, $\underline{X}'$ is the class containing all and only the denials of members of $\underline{X}$. (Warning: $\underline{X}'$ is *not* the complement of $\underline{X}$.) Acting under the philosophical conviction that we have no *semantic* grounds for distinguishing between "positive" and "negative" propositions, we instead simply treat "is the negation of" as a symmetrical relation, and we reflect this view in the conditions that (i) if $\underline{p}$ is the denial of $\underline{q}$, then $\underline{q}$ is the denial of $\underline{p}$, and (ii) if $\underline{X} = \underline{Y}'$, then $\underline{X}' = \underline{Y}$ (where "$=$" is class identity). Then from $\underline{Y}' = \underline{Y}'$ it follows that $\underline{Y}'' = \underline{Y}$.

We now introduce a relation between classes of propositions, "$\underline{X}$ cons $\underline{Y}$", to be read "the proposition that all the members of $\underline{X}$ are true has as a logical consequence the proposition that at least one of the members of $\underline{Y}$ is true." We postulate that "cons" satisfies conditions (1)—(5) for every $\underline{X}$, $\underline{Y}$, and $\underline{Z}$:

(1) If $\underline{X}$ and $\underline{Y}$ share a member, then $\underline{X}$ cons $\underline{Y}$.

(2) If $\underline{X}$ cons $\underline{Y}$, and $\{\underline{p}\}$ cons $\underline{Z}$ for every $\underline{p}$ in $\underline{Y}$, then $\underline{X}$ cons $\underline{Z}$.

(3) If $\underline{X}$ cons $\{\underline{p}\}$ for every $\underline{p}$ in $\underline{Y}$, and $\underline{Y}$ cons $\underline{Z}$, then $\underline{X}$ cons $\underline{Z}$[1]).

(4) If $\underline{X}$ cons $\underline{Y}$, then $\underline{Y}'$ cons $\underline{X}'$.

We also postulate that for each class $\underline{Z}$ of propositions, there is a proposition $\underline{p}$ satisfying the following condition: for every $\underline{X}$ and $\underline{Y}$,

(5′) $\underline{X} \cup \{\underline{p}\}$ cons $\underline{Y}$ iff for every $\underline{q}$ in $\underline{Z}$, $\underline{X} \cup \{\underline{q}\}$ cons $\underline{Y}$.

("iff" abbreviates "if and only if.")

We first show that, given $\underline{Z}$, the proposition $\underline{p}$ of (5′) is unique. For suppose that for every $\underline{X}$ and $\underline{Y}$,

(5″) $\underline{X} \cup \{\underline{r}\}$ cons $\underline{Y}$ iff for every $\underline{q}$ in $\underline{Z}$, $\underline{X} \cup \{\underline{q}\}$ cons $\underline{Y}$.

Then we have immediately that, for every $\underline{X}$ and $\underline{Y}$,

(6) $\underline{X} \cup \{\underline{r}\}$ cons $\underline{Y}$ iff $\underline{X} \cup \{\underline{p}\}$ cons $\underline{Y}$.

From which (taking $\underline{X}$ as void and $\underline{Y}$ as $\{\underline{r}\} \cup \underline{W}$, and then as $\{\underline{p}\} \cup \underline{W}$), we get by (1)

$$\{\underline{p}\} \text{ cons } \{\underline{r}\} \cup \underline{W}, \text{ and } \{\underline{r}\} \text{ cons } \{\underline{p}\} \cup \underline{W}.$$

Moreover, by (1) and the latter we have that

$$\text{for every } \underline{q} \text{ in } \{\underline{p}\} \cup \underline{W},\ \{\underline{q}\} \text{ cons } \{\underline{r}\} \cup \underline{W}, \text{ and}$$

$$\text{for every } \underline{q} \text{ in } \{\underline{r}\} \cup \underline{W},\ \{\underline{q}\} \text{ cons } \{\underline{r}\} \cup \underline{W},$$

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[1]) This property of "cons" is not independent; it follows from (2) and (4).

hence by (2)—(3), we have for every $\underline{X}$ and $\underline{W}$,

$$\underline{X} \text{ cons } \{\underline{r}\} \cup \underline{W} \text{ iff } \underline{X} \text{ cons } \{\underline{p}\} \cup \underline{W}. \tag{7}$$

$\underline{p}$ is then unique (in the sense that (6) and (7) both hold), and we introduce the notation $\underline{D}\underline{Z}$ (read "disjointly, $\underline{Z}$") for $\underline{p}$; so that (5′) may now be rewritten as

$$\underline{X} \cup \{\underline{D}\underline{Z}\} \text{ cons } \underline{Y} \text{ iff for every } \underline{q} \text{ in } \underline{Z},\ \underline{X} \cup \{\underline{q}\} \text{ cons } \underline{Y}. \tag{5}$$

In consequence of (1)—(5), we have properties (8)—(10) below for $\underline{D}$ (which, incidentally, we may think of as a function taking a class of propositions $\underline{X}$ into a proposition $\underline{D}\underline{X}$)[2].

$$\underline{X} \text{ cons } \underline{Y} \cup \underline{Z} \text{ iff } \underline{X} \text{ cons } \underline{Y} \cup \{\underline{D}\underline{Z}\}. \tag{8}$$

For by (1), we have $\{\underline{q}\}$ cons $\underline{Y} \cup \underline{Z}$ for every $\underline{q}$ in $\underline{Z}$; hence by (5) (with $\underline{X}$ void), $\{\underline{D}\underline{Z}\}$ cons $\underline{Y} \cup \underline{Z}$. But this fact and (1) yield that $\{\underline{q}\}$ cons $\underline{Y} \cup \underline{Z}$, for every $\underline{q}$ in $\underline{Y} \cup \{\underline{D}\underline{Z}\}$. Hence by (2), if $\underline{X}$ cons $\underline{Y} \cup \{\underline{D}\underline{Z}\}$ then $\underline{X}$ cons $\underline{Y} \cup \underline{Z}$. For the converse, suppose that $\underline{X}$ cons $\underline{Y} \cup \underline{Z}$. Now for every $\underline{q}$ in $\underline{Y} \cup \underline{Z}$, $\{\underline{q}\}$ cons $\underline{Y} \cup \{\underline{D}\underline{Z}\}$, since if $\underline{q}$ is in $\underline{Y}$, $\{\underline{q}\}$ cons $\underline{Y} \cup \{\underline{D}\underline{Z}\}$ by (1), and if $\underline{q}$ is in $\underline{Z}$, $\{\underline{q}\}$ cons $\underline{Y} \cup \{\underline{D}\underline{Z}\}$ by (5) and (1). Then by (2), $\underline{X}$ cons $\underline{Y} \cup \{\underline{D}\underline{Z}\}$, as required.

$$\underline{X} \cup \underline{Y}' \text{ cons } \underline{Z} \text{ iff } \underline{X} \cup \{\underline{D}\,\underline{Y}\}' \text{ cons } \underline{Z}. \tag{9}$$

For every $\underline{q}$ in $\underline{X} \cup \underline{Y}'$, by (1) $\{\underline{q}\}'$ cons $\underline{X}' \cup \underline{Y}$, and hence by (8), $\{\underline{q}\}'$ cons $\underline{X}' \cup \{\underline{D}\,\underline{Y}\}$; so by (4) $\underline{X} \cup \{\underline{D}\,\underline{Y}\}'$ cons $\{\underline{q}\}$, for every $\underline{q}$ in $\underline{X} \cup \underline{Y}'$. Hence by (3), if $\underline{X} \cup \underline{Y}'$ cons $\underline{Z}$, then $\underline{X} \cup \{\underline{D}\,\underline{Y}\}'$ cons $\underline{Z}$. The converse is similar.

