Tautological Entailments

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In *The Pure Calculus of Entailment*[1] we offered a formal theory concerning the relation between A and B when B *follows logically from* A. We showed there that the calculus E_I avoided certain fallacies of modality and fallacies of relevance which are allowed as valid inferences by those who hold that material, intuitionistic, and strict "implication" relations are implication relations. We take it as our problem in this paper to extend the previous results by finding plausible criteria for picking out from among *first-degree entailments* (i.e., entailments of the form A→B, where A and B are purely truth-functional) those that are valid. We refer to such valid entailments as *tautological entailments.*

Clearly none of the "implication" relations mentioned above will do as a criterion, since if any of these conditions were sufficient for entailment, we would have A&∼A→B. But as the informal discussion of the papers cited in note 1 would indicate, we regard a contradiction A&∼A as in general irrelevant to an arbitrary proposition B, and we accordingly think of the principle "(A and not-A) implies B" as embodying a fallacy of relevance. On the other hand, "(A and not-A) implies A" seems true to our preanalytic idea of conjunction, since it is a special case of the plausible principle "(A and B) implies A." What is wanted, *inter alia,* is a way of distinguishing these cases.

Von Wright (27) has proposed a criterion of the sort we seek: "A entails B, if and only if, by means of logic, it is possible to come to know the truth of A ⊃ B without coming to know the falsehood of A or the truth of B." Geach (14), following von Wright, proposes a slightly different criterion: "I maintain that A entails B if and only if there is an *a priori* way of getting to know that A ⊃ B which is not a way of getting to know whether A or whether B." These proposals seem to us to be on the right track, but they need improvement, for two reasons.

In the first place, the expression "come to know" is vague. One might imagine a person's "coming to know" the truth of A ⊃ Bv∼B without coming to know the truth of Bv∼B owing to the fact (say) that the formula was fed into a computer programmed to test tautologies. Strawson (26) finds a similar difficulty:

"It appears that von Wright has overlooked the implications of one fa-

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miliar way of arriving at the paradoxes. Consider $p \supset (q \vee \sim q)$. Von Wright would of course wish to deny that p entails $q \vee \sim q$. Now the following is demonstrable independently of demonstrating the falsity of $q \vee \sim q$ or, of course, the truth of p:

(1) $p \supset ((p \& q) \vee (p \& \sim q))$;

for in the truth-table proof of (1) there is only one 'unmixed' column (i.e. column consisting purely of Ts or Fs) which is the last column showing the whole expression to be a tautology. Still more obviously the following are demonstrable independently of demonstrating the truth of p or the falsity of qvq:

(2) $((p \& q) \vee (p \& \sim q)) \supset (p \& (q \vee \sim q))$

(3) $(p \& (q \vee \sim q)) \supset (q \vee \sim q)$

(4) $(p \supset q) \supset [(q \supset r) \supset ((r \supset s) \supset (p \supset s))]$.

For although (2) and (3) both contain $q \vee \sim q$, they are respectively substitution instances of $((p \& q) \vee (p \& r)) \supset (p \& (q \vee r))$ and of $(p \& q) \supset q$. Hence, by substituting the second halves of (1), (2), and (3) for q, r, and s respectively in (4) and by repeated applications of Modus Ponens, we obtain a demonstration of $p \supset (q \vee \sim q)$ which is independent of demonstrating the falsity of p or the truth of $q \vee \sim q$. Consequently, on von Wright's definition, p entails $q \vee \sim q$. But this is one of the paradoxical cases which his theory is intended to avoid."

In reply, von Wright (28) distinguishes several senses of "coming to know," in one of which, he claims, Strawson's alleged counterexample is not a counterexample. But we do not pursue these distinctions, because, as von Wright indicates, the situation remains in an unsatisfactorily vague state.

Smiley (25), while retaining the spirit of von Wright's proposal, modifies it in such a way as to eliminate the vagueness. He holds that $A_1 \& \ldots \& A_n$ should entail B just in case $(A_1 \& \ldots \& A_n) \supset B$ is a substitution instance of a tautology $(A_1' \& \ldots \& A_n') \supset B'$, such that neither B' nor the denial of $A_1' \& \ldots \& A_n'$ is provable. He cites as an example: " . . . for any A, $A \& \sim A$ entails A, because $A \& \sim A \rightarrow A$ is a substitution instance of $A \& B \rightarrow A$; but $A \& \sim A$ does not entail just any B, because there is in general no way of deriving $A \& \sim A \rightarrow B$ from an implication which is itself tautologous but whose antecedent is not self-contradictory."

Smiley's criterion gives rise to a definition of entailment which is effectively decidable, and seems also to capture the intent of von Wright and Geach. But there is an application of it which leads to a second objection (as do the proposals of von Wright and Geach, under at least one interpretation). Since $A \rightarrow A \& (B \vee \sim B)$ satisfies the criterion, and $A \& (B \vee \sim B) \rightarrow$

$B v \sim B$ does also, we find that the paradox $A \rightarrow B v \sim B$ can be avoided only at the price of giving up transitivity of entailment. This unwelcome course has in fact been recommended by Lewy (19), Geach (14), and Smiley (25). Smiley considers the matter as follows:

"It is true that 'connexion of meanings' is not as simple as might be thought: 'it has been plausibly argued that any proposition asserts (at least implicitly) something about all objects whatsoever. "Grass is green," for instance, says among other things that it is not the case that *grass is not-green and roses are red*, and so on. This follows simply from the fact that any proposition constitutes a denial of some other propositions and therefore of all conjunctions of which these propositions are members.' [Bennett (11).] But to conclude from this that 'thus there *is* a connexion of meanings between *any two* propositions; and a necessary or impossible proposition has with *any* other proposition a connexion of meanings such as will validate one or other of the paradoxical inferences' is to assume that 'connexion of meanings' is a transitive relation, and it is only necessary to examine the derivation of one of the paradoxical principles to see that it is not."

It is of course correct that 'connexion of meanings' is not transitive, at least under one interpretation: there is a meaning connection between A and A&B, and also between A&B and B — but there need be no connection of meaning between A and B. And what this shows is that connection of meaning, though necessary, is not a sufficient condition for entailment, since the latter relation *is* transitive. Any criterion according to which entailment is not transitive, is *ipso facto* wrong. It seems in fact incredible that anyone should admit that A entails B, and that B entails C, but feel that some *further* argument was required to establish that A entails C. What better evidence for $A \rightarrow C$ could one want?

The failure of these proposals arises from an attempt to apply them to *all* formulas. For there is a class of entailments for which Smiley's criterion is absolutely unarguably both a necessary and a sufficient condition; namely, the class of *primitive entailments*, which, after introducing some auxiliary notions, we proceed to define.

An *atom* is a propositional variable or the negate of a propositional variable. A *primitive disjunction* is a disjunction $A_1 v A_2 \ldots v A_m$, where each disjunct A_i is an atom. A *primitive conjunction* is a conjunction $B_1 \& B_2 \ldots \& B_n$, each B_j being an atom. $A \rightarrow B$ is a *primitive entailment* if A is a primitive conjunction and B is a primitive disjunction. We take it as obvious that if A and B are both atoms, then $A \rightarrow B$ should be a valid entailment if and only if A and B are the same atom; e.g., we would want $p \rightarrow p$ and $\sim p \rightarrow \sim p$ but not $p \rightarrow q$ or $p \rightarrow \sim p$. We think it equally obvious that if $A_1 \& A_2 \ldots \& A_m$ is a primitive conjunction and $B_1 v \ldots v B_n$ is a primitive disjunction, then

$A_1\& \ldots \&A_m \rightarrow B_1 v \ldots vB_n$ should be a valid entailment if and only if some atom A_i is the same as some atom B_j; e.g., we want p&q→qvr, and ~p&q&r→sv~pv~r, but neither p&~q→r, nor ~p→qvpvr, nor (it need hardly be added) p&~p→q. We shall say that a primitive entailment A→B is *explicitly tautological*, if some (conjoined) atom of A is identical with some (disjoined) atom of B. Such entailments may be thought of as satisfying the classical dogma that for A to entail B, B must be "contained" in A.[2]