And we state finally without proof,

$$\underline{X} \text{ cons } \underline{Y} \cup \{\underline{D}\underline{Z}\}' \text{ iff for every } \underline{q} \text{ in } \underline{Z},\ \underline{X} \text{ cons } \underline{Y} \cup \{\underline{q}\}'. \tag{10}$$

We now introduce the notion of a *frame*, i.e., a triple $\langle \underline{P}, \underline{I}, \underline{F} \rangle$ (where $\underline{P}$ is a non-empty class of propositions, $\underline{I}$ is a non-empty class of individuals, and $\underline{F}$ is the set of all functions taking n-tuples of individuals into members of $\underline{P}$), such that $\underline{P}$ satisfies the following two conditions for every non-empty subset $\underline{X}$ of $\underline{P}$:

$$\text{if } \underline{X} \subseteq \underline{P}, \text{ then } \underline{X}' \subseteq \underline{P}, \text{ and} \tag{11}$$

$$\text{if } \underline{X} \subseteq \underline{P}, \text{ then } \underline{D}\underline{X} \in \underline{P}. \tag{12}$$

An *assignment* of values for a wff A of LEQ_1 from $\langle \underline{P}, \underline{I}, \underline{F} \rangle$ consists of a function which (i) gives to each free individual variable of A an element of $\underline{I}$, (ii) gives to each function variable of A an element of $\underline{F}$, and (iii) gives to each propositional variable of A an element of $\underline{P}$. (We let $\underline{p}$ range over $\underline{P}$, $\underline{i}$ range over $\underline{I}$, and $\underline{f}$ range over one-argument functions in $\underline{F}$; $\underline{f}(\underline{i})$ denotes the value in $\underline{P}$ assumed by $\underline{f}$ in $\underline{F}$ for the argument $\underline{i}$ in $\underline{I}$.) Then each wf part of A assumes a propositional value in $\underline{P}$ as follows:

A propositional variable B assumes as value the proposition $\underline{p}$ in $\underline{P}$ for the value $\underline{p}$ in $\underline{P}$ of B.

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[2] Dually, we could introduce a function CZ (read "conjointly, Z") satisfying the condition: X cons $Y \cup \{CZ\}$ iff for every q in Z, X cons $Y \cup \{q\}$; and then continue the development in a similar (dual) way.

$F(x_1, \ldots, x_n)$ assumes as value the proposition $\underline{p}$ in $\underline{P}$ into which the function assigned to F takes the individuals assigned to $x_1, \ldots, x_n$.

If, for a system of values of the free variables of B, B assumes as value the proposition $\underline{p}$, then $\overline{B}$ assumes as value the denial of $\underline{p}$.

If moreover the value assumed by C is $\underline{q}$, then $B \vee C$ assumes as value in $\underline{P}$ the value $\underline{D}\,\{\underline{p}, \underline{q}\}$.

And if, for a system of values for the free variables of $\exists x B$, B assumes the value $\underline{f}$ in $\underline{F}$, then $\exists x B$ assumes as value the proposition $\underline{D}\,\underline{X}$, where $\underline{X}$ is the range of values assumed by $\underline{f}$ (i.e., $\underline{X}$ has $\underline{f}(\underline{i})$ as a member for every $\underline{i}$ in $\underline{I}$).

We see inductively that if values from $\langle \underline{P}, \underline{I}, \underline{F} \rangle$ are assigned to all free variables in $J \to K$, then for this assignment the wffs in J and K all assume propositions in $\underline{P}$ as values. For a given assignment of values, we let $\underline{X}_J\,[\underline{Y}_K]$ be the class of propositions assumed by $J\,[K]$, as values. We then call $J \to K$ *true in* $\langle \underline{P}, \underline{I}, \underline{F} \rangle$, for an assignment of values from $\langle \underline{P}, \underline{I}, \underline{F} \rangle$, if $\underline{X}_J$ cons $\underline{Y}_K$. And $J \to K$ is *valid in* $\langle \underline{P}, \underline{I}, \underline{F} \rangle$ if $J \to K$ is true in $\langle \underline{P}, \underline{I}, \underline{F} \rangle$ for every assignment from $\langle \underline{P}, \underline{I}, \underline{F} \rangle$. Finally, $J \to K$ is *valid*, if it is valid in every frame.

III. Consistency

Theorem. LEQ_1 is consistent (i.e., if $J \to K$ is provable in LEQ_1, then $J \to K$ is valid).

Proof. The axioms of LEQ_1 are all valid in every frame (by (1)), and the rules all preserve this property. We consider four cases as examples.

Case (R1). If $J \to K$ is a consequence of $L \to K$ by R1, then in every frame $\underline{X}_J = \underline{X}_L$; hence if $\underline{X}_L$ cons $\underline{Y}_K$, then $\underline{X}_J$ cons $\underline{Y}_K$.

Case (R6). Suppose that $J \to K, \overline{A}, L$ and $J \to K, \overline{B}, L$ are both valid. Then $\underline{X}_J$ cons $\underline{Y}_K \cup \{\underline{p}\}' \cup \underline{Z}_L$ and $\underline{X}_J$ cons $\underline{Y}_K \cup \{\underline{q}\}' \cup \underline{Z}_L$, where $A\,[B]$ has the value $\underline{p}\,[\underline{q}]$. Then for every $\underline{r}$ in $\{\underline{p}, \underline{q}\}'$, $\underline{X}_J$ cons $\underline{Y}_K \cup \{\underline{r}\} \cup \underline{Z}_L$, so (10) applies, and we have $\underline{X}_J$ cons $\underline{Y}_K \cup \{\underline{D}\{\underline{p}, \underline{q}\}\}' \cup \underline{Z}_L$. Since nothing in the argument depends on the choice of frame or assignment, we conclude that $J \to K, \overline{A \vee B}, L$ is valid.

Case (R7). Suppose that $J, Ax, K \to L$ is valid. Choose a frame, and an assignment of values to all variables except x. Since x does not occur in J, K, or L, this assignment assigns propositional values to every member of J, K, and L, and moreover gives a value $\underline{f}$ to Ax. Since $J, Ax, K \to L$ is valid, $J, Ax, K \to L$ is true for every assignment of values to x; i.e., $\underline{X}_J \cup \{\underline{p}\} \cup \underline{Y}_K$ cons $\underline{Z}_L$ for every $\underline{p}$ in the range $\underline{W}$ of $\underline{f}$. But then by (5), $\underline{X}_J \cup \{\underline{D}\,\underline{W}\} \cup \underline{Y}_K$ cons $\underline{Z}_L$; i.e., $J, \exists x\, Ax, K \to L$ is valid.

Case (R9). Suppose that $J, \overline{Ay}, K, \overline{\exists x\, Ax} \to L$ is valid. Then $\underline{X}_J \cup \{\underline{p}\}' \cup \underline{Y}_K \cup \{\underline{D}\,\underline{W}\}'$ cons $\underline{Z}_L$, where Ay assumes the value $\underline{p}$, and where $\underline{W}$ is the range of values assumed by $\underline{f}$ for every $\underline{i}$ in $\underline{I}$, $\underline{f}$ being the value of Ax. But $\underline{X}_J \cup \{\underline{p}\}' \cup \underline{Y}_K \cup \{\underline{D}\,\underline{W}\}'$ cons $\underline{Z}_L$ iff (by (9)) $\underline{X}_J \cup \{\underline{p}\}' \cup \underline{Y}_K \cup \underline{W}'$ cons $\underline{Z}_L$, iff (since $\underline{p} \in \underline{W}$) $\underline{X}_J \cup \underline{W}' \cup \underline{Y}_K$ cons $\underline{Z}_L$, iff (by (9) again) $\underline{X}_J \cup \{\underline{D}\,\underline{W}\}' \cup \underline{Y}_K$ cons $\underline{Z}_L$. So $J, \overline{\exists x\, Ax}, K \to L$ is valid as required.

Other cases are similar.

IV. Completeness

Theorem. LEQ_1 is complete (i.e., if $J \to K$ is valid, then $J \to K$ is provable in LEQ_1).

Proof. After introducing some terminology, we begin by describing a uniform proof procedure for LEQ_1.

If A, in a sequent $J, A, K \to L$ or $J \to K, A, L$, is a propositional variable or a negated propositional variable, or has the form $F(x_1, \dots, x_n)$ or $\overline{F(x_1, \dots, x_n)}$, then we call A an *atom* of the sequent; otherwise A is *non-atomic.*

Without loss of generality (in view of the rule for alphabetic change of bound variables), we restrict attention to sequents in which no individual variable occurs both bound and free, and if $J \to K$ is a candidate for the proof procedure, then $V (= \{v_1, v_2, \dots\})$ is the set of all variables (in alphabetic order) which do not occur bound in $J \to K$.

Now given a candidate sequent $J \to K$ of LEQ_1, we attempt to construct a dendriform proof. The tree consists of branches, and a *full normal branch*[3]) is defined as a sequence $S_0, \dots, S_i, \dots$ (possibly finite, possibly infinite) of sequents, satisfying the following conditions:

S_0 is $J \to K$.