It is clear that explicitly tautological entailments satisfy the requirements of von Wright, Geach, and Smiley: every explicitly tautological entailment answers to a material "implication" which is a substitution instance of a tautologous material "implication" with noncontradictory antecedent and nontautologous consequent; and evidently we may ascertain the truth of the entailment without coming to know the truth of the consequent or the falsity of the antecedent. Certainly all explicitly tautological entailments are valid, and we see absolutely no way in which the stock of valid primitive entailments could plausibly be enlarged; we take it therefore that explicitly tautological entailmenthood is both necessary and sufficient for the validity of a primitive entailment.

Let us now return to a consideration of the (nonprimitive) entailment A→A&(Bv~B), which, as was pointed out before, satisfies the criteria of von Wright, Geach, and Smiley. Lewy (19) remarks (in effect) that A→A&(Bv~B) seems "very nearly, if not quite," as counterintuitive as A→(Bv~B). We agree in substance with Lewy, but we think his estimate is too high: A→A&(Bv~B) is exactly 50 per cent as counterintuitive as A→(Bv~B). That is, A→A&(Bv~B) is valid just in case both the primitive entailments A→A and A→(Bv~B) are valid; the former is valid, but the latter is not—hence A→A&(Bv~B) does not represent a valid inference. Dually, (A&~A)vB→B is valid if and only if both A&~A→B and B→B are valid; and again one is valid and the other not.

These considerations suggest criteria for evaluating certain first-degree entailments other than primitive ones: A→B&C is valid if and only if both A→B and A→C are valid; and AvB→C is valid if and only if A→C and B→C are both valid. This gives us a technique for evaluating entailments in *normal form*, i.e., entailments A→B having the form $A_1 v \ldots vA_n \rightarrow B_1 \& \ldots \&B_m$, where each A_i is a primitive conjunction and each B_j is a primitive disjunction. Such an entailment is valid just in case each $A_i \rightarrow B_j$ is explicitly tautological. For example, (p&q)v~p→(~pvp)&(~pvq), (p&q)v(p&r)→p&(qvr), (pvq)vr→pv(qvr), and p&q→q&(rvp), are all valid entailments in normal form; but the following are invalid: (p&~p)vq→q, and p→p&(qv~q).

The proposal as stated is still not complete, however, since there are combinations of disjunction and conjunction which the rule fails to cover; there is no way to apply it directly to $A\&(\sim AvB)\rightarrow B$ or $\sim A\rightarrow\sim\sim(A\&B)$, for example. But all that would be required to make the criterion everywhere applicable is the ability to convert any first-degree entailment into normal form. This in turn will require converting truth-functional formulas into disjunctive and conjunctive normal forms (i.e., to disjunctions of one or more primitive conjunctions, and to conjunctions of one or more primitive disjunctions).

We therefore propose adding the following replacement rules (all of which we take to preserve validity), which enable us to find, for any first-degree entailment, an equivalent entailment in normal form:

Commutativity: replace a part A&B by B&A; replace a part AvB by BvA;
Associativity: replace a part (A&B)&C by A&(B&C), and conversely; replace a part (AvB)vC by Av(BvC), and conversely;
Distributivity: replace a part A&(BvC) by (A&B)v(A&C), and conversely; replace a part Av(B&C) by (AvB)&(AvC) and conversely;
Double negation: replace a wf part A by ∼∼A, and conversely;
De Morgan's laws: replace a part ∼(A&B) by ∼Av∼B, and conversely; replace a part ∼(AvB) by ∼A&∼B, and conversely.

We propose then to call an entailment A→B, where A and B are purely truth-functional, a *tautological entailment,* if A→B has a normal form[3] $A_1v\ldots vA_n\rightarrow B_1\&\ldots\&B_m$ such that each $A_i\rightarrow B_j$ is an explicitly tautological entailment.