If S_i is an axiom, or if S_i contains no non-atomic formulas, then the branch terminates at S_i; otherwise we obtain S_{i+1} by an L-rule or an R-rule below (beginning with an L-rule, if possible, and alternating between applications of L-rules and R-rules as much as possible):

L-rules: If S_i has the form $J, A, K \to L$, where A is the leftmost non-atomic formula in the antecedent, then

(a) if A is $\overline{\overline{B}}$, then S_{i+1} is $J, B, K \to L$;

(b) if A is $B \vee C$, then S_{i+1} may be either $J, B, K \to L$ or $J, C, K \to L$;

(c) if A is $\overline{B \vee C}$, then S_{i+1} is $J, \overline{B}, \overline{C}, K \to L$;

(d) if A is $\exists x\, Bx$, then S_{i+1} is $J, Bv_k, K \to L$, where v_k is the first individual variable in V which occurs in no S_j for $j \leq i$; and Bv_k is the result of writing v_k for free x throughout Bx; and

(e) if A is $\overline{\exists x\, Bx}$, then S_{i+1} is $J, \overline{Bv_k}, K, \overline{\exists x\, Bx} \to L$, where v_k is the first individual variable in V such that $\overline{Bv_k}$ occurs in the antecedent of no S_j for $j \leq i$ (where again Bv_k is the result of writing v_k for free x throughout Bx).

R-rules. If S_i has the form $J \to K, A, L$, where A is the leftmost non-atomic formula in the consequent, then

(a) if A is $\overline{\overline{B}}$, then S_{i+1} is $J \to K, B, L$;

(b) if A is $B \vee C$, then S_{i+1} is $J \to K, B, C, L$;

(c) if A is $\overline{B \vee C}$, then S_{i+1} is $J \to K, \overline{B}, L$, or $J \to K, \overline{C}, L$;

(d) if A is $\exists x\, Bx$, then S_{i+1} is $J \to K, Bv_k, L, \exists x\, Bx$, where v_k is the first individual variable in V such that Bv_k occurs in the consequent of no S_j for $j \leq i$.

(e) if A is $\overline{\exists x\, Bx}$, then S_{i+1} is $J \to K, \overline{Bv_k}, L$, where [as in L-rule (d)].

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[3]) The terminology is that of KANGER 1957, from whom the ideas of Lemmas I and II below were adapted.

We introduce the following notation. Where ϕ is a full normal branch for $J \rightarrow K$, we use $L(\phi)$ to refer to the class of formulas occurring in the antecedents of sequents in ϕ, and $R(\phi)$ for the class of formulas occurring in consequents.

Lemma I. If ϕ is a full normal branch for $J \rightarrow K$ not terminating in an axiom, then there is no atom A which is in both $L(\phi)$ and $R(\phi)$.

Proof. The rules for constructing full normal branches guarantee that no atoms are lost in passing up the branch. Hence if $L(\phi)$ and $R(\phi)$ shared an atom, the branch would terminate in an axiom.

Lemma II. If ϕ is a full normal branch not terminating in an axiom, and if (1) $\overline{\overline{A}}$, (2) $A \vee B$, (3) $\overline{A \vee B}$, (4) $\exists x\, A x$, (5) $\overline{\exists x\, A x}$ occurs in $L(\phi)$, then so does (1) A, (2) A [or B], (3) $\overline{A}$ and $\overline{B}$, (4) $A v_{\underline{k}}$ for some $v_{\underline{k}}$, (5) $\overline{A v_{\underline{k}}}$ for every $v_{\underline{k}}$; and if (1) $\overline{\overline{A}}$, (2) $A \vee B$, (3) $\overline{A \vee B}$, (4) $\exists x\, A x$, (5) $\overline{\exists x\, A x}$ occurs in $R(\phi)$, then so does (1) A, (2) A and B, (3) $\overline{A}$ [or $\overline{B}$], (4) $A v_{\underline{k}}$ for every $v_{\underline{k}}$, (5) $\overline{A v_{\underline{k}}}$ for some $v_{\underline{k}}$.

Proof. By rules for branch construction.

We come now to the principal Lemma (required for the proof of the completeness of LEQ_1), which shows that validity depends in an essential way on the *atoms* in a full normal branch.

Lemma III. Let ϕ be a full normal branch for $J \rightarrow K$ not terminating in an axiom, and consider an assignment of values to the variables in ϕ from a frame $\langle \underline{P}, \underline{I}, \underline{F} \rangle$. Then if there exist subsets M of $L(\phi)$ and N of $R(\phi)$ such that $\underline{\underline{X}}_M$ cons $\underline{Y}_N$ under this assignment, then there exist subsets M_0 of the atoms of $L(\phi)$ and N_0 of the atoms of $R(\phi)$ such that $\underline{\underline{X}}_{M_0}$ cons $\underline{Y}_{N_0}$, under the same assignment.

Proof. We define the *grade of a formula* or class of formulas as follows. If a wff A is an atom, then it has grade zero; if A has grade $\underline{n}$, then $\exists x A$ has grade $\underline{n} + 1$; if A has grade $\underline{n}$, then $\overline{A}$ has grade $\underline{n} + 1$ (unless $\overline{A}$ is an atom); and if A has grade $\underline{n}$ and B has grade $\underline{m}$, then $A \vee B$ has grade $\max(\underline{m}, \underline{n}) + 1$. And the *grade of a subset* of $L(\phi)$ or $R(\phi)$ is the maximum grade among the wffs of that subset.

The proof is by induction, first on the grade $\underline{n}$ of N, and then on the grade $\underline{m}$ of M. Assume that for M_0 of grade zero, and for every N^* of grade less than $\underline{n}$, the Lemma holds. We show that it holds also where N is of grade $\underline{n}$. The Lemma is trivial for N of grade zero, and we consider N of grade $\underline{n} > 0$. Given N, we define N^* as follows. For every A in N:

(0) if A has grade less than $\underline{n}$, A is in N^*. Otherwise A has grade $\underline{n}$, and

(i) A has the form $\overline{\overline{B}}$. Then B is in N^*.

(ii) A has the form $B \vee C$. Then both B and C are in N^*.

(iii) A has the form $\overline{B \vee C}$. Then if $\overline{B}[\overline{C}]$ is in $R(\phi)$ (by Lemma II at least one will be), then $\overline{B}[\overline{C}]$ is in N^*.

(iv) A has the form $\exists x\, Bx$. Then $Bv_{\underline{i}}$ is in N^* for every $v_{\underline{i}}$.

(v) A has the form $\overline{\exists x\, Bx}$. Then if $\overline{Bv_{\underline{i}}}$ is in $R(\phi)$ (by Lemma II, $\overline{Bv_{\underline{i}}}$ is in $R(\phi)$ for some $v_{\underline{i}}$), then $\overline{Bv_{\underline{i}}}$ is in N^*.

We now establish that for every A in N, $\underline{\underline{X}}_{\{A\}}$ cons $\underline{\underline{Y}}_{N*}$. For the following cases, we suppose that the value assumed by $B[C]$ is $\underline{p}[\underline{q}]$, and that for the assignment to all the free variables of Bx except x, Bx assumes the value $\underline{f}$ (in $\underline{F}$). Moreover, for each v_i in V, v_i assumes the value $\underline{i}$ in $\underline{I}$. Then we consider cases as follows.

(0) A is in N^*. Then $\underline{\underline{X}}_{\{A\}}$ cons $\underline{\underline{Y}}_{N*}$ by (1).

For the remaining cases, A is in not N^*.

(i) A has the form $\overline{\overline{B}}$. Then $\underline{\underline{X}}_{\{A\}}$ is $\underline{p}''$, i.e., $\underline{p}$. But B also has the value $\underline{p}$. and B is in N^* by the definition of N^*; hence $\underline{\underline{X}}_{\{A\}}$ cons $\underline{\underline{Y}}_{N*}$ by (1).

(ii) A has the form $B \vee C$. Then $\underline{\underline{X}}_{\{A\}}$ is $\underline{D}\{\underline{p}, \underline{q}\}$, and by the definition of N^*, both B and C are in N^*. Hence both $\underline{p}$ and $\underline{q}$ are in $\underline{\underline{Y}}_{N*}$, so by (1), $\underline{p}$ cons $\underline{\underline{Y}}_{N*}$ and $\underline{q}$ cons $\underline{\underline{Y}}_{N*}$. But then by (5), $\underline{D}\{\underline{p}, \underline{q}\}$ cons $\underline{\underline{Y}}_{N*}$; i.e., $\underline{\underline{X}}_{\{A\}}$ cons $\underline{\underline{Y}}_{N*}$.

(iii) A has the form $\overline{B \vee C}$. Then $\underline{\underline{X}}_{\{A\}}$ is $\{\underline{D}\{\underline{p}, \underline{q}\}\}'$, and by the definition of N^*, either $\overline{B}$ or $\overline{C}$ is in N^*. Hence either $\underline{p}'$ or $\underline{q}'$ is in $\underline{\underline{Y}}_{N*}$. Then by (1), $\{\underline{p}', \underline{q}'\}$ cons $\underline{\underline{Y}}_{N*}$; and by (9), $\{\underline{D}\{\underline{p}, \underline{q}\}\}'$ cons $\underline{\underline{Y}}_{N*}$; i.e., $\underline{\underline{X}}_{\{A\}}$ cons $\underline{\underline{Y}}_{N*}$.

(iv) A has the form $\exists x\, Bx$. Then $\underline{\underline{X}}_{\{A\}}$ is $\underline{D}\underline{\underline{Z}}$, where $\underline{\underline{Z}}$ is the range of values of the value $\underline{f}$ of Bx. By the definition of N^*, Bv_i is in N^* for every v_i. By (1), $\{\underline{f}(\underline{i})\}$ cons $\underline{\underline{Y}}_{N*}$ for every $\underline{f}(\underline{i})$ in $\underline{\underline{Z}}$; hence by (5), $\underline{D}\underline{\underline{Z}}$ cons $\underline{\underline{Y}}_{N*}$; i.e., $\underline{\underline{X}}_{\{A\}}$ cons $\underline{\underline{Y}}_{N*}$.

(v) A has the form $\overline{\exists x\, Bx}$. Then $\underline{\underline{X}}_{\{A\}}$ is $\{\underline{D}\underline{\underline{Z}}\}'$, where $\underline{\underline{Z}}$ is as in (iv). By the definition of N^*, $\overline{Bv_i}$ is in N^* for some v_i. By (1), $\underline{\underline{Z}}'$ cons $\underline{\underline{Y}}_{N*}$. Hence by (9), $\{\underline{D}\underline{\underline{Z}}\}'$ cons $\underline{\underline{Y}}_{N*}$; i.e., $\underline{\underline{X}}_{\{A\}}$ cons $\underline{\underline{Y}}_{N*}$.

It follows then by (2) that if $\underline{\underline{X}}_{M_0}$ cons $\underline{\underline{Y}}_N$, then $\underline{\underline{X}}_{M_0}$ cons $\underline{\underline{Y}}_{N*}$, where, by Lemma II, N^* is a subclass of $R(\phi)$, and is of grade less than $\underline{n}$. But by the inductive hypothesis, if $\underline{\underline{X}}_{M_0}$ cons $\underline{\underline{Y}}_{N*}$, then $\underline{\underline{X}}_{M_0}$ cons $\underline{\underline{Y}}_{N_0}$ for some subset N_0 of the atoms of $R(\phi)$; hence if $\underline{\underline{X}}_{M_0}$ cons $\underline{\underline{Y}}_N$, then $\underline{\underline{X}}_{M_0}$ cons $\underline{\underline{Y}}_{N_0}$.

We wish now to establish that if $\underline{\underline{X}}_M$ cons $\underline{\underline{Y}}_N$, then $\underline{\underline{X}}_{M_0}$ cons $\underline{\underline{Y}}_{N_0}$, for some subset M_0 of the atoms of $L(\phi)$. Assume then that for every N, and for every M^* of grade less than m, the Lemma holds. We show that it holds also where M is of grade m. The previous induction secured the Lemma for M of grade zero, and we now consider M of grade $m > 0$.

We define M^* as follows. For every A in M:

(0) if A has grade less than m, then A is in M^*. Otherwise A has grade m, and

(i) A has the form $\overline{\overline{B}}$. Then B is in M^*.

(ii) A has the form $B \vee C$. Then if $B[C]$ is in $L(\phi)$ (by Lemma II at least one will be), then $B[C]$ is in M^*.

(iii) A has the form $\overline{B \vee C}$. Then both $\overline{B}$ and $\overline{C}$ are in M^*.

(iv) A has the form $\exists x\, Bx$. Then if Bv_i is in $L(\phi)$ (by Lemma II it will be in $L(\phi)$ for some v_i), then Bv_i is in M^*.

(v) A has the form $\overline{\exists x\, Bx}$. Then $\overline{Bv_i}$ is in M^* for every v_i.

Then by methods exactly dual to the foregoing, we establish that for every A in M, $\underline{\underline{X}}_{M*}$ cons $\underline{\underline{Y}}_{\{A\}}$. Then with the help of (3) and the two inductions, the Lemma is secured.

Returning now to the completeness of LEQ_1, we first note that if $J \rightarrow K$ is unprovable, then some branch of the full normal tree for $J \rightarrow K$ fails to terminate in an axiom. Then we want a frame $\langle \underline{P}, \underline{I}, \underline{F} \rangle$ from which we can assign values to variables in $J \rightarrow K$, in such a way that $\underline{X}_J \ \overline{\text{cons}}\ \underline{Y}_K$ (i.e., it is not the case that $\underline{X}_J$ cons $\underline{Y}_K$). Where ϕ is a full normal branch for $J \rightarrow K$ not terminating in an axiom, it suffices to find an assignment such that for every subset M_0 of the atoms of $L(\phi)$ and N_0 of the atoms of $R(\phi)$, $\underline{X}_{M_0} \ \overline{\text{cons}}\ \underline{Y}_{N_0}$. For by Lemma III, it will follow that for all subsets M of $L(\phi)$ and N of $R(\phi)$, $\underline{X}_M \ \overline{\text{cons}}\ \underline{Y}_N$; and in particular $\underline{X}_J \ \overline{\text{cons}}\ \underline{Y}_K$.

We now introduce the notion of an *atomic frame*, defined as follows.

If $\underline{Y}$ is a class of propositions, and $\underline{P}$ is the closure of $\underline{Y}$ under $'$ and $\underline{D}$, we will say that $\underline{P}$ is *generated* by $\underline{Y}$. We postulate the existence of *atomic classes* $\underline{Y}$ of propositions, satisfying the condition that for all subsets $\underline{X}$ and $\underline{W}$ of $\underline{Y} \cup \underline{Y}'$, $\underline{X}$ cons $\underline{W}$ if *and only if* $\underline{X} \cap \underline{W} \neq 0$.[4] If $\underline{P}$ is generated by an atomic class of propositions, we say that $\langle \underline{P}, \underline{I}, \underline{F} \rangle$ is an *atomic frame.*

Now let ϕ be a full normal branch, which does not terminate in an axiom, and assign values to the constants and variables in ϕ as follows.

We choose an atomic frame $\langle \underline{P}_1, \underline{I}_1, \underline{F}_1 \rangle$, where $\underline{P}_1$ is generated by an atomic class $\{\underline{p}, \underline{q}, \underline{r}\}$ of propositions; $\underline{I}_1$ is the natural numbers (and $\underline{F}_1$ is a set of functions taking $\underline{n}$-tuples of natural numbers into $\underline{P}_1$).

Then to each individual variable v_i we give as value the natural number $\underline{i}$.

If A is a propositional variable, then:

(i) If neither A nor $\overline{A}$ occurs in $R(\phi)$, give A the value $\underline{p}$.

(ii) If A is not given a value by (i), and if neither A nor $\overline{A}$ occurs in $L(\phi)$, give A the value $\underline{q}$.

(iii) If A occurs in $L(\phi)$ and $\overline{A}$ occurs in $R(\phi)$, give A the value $\underline{r}$.

(iv) If $\overline{A}$ occurs in $L(\phi)$ and A occurs in $R(\phi)$, give A the value $\underline{r}'$.

And to each function letter F we assign as value that function $\underline{f}$ taking the $\underline{n}$-tuple $\underline{i}_1, \ldots, \underline{i}_n$ into

(i) $\underline{p}$, if neither $F(v_{i_1}, \ldots, v_{i_n})$ nor its negate occurs in $R(\phi)$;

(ii) $\underline{q}$, if $F(v_{i_1}, \ldots, v_{i_n})$ has not been given a value by (i), and if neither $F(v_{i_1}, \ldots, v_{i_n})$ nor its negate occurs in $L(\phi)$;

(iii) $\underline{r}$, if $F(v_{i_1}, \ldots, v_{i_n})$ occurs in $L(\phi)$ and $\overline{F(v_{i_1}, \ldots, v_{i_n})}$ occurs in $R(\phi)$; and

(iv) $\underline{r}'$, if $\overline{F(v_{i_1}, \ldots, v_{i_n})}$ occurs in $L(\phi)$ and $F(v_{i_1}, \ldots, v_{i_n})$ occurs in $R(\phi)$.