We propose tautological entailmenthood as a necessary and sufficient condition for the validity of first-degree entailments. (The property is obviously decidable.)

As an example, we show that $(p\supset q)\&(q\supset r)\rightarrow(p\supset r)$ is invalid. By the definition of "⊃," we have $(\sim pvq)\&(\sim qvr)\rightarrow\sim pvr$ which has a normal form, $(\sim p\&\sim q)v(\sim p\&r)v(q\&\sim q)v(q\&r)\rightarrow\sim pvr$. But $q\&\sim q\rightarrow\sim pvr$ is not an explicitly tautological entailment; hence the candidate fails.

The foregoing example shows that material "implication" is not transitive, if by saying that R is *transitive* we mean that ARB and BRC jointly entail ARC.

The differences between tautological entailments and other alleged implication relations between truth-functions can be brought out clearly by considering the primitive entailments to which an arbitrary candidate A→B reduces. If we regard A→B as valid only when each $A_i\rightarrow B_j$ is an explicitly tautological entailment, then A→B is a valid entailment. If we also call A→B "valid" when each $A_i\rightarrow B_j$ either (i) is an explicitly tautological entailment or (ii) contains atoms C and ∼C conjoined in the antecedent, then the first-degree fragment of the propositional calculus of Fitch (13) is complete and sound for this definition of "validity." And finally if we add to (i) and (ii) the condition (iii) that $A_i \to B_j$ is "valid" if B_j contains C and ~C disjointly, then the arrow reduces to material "implication." (So far as we know, no one has investigated the system obtained by taking as "valid" primitive entailments satisfying (i) and (iii) only.)

Tautological Entailments and E

The system E of entailment (3) may be captured axiomatically as follows:

Axiom schemata for the system E of entailment:

Entailment.

E.1 A→A→B→B
E.2 A→B→.B→C→.A→C
E.3 (A→.A→B)→.A→B

Conjunction.

E.4 A&B→A
E.5 A&B→B
E.6 (A→B)&(A→C)→.A→(B&C)

Relating modality and conjunction.

E.7 NA&NB→N(A&B) [$NA =_{df}$ A→A→A.]

Disjunction.

E.8 A→AvB
E.9 B→AvB
E.10 (A→C)&(B→C)→.(AvB)→C

Relating conjunction and disjunction.

E.11 A&(BvC)→(A&B)vC

Negation.

E.12 A→~A→~A
E.13 A→~B→.B→~A
E.14 ~~A→A

Rules:

Modus ponens: If A→B is asserted, then from A to infer B.
Adjunction: From A and B to infer A&B.

It develops that the system E is sound and complete relatively to tautological entailments in the following sense:

Theorem. A first-degree entailment A→B (that is, where A and B contain only variables, ~, v, and &) is provable in E if and only if A→B is a tautological entailment (Belnap, 10).

Proof. By methods of Ackermann (1), it can easily be shown that a first-degree entailment $A \to B$ is provable in E just in case a normal form $A_1 \vee \ldots \vee A_m \to B_1 \& \ldots \& B_n$ of $A \to B$ is provable in E. (All replacement rules used in defining tautological entailments are provable as co-entailments in E, and E has a derivable replacement rule.) Then by E.2, E.4–E.6, and E.8–E.10, $A \to B$ is provable in E just in case each $A_i \to B_j$ is provable in E; and $A \to B$ is a tautological entailment just in case each $A_i \to B_j$ is an explicitly tautological entailment. Hence it will suffice to show that each primitive entailment $A_i \to B_j$ is provable if and only if it is an explicitly tautological entailment. It is trivial that all explicitly tautological primitive entailments are provable in E, and we consider the converse. For this we need the accompanying matrices, which satisfy the axioms and rules of E.

A→B

	−3	−2	−1	−0	+0	+1	+2	+3
−3	+3	+3	+3	+3	+3	+3	+3	+3
−2	−3	+2	−3	+2	−3	−3	+2	+3
−1	−3	−3	+1	+1	−3	+1	−3	+3
−0	−3	−3	−3	+0	−3	−3	−3	+3
+0	−3	−2	−1	−0	+0	+1	+2	+3
+1	−3	−3	−1	−1	−3	+1	−3	+3
+2	−3	−2	−3	−2	−3	−3	+2	+3
+3	−3	−3	−3	−3	−3	−3	−3	+3

~A

−3	+3
−2	+2
−1	+1
−0	+0
+0	−0
+1	−1
+2	−2
+3	−3

A&B

	−3	−2	−1	−0	+0	+1	+2	+3
−3	−3	−3	−3	−3	−3	−3	−3	−3
−2	−3	−2	−3	−2	−3	−3	−2	−2
−1	−3	−3	−1	−1	−3	−1	−3	−1
−0	−3	−2	−1	−0	−3	−1	−2	−0
+0	−3	−3	−3	−3	+0	+0	+0	+0
+1	−3	−3	−1	−1	+0	+1	+0	+1
+2	−3	−2	−3	−2	+0	+0	+2	+2
+3	−3	−2	−1	−0	+0	+1	+2	+3

AvB

	−3	−2	−1	−0	+0	+1	+2	+3
−3	−3	−2	−1	−0	+0	+1	+2	+3
−2	−2	−2	−0	−0	+2	+3	+2	+3
−1	−1	−0	−1	−0	+1	+1	+3	+3
−0	−0	−0	−0	−0	+3	+3	+3	+3
+0	+0	+2	+1	+3	+0	+1	+2	+3
+1	+1	+3	+1	+3	+1	+1	+3	+3
+2	+2	+2	+3	+3	+2	+3	+2	+3
+3	+3	+3	+3	+3	+3	+3	+3	+3

Let $A_i \to B_j$ be a primitive entailment which is not explicitly tautological. Then assign values from the accompanying matrix to the variables in $A_i \to B_j$ as follows:

(i) If the propositional variable p occurs in A_i but not in B_j, give p the value $+1$;

(ii) If p occurs in B_j but not in A_i, give p the value $+2$;

(iii) If p occurs in A_i, and $\sim p$ occurs in B_j, give p the value $+3$; and

(iv) If $\sim p$ occurs in A_i, and p occurs in B_j, give p the value -3.

(We note that (iii) and (iv) cannot conflict, since $A_i \to B_j$ is not an explicitly tautological entailment, and hence A_i and B_j share no atoms.)

Under this assignment, A_i always assumes a value ± 1 or $+3$, and B_j always assumes a value ± 2 or -3. Hence $A_i \rightarrow B_j$ assumes the value -3, and is therefore unprovable (plus values being designated).

The matrices also enable us to show that E satisfies a plausible necessary condition for avoidance of fallacies of relevance:

Theorem. A→B is provable in E only if A and B share a variable (Belnap, 8, 9).

If A and B fail to share a variable, then we may assign all variables of A the value $+1$, so that A takes the value ± 1, and we may assign all variables of B the value $+2$, so that B takes the value ± 2. Then A→B takes the value -3, and is therefore unprovable.

The matrices above may be given the following partial interpretation.[4] Choose A and B in such a way that ~A→B, ~B→B, ~AB→~B, ~BA→~A, and ~(A~A→Bv~B). (These conditions may be secured by letting B be ~CvCD, and A be ~DvCD, where C is pv~p and D is qv~q; and where p is "Napoleon was born in Corsica," and q is "666 is a perfect number" and hence ~(p~p→qv~q).) Then we assign propositional values to **v**(i) to the numbers i as follows:

v(-3) = ~A&~B	**v**$(+0)$ = A&B
v(-2) = ~A	**v**$(+1)$ = B
v(-1) = ~B	**v**$(+2)$ = A
v(-0) = ~Av~B	**v**$(+3)$ = AvB

This gives a complete interpretation of the tables for conjunction, disjunction, and negation, and a partial interpretation of the table for entailment, as follows. For conjunction, $i \& j = k$ just in case **v**(i)&**v**(j)→**v**(k) and **v**(k)→**v**(i)&**v**(j) are tautological entailments. Similarly for disjunction and negation. For entailments, we read plus values in the matrix as "true," minus values as "false."