(The possibility of this assignment is guaranteed by Lemma I.)

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[4] The atoms of an atomic frame are thus *logically independent*; as an example we mention the frame generated by {the proposition that MICHELANGELO designed the Sistine Chapel, the proposition that SCOTT was the author of Waverly}. No one really believes that there is any logical connection between these two propositions, though those who believe that material "implication" is an implication relation must maintain that "If MICHELANGELO designed the Sistine Chapel, then SCOTT was the author of Waverly," is true. We think it is false, and self-evidently so. Certainly there *is* a sense in which the two propositions are logically independent of one another — and that is the sense invoked here.

For further consideration of the underlying philosophical issues, see ANDERSON and BELNAP 1960b.

Then for this assignment all the atoms in $L(\phi)$ take $\underline{p}$, $\underline{p}'$, or $\underline{r}$ as values' and all the atoms of $R(\phi)$ take the values $\underline{q}$, $\underline{q}'$, or $\underline{r}'$. Hence if $\underline{Z}$ is any subset of $\underline{X}_{M_0}$, and $\underline{W}$ is any subset of $\underline{Y}_{N_0}$, then $\underline{Z}\ \overline{\text{cons}}\ \underline{W}$ by the atomicity of $\langle \underline{P}_1, \underline{I}_1, \underline{F}_1 \rangle$; hence by Lemma III, $\underline{X}_J\ \overline{\text{cons}}\ \underline{Y}_K$.

And the completeness of LEQ_1 is then secured. If every branch terminates in an axiom, then $J \rightarrow K$ is provable, the tree constituting a proof. And if some branch fails to terminate in an axiom, then $J \rightarrow K$ is invalid in the atomic frame generated by $\{\underline{p}, \underline{q}, \underline{r}\}$.

V. Undecidability

Theorem. LEQ_1 is undecidable.

Proof. Any decision procedure for LEQ_1 could be made to yield a decision procedure for the classical first order functional calculus, as follows. Belnap 1959a provides a proof technique for formulas C of the first-order functional calculus, in which all rules are entailments, and all axioms are of the form ΠA, where Π is a sequence (possibly void) of universal quantifiers, and A is a (perhaps multiple) disjunction containing atoms B and $\overline{B}$ as disjuncts. Moreover, the positive atoms B are all subformulas of the formula C; hence if C is a valid formula of the first order predicate calculus, then the positive atoms B occurring in axioms leading to C are positive atomic subformulas of C. Now $\Pi(B \vee \overline{B}) \rightarrow \Pi'(D_1 \vee B \vee D_2 \vee \overline{B} \vee D_3)$ is a valid entailment in LEQ_1, where the D_i are arbitrary, and Π' contains all the quantifiers in Π (and perhaps some others). It follows that the conjunction

$$\Pi_1(B_1 \vee \overline{B}_1) \;\&\; \cdots \;\&\; \Pi_k(B_k \vee \overline{B}_k)$$

containing $\Pi_i(B_i \vee \overline{B}_i)$ (where Π_i contains a universal quantifier for each variable x in B_i) for each atom B_i in C, entails a conjunction of all the axioms leading to C in Belnap's treatment. And since the arrow of $J \rightarrow K$ in LEQ_1 is transitive when J and K each contain a single wff, it follows that the conjunction above entails the candidate C, if C is valid. Hence C is provable in the first order functional calculus iff

$$\Pi_1(B_1 \vee \overline{B}_1) \;\&\; \cdots \;\&\; \Pi_k(B_k \vee \overline{B}_k) \rightarrow C$$

is provable in LEQ_1, and the decision problems are equivalent. Hence LEQ_1 is undecidable.

VI. Löwenheim-Skolem Theorem

In the preceding section we defined *true* and *valid* for sequents $J \rightarrow K$ independently of the truth or falsity of the formulas in J and the formulas in K, a fact which reflects the intensional character of first degree entailments: the truth or falsity of such an entailment depends on a connection of meaning between antecedent and consequent, and is not simply a truth-function of them. But of course it is also possible to define truth for wffs, and, with a view to providing an appropriate version of the Löwenheim-Skolem theorem for LEQ_1, we now define *true* and *valid* for wffs (as opposed to sequents) of the system.

Given a frame $\langle \underline{P}, \underline{I}, \underline{F} \rangle$, we assume that for each $\underline{p}$ in $\underline{P}$, $\underline{p}$ is true if and only if it is not false; hence $\underline{p}$ is true iff $\underline{p}'$ is false. Moreover, $\underline{D}\underline{X}$ is true iff some member of $\underline{X}$ is true.

We say that A is *true in* $\langle \underline{P}, \underline{I}, \underline{F} \rangle$, for an assignment of values to the variables in A from $\langle \underline{P}, \underline{I}, \underline{F} \rangle$, if and only if the proposition assumed by A as value (under the valuation of section II) is true. A is *valid in* $\langle \underline{P}, \underline{I}, \underline{F} \rangle$ if A is true in $\langle \underline{P}, \underline{I}, \underline{F} \rangle$ for every assignment from $\langle \underline{P}, \underline{I}, \underline{F} \rangle$; and A is *valid* if A is valid in every frame.

We see at once that this definition coincides with the classical one for special frames $\langle \underline{P}_2, \underline{I}, \underline{F}_2 \rangle$, where $\underline{I}$ may be arbitrary, but $\underline{P}_2$ is generated by a class of propositions containing just $\underline{D}\{\underline{p}, \underline{p}'\}$ (for some $\underline{p}$). Since $\{\underline{D}\{\underline{p}, \underline{p}'\}\}'$ cons $\{\underline{D}\{\underline{p}, \underline{p}'\}\}$, it is easy to see inductively that every proposition in the resulting $\underline{P}_2$ is equivalent (in the sense of "cons") either with $\underline{D}\{\underline{p}, \underline{p}'\}$ or with $(\underline{D}\{\underline{p}, \underline{p}'\})'$; hence $\underline{P}_2$ contains exactly two propositions, one of which is true, and the other false. (We note that for an assignment of values from $\langle \underline{P}_2, \underline{I}, \underline{F}_2 \rangle$, $J \to K$ is true iff the conjunction of J materially "implies" the disjunction of K; that is, $\underline{X}_J$ cons $\underline{Y}_K$ iff $\underline{D}\{\underline{X}_{J'} \cup \underline{Y}_K\}$.)

Lemma. If A is true [false] for a given assignment Q of values from an arbitrary $\langle \underline{P}, \underline{I}, \underline{F} \rangle$, then it is true [false] in $\langle \underline{P}_2, \underline{I}, \underline{F}_2 \rangle$ for the following assignment Q^*:

(i) if Q assigns a true proposition to a variable B, then Q^* assigns $\underline{D}\{\underline{p}, \underline{p}'\}$ to B, and otherwise Q^* assigns $(\underline{D}\{\underline{p}, \underline{p}'\})'$ to B;

(ii) if Q assigns the individual $\underline{i}$ to the variable x, then Q^* does likewise; and

(iii) if Q assigns the function $\underline{f}$ to the $\underline{n}$-ary function variable F, then Q^* assigns the function $\underline{f}^*$ taking $\underline{i}_1, \ldots, \underline{i}_n$ into $\underline{D}\{\underline{p}, \underline{p}'\}$ if $\underline{f}(\underline{i}_1, \ldots, \underline{i}_n)$ is true, and into $(\underline{D}\{\underline{p}, \underline{p}'\})'$ otherwise.

Proof. Obvious from truth-conditions for atoms, and for formulas of the form $\overline{B}$, $B \vee C$, and $\exists x B$.

The relation between validity in general, and validity in $\langle \underline{P}_2, \underline{I}, \underline{F}_2 \rangle$ is then stated in the following

Theorem. A is valid if and only if A is valid in every frame $\langle \underline{P}_2, \underline{I}, \underline{F}_2 \rangle$.

Proof. It is trivial that if A is valid, it is valid in $\langle \underline{P}_2, \underline{I}, \underline{F}_2 \rangle$, and the converse follows immediately from the Lemma.

Our object now is to show that if a set of sequents is simultaneously satisfiable, it is so in a frame with denumerably many individuals, and for this we need definitions of satisfiability for $J \to K$. This in turn requires us to place an additional condition on the "cons" relation, to wit:

(13) If all the propositions in $\underline{X}$ are true, and all the propositions in $\underline{Y}$ are false, then $\underline{X}$ $\overline{\text{cons}}$ $\underline{Y}$[5]).