We note in passing that necessary and sufficient conditions for ~A→A among first-degree entailments are as follows:

Theorem. Let B be a truth-function. Then ~B→B is valid if and only if a conjunctive normal form $B_1 \& \ldots \& B_n$ of B has the property that for each B_i and B_j, there is an atom p such that p occurs in B_i $\{B_j\}$ and ~p occurs in B_j $\{B_i\}$.

The proof is left to the reader.

Truth-Functional Fallacies of Relevance

As logicians have always taught, logic is a *formal* matter, and the validity of an inference has nothing to do with the truth or falsity of premise(s) or conclusion. But the view that material "implication" is an implication relation flies squarely in the face of this teaching, and leads directly and immediately to fallacies of relevance. $A\&{\sim}A{\rightarrow}B$ *has* been defended on the ground that although it is useless, it is harmless, since the antecedent can never be realized. We grant that it is harmless in this sense, but still contend that it is harmful in another sense, namely, in being false. To be sure, there *is* a (somewhat odd) sense in which we "lose control" in the presence of contradictions. Namely: we define a *manifest repugnancy* as a primitive conjunction $A_1\& \ldots \&A_n$, having the property that for each A_i, if A_i is a propositional variable, then for some A_j, $A_j = {\sim}A_i$, and if A_i is the negate of a propositional variable, then for some A_j, $A_i = {\sim}A_j$. An example is $p\&{\sim}p\&q\&{\sim}q\&r\&{\sim}r$. And for such expressions we have the following:

Theorem. Manifest repugnancies entail every truth function to which they are analytically relevant.

Proof. We say that A is analytically relevant to B if all variables of B occur in A (see note 2). The theorem then states that a manifest repugnancy entails every truth-functional compound of its own variables. And this may be readily seen as follows. Let A be a manifest repugnancy, and let B be any truth-function of the variables in A. Rewrite B equivalently in conjunctive normal form $B_1\& \ldots \&B_n$. Then each B_i contains at least one of the atoms in A; hence each $A{\rightarrow}B_i$ is an explicitly tautological entailment.

Dually, we have that every truth-functional expression entails a (very weak) tautology, consisting of a disjunction of various special cases of $A\vee{\sim}A$. But admitting these obvious logical truths is a far cry from admitting that a contradiction entails *any* old thing.

It is of course sometimes said that the implication we *use* admits that false or contradictory propositions imply anything you like, and we are given the example "If Truman wins, I'll be a monkey's uncle." But it seems to us unsatisfactory to dignify as a principle of logic what is obviously no more than a rhetorical figure of speech, and a facetious one at that; one might as well cite Cicero's use of *praeteritio* as evidence that one can do and not do the same thing at the same time (and in the same respect).

Lewis, however, has explicitly argued that the paradoxes of strict "implication" "state a fact about deducibility" (18, p. 251) and has presented "independent proofs" of their validity. Since there is a clear opposition between our position and that of Lewis (and practically everyone else), we will examine one of these proofs in detail.

The argument concerns $A{\sim}A{\rightarrow}B$, and has two steps, (i) "A entails B" or "$A{\rightarrow}B$" *means* that B is deducible from A "by some mode of inference which is valid" (18, p. 248), and (ii) there is a "valid mode of inference" from $A{\sim}A$ to B.

We may accept (i) without cavil. Arguments for (ii), that is, for the proposition that there is a valid mode of inference from a contradiction to any arbitrary proposition, were known to several logicians flourishing circa the year 1350, and are found in extant writings of the astute Bishop of Halberstadt, Albert of Saxony (see Boehner (12), pp. 99–100). Lewis and Langford's presentation of the argument (18, p. 250) does not differ significantly from Albert's, although it is almost certain that the modern appearance of the argument represents a rediscovery rather than a continuity of tradition. The argument has also been accepted by a variety of other modern logicians—e.g., Popper (21), pp. 407–10, and (22)—and indeed, as Bennett points out (11, p. 451), "this acceptance has not been an entirely academic matter. William Kneale [17] and Popper [23] have both used the paradoxes as integral parts of their respective accounts of the nature of logic."

Though departing in insignificant detail from Lewis' own, the following is a convenient presentation of the argument. Grant that the following are 'valid modes of inference':

1. from A&B to infer A,
2. from A&B to infer B,
3. from A to infer AvB, and
4. from AvB and ~A to infer B.

The argument then proceeds in this way:

(a)	A&~A	premise
(b)	A	from (a) by 1
(c)	AvB	from (b) by 3
(d)	~A	from (a) by 2
(e)	B	conclusion: from (c) and (d) by 4.

Than which nothing could be simpler: if the four rules above are "valid modes of inference" and if "A→B" means that there is a valid mode of inference from A to B, then a contradiction such as A&~A surely does entail any arbitrary proposition, B, whence A&~A→B represents a fact about deducibility.

We agree with those who find the argument from (a) to (e) self-evidently preposterous, and from the point of view we advocate it is immediately obvious where the fallacious step occurs: namely, in passing from (c) and (d) to (e). The principle 4 (from AvB and ~A to infer B), which commits a fallacy of relevance, is not a tautological entailment. We therefore reject 4 as an entailment, and as a valid principle of inference.

We seem to have been pushed into one of the "peculiar positions" of which Prior speaks (24, p. 195), for we are explicitly denying that the principle of the disjunctive syllogism—or detachment for material "implication"—

is a "valid mode of inference." The validity of this form of inference is something Lewis never doubts (see, for example, Lewis and Langford (18), pp. 242–43) and is something which has perhaps never been seriously questioned before (though the possibility of dispensing with the disjunctive syllogism is raised by Smiley (25)). Nevertheless, we do hold that the inference from $\sim A$ and $A \vee B$ to B is an error: it is a simple inferential mistake. Such an inference commits nothing less than a fallacy of relevance. We shall first anticipate possible misinterpretations of this thesis and then proceed to an "independent proof" of the invalidity of $\sim A(A \vee B) \rightarrow B$.

In the first place, we do not deny that the inference from $\vdash \sim A$ and $\vdash A \vee B$ to $\vdash B$ is valid, where "$\vdash A$" means 'A is a theorem of the two-valued propositional calculus.' However, from this it does not follow that B follows from $\sim A$ and $A \vee B$, nor does it follow that if $\vdash \sim A$ and $\vdash A \vee B$ then B follows from $\sim A$ and $A \vee B$. We even admit that if $\vdash B$ then B is necessarily true, and still hold that the argument from $\sim A$ and $A \vee B$ to B is invalid even when $\vdash \sim A$ and $\vdash A \vee B$ (and hence $\vdash B$). Such a claim would be senseless on Lewis' doctrine, for to admit B is necessarily true is to admit that *any* argument for B is valid.

Secondly, we do not say that the inference from $\sim A$ and $A \vee B$ to B is invalid for *all* choices of A and B; it will be valid at least when A entails B (in which case $\sim A$ is not required) or when $\sim A$ entails B (in which case $A \vee B$ is not required); more generally, it will be valid when $A \sim A$ entails B (in which case the disjoined premise B is not required).

Furthermore, in rejecting the principle of the disjunctive syllogism, we intend to restrict our rejection to the case in which the "or" is taken truth-functionally. In general and with respect to our ordinary reasonings this would not be the case; perhaps always when the principle is used in reasoning one has in mind an intensional meaning of "or," where there is relevance between the disjuncts. But for the intensional meaning of "or," it seems clear that the analogues of $A \rightarrow A \vee B$ are invalid, since this would hold only if B was relevant to A; hence, there is a sense in which the real flaw in Lewis' argument is not a fallacy of relevance but rather a fallacy of ambiguity: the passage from (b) to (c) is valid only if the "v" is read truth-functionally, while the passage from (c) and (d) to (e) is valid only if the "v" is taken intensionally. We shall further consider the intensional "or" below.