Then we say that $J \to K$ is *satisfiable* if there exists an assignment from a frame $\langle \underline{P}, \underline{I}, \underline{F} \rangle$, such that for this assignment $\underline{X}_J$ cons $\underline{Y}_K$; and $J \to K$ is *unsatisfiable* if for every assignment from every frame, $\underline{X}_J$ $\overline{\text{cons}}$ $\underline{Y}_K$.

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[5]) For present purposes, (13) could be weakened to (13′): if each member p of X is such that $\{p\}'$ cons $\{p\}$ and each member q of Y is such that $\{q\}$ cons $\{q\}'$, then X cons Y.

Löwenheim-Skolem theorem for LEQ_1. Let S be a set of sequents, simultaneously satisfiable in $\langle \underline{P}, \underline{I}, \underline{F} \rangle$. Then the members of S are simultaneously satisfiable in $\langle \underline{P}, \underline{I}_1, \underline{F} \rangle$, where $\underline{I}_1$ is denumerable.

Proof. For a satisfying assignment in $\langle \underline{P}, \underline{I}, \underline{F} \rangle$, each sequent $J \rightarrow K$ in S must have either a false A in J or a true B in K, by (13). Consider the set T containing the negation $\overline{A}$ of every A such that A is a false member of some antecedent of a sequent in S, and containing also every B such that B is a true member of the consequent of some member of S. All members of T are true under the assignment. Then choose a proposition $\underline{p}$ in $\underline{P}$, and consider the sub-frame $\langle \underline{P}_2^*, \underline{I}, \underline{F}_2^* \rangle$ of $\langle \underline{P}, \underline{I}, \underline{F} \rangle$, generated by $\underline{D}\{\underline{p}, \underline{p}'\}$. By the Lemma, the members of T are simultaneously satisfiable in $\langle \underline{P}_2^*, \underline{I}, \underline{F}_2^* \rangle$. But then (choosing $\underline{D}\{\underline{p}, \underline{p}'\}$ as classical *falsehood*), the sequents of T are simultaneously satisfiable in the classical sense, and the classical Löwenheim-Skolem theorem applies. Hence the members of T are simultaneously satisfiable in a frame $\langle \underline{P}_2^*, \underline{I}_1, \underline{F}_2^* \rangle$ with denumerable $\underline{I}_1$. But if $\underline{X}$ and $\underline{Y}$ are classes of propositions from $\underline{P}_2^*$, and either $\underline{X}$ contains $(\underline{D}\{\underline{p}, \underline{p}'\})'$ or $\underline{Y}$ contains $\underline{D}\{\underline{p}, \underline{p}'\}$, it is easy to see that $\underline{X}$ cons $\underline{Y}$. Hence in $\langle \underline{P}_2^*, \underline{I}_1, \underline{F}_2^* \rangle$, for every $J \rightarrow K$ in S, $\underline{X}_J$ cons $\underline{Y}_K$; and the sequents of S are simultaneously satisfied in $\langle \underline{P}_2^*, \underline{I}_1, \underline{F}_2^* \rangle$ — hence also in $\langle \underline{P}, \underline{I}_1, \underline{F} \rangle$.

By similar arguments it can also be shown that:

Theorem. $J \rightarrow K$ is unsatisfiable iff J is valid and K is unsatisfiable.

(We remark that then $J \rightarrow K$ is unsatisfiable if and only if the material "implication" from the members of J taken conjointly, to the members of K taken disjointly, is also unsatisfiable.)

VII. LEQ_1 and EQ

The system EQ (Anderson 1959) has the following axioms (with wffs as usual, and with dots replacing parentheses in accordance with the conventions of Church 1956):

(1) $A \rightarrow A \rightarrow B \rightarrow B$

(2) $A \rightarrow B \rightarrow . B \rightarrow C \rightarrow . A \rightarrow C$

(3) $(A \rightarrow . A \rightarrow B) \rightarrow . A \rightarrow B$

(4) $A \,\&\, B \rightarrow A$

(5) $A \,\&\, B \rightarrow B$

(6) $(A \rightarrow B) \,\&\, (A \rightarrow C) \rightarrow . A \rightarrow (B \,\&\, C)$

(7) $NA \,\&\, NB \rightarrow N(A \,\&\, B) \quad [NA =_{df} A \rightarrow A \rightarrow A\ .]$

(8) $A \rightarrow A \vee B$

(9) $B \rightarrow A \vee B$

(10) $(A \rightarrow C) \,\&\, (B \rightarrow C) \rightarrow . (A \vee B) \rightarrow C$

(11) $A \,\&\, (B \vee C) \rightarrow (A \,\&\, B) \vee C$

(12) $A \rightarrow \overline{\overline{A}} \rightarrow \overline{A}$

(13) $A \rightarrow \overline{B} \rightarrow . B \rightarrow \overline{A}$

(14) $$\overline{\overline{A}} \to A$$

(15) $$(x)\,A\,x \to A\,y$$

(16) $$(x)\,(B \to A) \to .\; B \to (x)\,A$$

(17) $$(x)\,(A \vee B) \to .\; (x)A \vee B$$

(18) $$(x)\,(A \to C) \to .\; (x)A \to (x)\,C$$

(19) $$(x)\,A \;\&\; (x)C \to (x)\,(A \;\&\; C)$$

(20) $$(x)\,NA \to N(x)\,A$$

And finally, if A is an axiom, so is $(x)A$. (In (15)—(20), A may contain x free, $A\,y$ is like $A\,x$ except that all free occurrences of x in $A\,x$ are replaced by free occurrences of y in $A\,y$, B does not contain x free, and C is any wff.)

The rules are *modus ponens* (for the arrow) and *adjunction*.

The formulation above takes entailment, conjunction, disjunction, negation, and universal quantification as primitive; but the axioms for conjunction are redundant, since conjunction may be defined in terms of disjunction and negation in the usual way, and shown to have the right properties. And existential quantification may also be defined in EQ. Similarly in LEQ_1, which takes entailment, negation, disjunction, and existential quantification as primitive, we may define conjunction and universal quantification. We shall assume that all these definitions have been made, and that $J \to K$ in LEQ_1 is to be interpreted in EQ as $A \to B$, where A is a conjunction of all the wffs in J, and B is a disjunction of all the wffs in K. (Since $C \& D$ $[C \vee D, (C \& D) \& E, (C \vee D) \vee E]$ is intersubstitutable with $D \& C$ $[D \vee C, C \& (D \& E), C \vee (D \vee E)]$ in both systems, it makes no difference how parentheses are associated in A and B.)

Theorem. Where A is a conjunction of all wffs in J, and B is a disjunction of all wffs in K, $A \to B$ is provable in EQ iff $J \to K$ is provable in LEQ_1.

Proof. It is easy to show that all the axioms and rules of LEQ_1 are, under the correspondence between wffs of EQ and sequents of LEQ_1, theorems and rules of EQ; hence EQ contains LEQ_1. For the proof of the converse we require the matrices on p. 315 (BELNAP 1959), in which $+$ values are designated.

The proof then proceeds as follows. We first show that, under an evaluation procedure to be explained, all theorems of EQ take designated values for all assignments from the matrices. Then we show that if a sequent $J \to K$ is unprovable in LEQ_1, then it can be made to assume an undesignated value from the matrix. It will follow then that if $A \to B$ is provable in EQ then it is also provable in LEQ_1; and the equivalence of the two systems (in the required sense) will be established.

Consider then a quasi-frame $\langle \underline{P}, \underline{I}, \underline{F} \rangle$ where $\underline{P}$ contains the eight "propositions" $-3, \ldots, +3$ of the matrices on p. 315, $\underline{I}$ is the set of non-negative integers, and $\underline{F}$ is the set of functions taking $\underline{n}$-tuples of members of $\underline{I}$ into $\underline{P}$. Given a wff A, we may then assign values from $\underline{P}$ to the propositional variables in A, values from $\underline{I}$ to the individual variables in A, and values from $\underline{F}$ to the function variables in A, and then compute the value in $\underline{P}$ of A as follows.

$A \to B$

	−3	−2	−1	−0	+0	+1	+2	+3
−3	+3	+3	+3	+3	+3	+3	+3	+3
−2	−3	+2	−3	+2	−3	−3	+2	+3
−1	−3	−3	+1	+1	−3	+1	−3	+3
−0	−3	−3	−3	+0	−3	−3	−3	+3
+0	−3	−2	−1	−0	+0	+1	+2	+3
+1	−3	−3	−1	−1	−3	+1	−3	+3
+2	−3	−2	−3	−2	−3	−3	+2	+3
+3	−3	−3	−3	−3	−3	−3	−3	+3

$\overline{A}$

−3	+3
−2	+2
−1	+1
−0	+0
+0	−0
+1	−1
+2	−2
+3	−3

$A \& B$

	−3	−2	−1	−0	+0	+1	+2	+3
−3	−3	−3	−3	−3	−3	−3	−3	−3
−2	−3	−2	−3	−2	−3	−3	−2	−2
−1	−3	−3	−1	−1	−3	−1	−3	−1
−0	−3	−2	−1	−0	−3	−1	−2	−0
+0	−3	−3	−3	−3	+0	+0	+0	+0
+1	−3	−3	−1	−1	+0	+1	+0	+1
+2	−3	−2	−3	−2	+0	+0	+2	+2
+3	−3	−2	−1	−0	+0	+1	+2	+3

$A \vee B$

	−3	−2	−1	−0	+0	+1	+2	+3
−3	−3	−2	−1	−0	+0	+1	+2	+3
−2	−2	−2	−0	−0	+2	+3	+2	+3
−1	−1	−0	−1	−0	+1	+1	+3	+3
−0	−0	−0	−0	−0	+3	+3	+3	+3
+0	+0	+2	+1	+3	+0	+1	+2	+3
+1	+1	+3	+1	+3	+1	+1	+3	+3
+2	+2	+2	+3	+3	+2	+3	+2	+3
+3	+3	+3	+3	+3	+3	+3	+3	+3

If A is a propositional variable, A assumes the value given to A under the assignment; and if A is $F(x_1, \ldots, x_n)$, then A assumes as value the value into which the function assigned to F takes the individuals assigned to $x_1, \ldots, x_n$.

If A has the form $\overline{B}$, $B \& C$, $B \vee C$, or $B \to C$, then the value of A is determined from the values of B and C by the matrices above.

If x is an individual variable, and A a wff in which x occurs free, then for a system of values for the free variables of $(x)A$, $(x)A$ assumes as value the value of the conjunction of those values assumed by A for all values of x. (And if x does not occur free in A, the value of $(x)A$ is the same as the value of A.)

(We remark that in calculating values of formulas in EQ, it follows from the definitions of disjunction and universal quantification, and from the condition above, that $(\exists x)A$ assumes as value the value of the disjunction of the values assumed by A for the various values of x.)

We leave to the reader the tedious but straightforward task of verifying that, under this evaluation procedure, all the axioms of EQ take designated values for every assignment to their variables, and that the rules of EQ preserve this property. It follows that all theorems of EQ satisfy the matrices.

Now let $J \to K$ be an unprovable sequent in LEQ_1. Then some full normal branch ϕ for $J \to K$ fails to terminate in an axiom. To the variables of this branch we assign values from the quasi-frame $\langle \underline{P}, \underline{I}, \underline{F} \rangle$ as follows (the possibility of this assignment being guaranteed by Lemma I of section IV):

If A is a propositional variable, then

(i) if A occurs in $L(\phi)$ but not in $R(\phi)$, give A the value $+1$;
(ii) if A occurs in $R(\phi)$ but not in $L(\phi)$, give A the value $+2$;
(iii) if A occurs in $L(\phi)$ and $\overline{A}$ occurs in $R(\phi)$, give A the value $+3$;
and
(iv) if $\overline{A}$ occurs in $L(\phi)$ and A occurs in $R(\phi)$, give A the value -3.

To each individual variable $v_{\underline{i}}$ give the value $\underline{i}$ from $\underline{I}$.

To each function letter F we assign as value that function $\underline{f}$ taking the $\underline{n}$-tuple $\underline{i}_1, \ldots, \underline{i}_{\underline{n}}$ into
(i) $+1$, if $F(x_1, \ldots, x_{\underline{n}})$ occurs in $L(\phi)$ but not in $R(\phi)$;
(ii) $+2$, if $F(x_1, \ldots, x_{\underline{n}})$ occurs in $R(\phi)$ but not in $L(\phi)$;
(iii) $+3$, if $F(x_1, \ldots, x_{\underline{n}})$ occurs in $L(\phi)$ and $\overline{F(x_1, \ldots, x_{\underline{n}})}$ occurs in $R(\phi)$;
and
(iv) -3, if $\overline{F(x_1, \ldots, x_{\underline{n}})}$ occurs in $L(\phi)$ and $F(x_1, \ldots, x_{\underline{n}})$ occurs in $R(\phi)$.

Now we claim that for this assignment, $J \rightarrow K$ takes an undesignated value. For (a) every member of $L(\phi)$ takes a value $+3$, $+1$, -1, or -0; and (b) every member of $R(\phi)$ takes the value $+2$, $+0$, -2, -3, $J \rightarrow K$ therefore taking an undesignated value. We prove (b), leaving (a) to the reader.

As a basis for the induction, all atoms in $R(\phi)$ assume values from $+2$, $+0$, -2, -3. Now suppose all members of $R(\phi)$ of length less than the length of A take one of those values. Then reference to the tables above shows that if A has the form $\overline{\overline{B}}$ $[B \vee C;\ \overline{B \vee C}]$, then A has one of the values $+2$, $+0$, -2, or -3. Hence by Lemma II and the inductive hypothesis, A has one of these values. And if A is $\exists x\, Bx$, then $Bv_{\underline{k}}$ occurs in $R(\phi)$ for every $\underline{k}$ (Lemma II); hence $Bv_{\underline{k}}$ takes only values $+2$, $+0$, -2, -3; hence by the table, $\exists x\, Bx$ is confined to the same values. Lastly, if A is $\overline{\exists x\, Bx}$, then for some $\underline{k}$, $\overline{Bv_{\underline{k}}}$ assumes $+2$, $+0$, -2, -3. Then $Bv_{\underline{k}}$ has a value -2, -0, $+2$, $+3$, whence $\exists x\, Bx$ has one of the latter values. Then $\overline{\exists x\, Bx}$ has $+2$, $+0$, -2, -3, as required.

Hence if $J \rightarrow K$ is unprovable in LEQ_1, it assumes an undesignated value, and is therefore unprovable in EQ; and the equivalence (in the appropriate sense) of the systems is established.

VIII. First Degree Formulas

A *first degree formula* of EQ is any wff in which no arrow occurs within the scope of an arrow. More generally, we define the *degree* of a formula of EQ as follows: any atom is of degree zero, and if A and B are of degree $\underline{m}$ and $\underline{n}$ respectively, then $\overline{A}$, $\exists x A$, and $(x)A$ are of degree $\underline{m}$, $A \vee B$ and $A \,\&\, B$ are of degree $\max(\underline{m}, \underline{n})$ and $A \rightarrow B$ is of degree $\max(\underline{m}, \underline{n}) + 1$. In view of the difficulty of providing appropriate semantics for EQ, it seems plausible that a start toward generalizing the results of the preceding sections might be made by considering first degree formulas. But even for this fragment of EQ difficulties arise almost at once, and our purpose in this section is to state some partial results, and raise some questions.

We define *validity* for first degree formulas as follows.

Given a frame $\langle \underline{P}, \underline{I}, \underline{F}\rangle$, and an assignment from $\langle \underline{P}, \underline{I}, \underline{F}\rangle$ to the free variables in a first degree formula A, we will say that A is *true* (in $\langle \underline{P}, \underline{I}, \underline{F}\rangle$, for the given assignment) under the following conditions:

If A is a propositional variable, then A is *true* iff the value $\underline{p}$ assigned from $\underline{P}$ to A is true.

If A has the form $F(x_1, \ldots, x_{\underline{n}})$, then A is true iff the proposition $\underline{p}$ into which the function $\underline{f}$ assigned to F takes the individuals $\underline{i}_1, \ldots, \underline{i}_{\underline{n}}$ assigned to the variables $x_1, \ldots, x_{\underline{n}}$, is true.

If A has the form $\overline{B}$, then A is true iff B is not.

If A has the form $B \,\&\, C$ $[B \vee C]$, then A is true iff both B and D are [either B or C is].

If A has the form $(x)Bx$ $[\exists x Bx]$, where x is an individual variable occurring free in Bx, then for the assignment to the free variables of $(x)Bx$ $[\exists x\, Bx]$, $(x)Bx$ $[\exists x\, Bx]$ is true iff for all values of x [for some value of x], Bx is true. (And if x does not occur free in Bx, then $(x)Bx$ $[\exists x\, Bx]$ is true iff Bx is true.)

Finally, if A has the form $B \rightarrow C$, then A is true iff $\underline{X}_B$ cons $\underline{Y}_C$, where $\underline{X}_B$ $[\underline{Y}_C]$ is he (unit) class of propositions assumed as values by B $[C]$.

Then A is *valid in* $\langle \underline{P}, \underline{I}, \underline{F}\rangle$ if A is true in $\langle \underline{P}, \underline{I}, \underline{F}\rangle$ for every assignment, and A is *valid* if A is valid in every frame.

We conjecture that a first degree formula is provable in EQ just in case it is valid under the definition above, but no proof has thus far been forthcoming. However, previous results enable us to establish the conjecture for various fragments of the system of first degree formulas.

Theorem. The fragment of EQ consisting of (perhaps multiply) disjoined first degree entailments is complete and consistent.

Proof. We treat each $A_{\underline{i}} \rightarrow B_{\underline{i}}$ in the $\underline{n}$-termed disjunction $A_1 \rightarrow B_1 \vee \cdots \cdots \vee A_{\underline{n}} \rightarrow B_{\underline{n}}$ separately, using the proof methods of section IV. If all the branches of the tree for some $A_{\underline{i}} \rightarrow B_{\underline{i}}$ terminate in axioms, then $A_{\underline{i}} \rightarrow B_{\underline{i}}$ is provable (and valid); hence the $\underline{n}$-termed disjunction is provable (and valid).

Now we show that if an $\underline{n}$-termed disjunction of first degree entailments is unprovable, then it is falsifiable in an atomic frame generated by $3\underline{n}$ atoms, from which it follows that the fragment of EQ in question is complete.

Let $A_1 \rightarrow B_1 \vee \cdots \vee A_{\underline{n}} \rightarrow B_{\underline{n}}$ be unprovable. Then each $A_{\underline{i}} \rightarrow B_{\underline{i}}$ has a tree containing at least one branch which fails to terminate in an axiom; let these branches be $\phi_1, \ldots, \phi_{\underline{n}}$. We now divide the $3\underline{n}$ atoms into three groups: $\underline{p}$-atoms $\underline{p}_1, \ldots, \underline{p}_{\underline{n}}$; $\underline{q}$-atoms $\underline{q}_1, \ldots, \underline{q}_{\underline{n}}$; and $\underline{r}$-atoms $\underline{r}_1, \ldots, \underline{r}_{\underline{n}}$. We first assign values to variables of ϕ_1, as in the proof in section IV, in such a way that all the atoms in $L(\phi)$ take $\underline{p}_1, \underline{p}'_1$, or $\underline{r}_1$ as values, and those in $R(\phi)$ take $\underline{q}_1, \underline{q}'_1$, and $\underline{r}'_1$, thus falsifying $A_1 \rightarrow B_1$ (by Lemma III of section IV). This assignment may have occasioned assignments to some of the atoms of ϕ_2, but we may be sure that these assignments do not satisfy $A_2 \rightarrow B_2$, in view of Lemma I of section IV. Hence we may assign values $\underline{p}_2, \underline{q}_2$, and $\underline{r}_2$ to variables of ϕ_2 in a similar way, with the result that all atoms not previously assigned of $L(\phi_2)$ take $\underline{p}_2, \underline{p}'_2$, or $\underline{r}_2$, and all unassigned atoms of $R(\phi_2)$ take $\underline{q}_2, \underline{q}'_2$ or $\underline{r}'_2$. Hence $A_2 \rightarrow B_2$ is also falsified; and obviously the procedure may be continued.

We note the following corollaries (which of course do not hold for material or strict "implication"):

Corollary: If S is a set of falsifiable first degree entailments, then the members of S are simultaneously falsifiable.

Corollary: $A_1 \to B_1 \vee \cdots \vee A_{\underline{n}} \to B_{\underline{n}}$ is provable just in case $A_{\underline{i}} \to B_{\underline{i}}$ is provable for some $\underline{i}$[6]).

Theorem. The fragment of EQ consisting of (perhaps multiply) disjoined negated first degree entailments is complete and consistent.

Proof. Let

$$(1) \qquad \overline{A_1 \to B_1} \vee \cdots \vee \overline{A_{\underline{n}} \to B_{\underline{n}}}$$

be a formula of the required sort, and consider

$$(2) \qquad (A_1 \,\&\, \overline{B_1}) \vee \cdots \vee (A_{\underline{n}} \,\&\, \overline{B_{\underline{n}}}) \,.$$

Suppose (1) is unprovable. Then since (2) entails (1) by $A \,\&\, \overline{B} \to \overline{A \to B}$ and properties of disjunction, (2) is unprovable. But then, since EQ contains the classical first order functional calculus (BELNAP 1959a), (2) is unprovable classically, and hence invalid. Then (2) has a falsifying assignment in the classical two-valued frame. Hence if we replace classical *truth* by $\underline{D}\{\underline{p}, \underline{p}'\}$ and classical *falsehood* by $(\underline{D}\{\underline{p}, \underline{p}'\})'$, then the formula has a falsifying assignment in the frame $\langle \underline{P}_2, \underline{I}, \underline{F}_2 \rangle$ of section VI. Hence to each $A_{\underline{i}} \,\&\, \overline{B_{\underline{i}}}$, this assignment *either* gives $(\underline{D}\{\underline{p}, \underline{p}'\})'$ to $A_{\underline{i}}$ *or* gives $\underline{D}\{\underline{p}, \underline{p}'\}$ to $B_{\underline{i}}$. But in both cases we have $\underline{X}_{\{A_{\underline{i}}\}}$ cons $\underline{Y}_{\{B_{\underline{i}}\}}$; hence $A_{\underline{i}} \to B_{\underline{i}}$ is true, and $\overline{A_{\underline{i}} \to B_{\underline{i}}}$ is false. Then $\overline{A_1 \to B_1} \vee \cdots \vee \overline{A_{\underline{n}} \to B_{\underline{n}}}$ is invalid[7]). It follows that the fragment is complete.

For consistency, assume that (1) is invalid. Then by an obvious generalization of the Lemma of section VI, (1) is falsifiable in $\langle \underline{P}_2, \underline{I}, \underline{F}_2 \rangle$. But then (1) is falsifiable in $\langle \underline{P}_2, \underline{I}, \underline{F}_2 \rangle$ with the arrow interpreted as material "implication". But every theorem of EQ is valid in $\langle \underline{P}_2, \underline{I}, \underline{F}_2 \rangle$ if the arrow is interpreted in this way, as is easily verified. Hence if (1) is invalid, it is unprovable.

Note that neither of the corollaries above hold for this case: the set $\{\overline{A \to \overline{A}}, \overline{\overline{A} \to A}\}$ and the formula $\overline{A \to \overline{A}} \vee \overline{\overline{A} \to A}$ are counter-examples to both.

But attempts to generalize further have thus far failed. We therefore list some open problems, solutions to which may provide steps toward finding satisfactory semantics for the full system EQ.

Let A be a quantifier-free formula of the first degree fragment of EQ.

(1) Is it true that A is valid if and only if it is valid in every atomic frame ?

(2) Is A valid iff A is valid in some frame $\langle \underline{P}, \underline{I}, \underline{F} \rangle$, with $\underline{P}$ finite (and of course $\underline{I}$ and $\underline{F}$ empty) ?

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[6]) Compare GÖDEL's conjecture (proved in MCKINSEY and TARSKI 1948) that $NA \vee NB$ is provable in $S4$ iff either A or B is provable in $S4$.

[7]) As is clear from the method of the proof, the fragment of EQ considered in the theorem could be expanded to include disjunctions of negated entailments together with expressions of degree zero.

(3) Is there some effective function of some effective property of A (e.g., of the number of variables in A), such that A is valid iff A is valid in a frame generated by $\underline{n}$ atomic propositions (where $\underline{n}$ is the value of the function for A)?

Two other related questions also remain open:

(4) Is the propositional fragment of EQ decidable?

(5) Does the rule (γ) of Ackermann 1956 hold for EQ; i.e., is it the case that whenever A and $\overline{A} \vee B$ are both provable, B is also provable? We notice that if the first degree formula fragment of EQ is complete, then (γ) holds for the fragment; but the more general problem remains untouched.

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