Our final remark concerns what Lewis might have meant by "some valid form of inference." It is hardly likely that he meant that a form of inference is valid if and only if either the premises are false or the conclusion true ("material validity"); more plausibly, he might have meant that a form of inference is valid if and only if it is *necessary* that either the premises are false or the conclusion true ("strict validity"). If this is what Lewis meant, then we agree at once that the inference from A and ${\sim}A \vee B$ to B is valid *in this sense*. However, if this is *all* that Lewis meant by "some valid form of inference," then his long argument for $A \& {\sim}A \rightarrow B$ is a quite unnecessary detour, for *in this sense* we should have agreed at once that there is a valid form of inference from $A \& {\sim}A$ to B: it is surely true that necessarily either the premise is false or the conclusion is true inasmuch as the premise is necessarily false. In short, Lewis' "independent proof" of $A \& {\sim}A \rightarrow B$ is convincing if "valid inference" is defined in terms of strict implication; but in that case it is superfluous and circular. And his argument serves a useful purpose only if "valid inference" is thought of in some other sense, in which case he has failed to prove—or even to argue for—his premises. Finally, should he wish to escape the horns of this dilemma by remarking that the various forms of inference used in the argument are valid in the sense of having always been accepted and used without question, then we should rest our case on the fallacy of ambiguity noted above.

Such a thesis so strongly stated will seem hopelessly naive to those logicians whose logical intuitions have been numbed through hearing and repeating the logicians' fairy tales of the past half century, and hence stands in need of further support. It will be insisted that to deny detachment for material and strict implication, as well as to deny the principle of the disjunctive syllogism, surely goes too far: 'from ${\sim}A$ and $A \vee B$ to infer B,' for example, is surely valid. For one of the premises states that at least one of A and B is true, and since the other premise, ${\sim}A$, says that A can't be the true one, the true one must be B (see Popper (22), p. 48). Our reply is to remark again that this argument commits a fallacy of ambiguity. There are indeed important senses of "or," "at least one," etc., for which the argument from ${\sim}A$ and A-or-B is perfectly valid, namely, senses in which there is a true relevance between A and B, for example, the sense in which "A-or-B" means precisely that ${\sim}A$ entails B. However, in *this* sense of "or," the inference from A to A-or-B is fallacious, and therefore this sense of "or" is not preserved in the truth-functional constant translated by the same word. As Lewis himself argued in some early articles, there are *intensional* meanings of "or," "not both," "at least one is true," etc., as well as of "if . . . then . . ." Those who claim that only an intensional sense of these words will support inferences are right—Lewis' only error was in supposing he captured this sense by tacking a modal operator onto a fundamentally truth-functional formula.

Nevertheless, the inference from ${\sim}A$ and A-or-B to B is sometimes valid even when the "or" is truth-functional, for it will be valid in every case in which $A{\sim}A \rightarrow B$. For example, although the decision procedure of the previous section shows that ${\sim}(AB) \rightarrow {\sim}AB \vee A{\sim}B \vee {\sim}A{\sim}B$ is not valid, nevertheless the inference from $\sim(AB)$ and $ABv{\sim}ABvA{\sim}Bv{\sim}A{\sim}B$ to infer $\sim ABvA{\sim}Bv{\sim}A{\sim}B$ is perfectly valid since, as is easily verified, $\sim(AB)AB \rightarrow \sim ABvA{\sim}Bv{\sim}A{\sim}B$. In general, if $A_1vA_2v \ldots vA_n$ $(n = 2^k)$ is a Boolean expansion, then the inference from $A_1vA_2v \ldots vA_n$ and $\sim A_1$ to $A_2v \ldots vA_n$ is valid, as the following considerations show.

Lemma. Let A, and $B_1 \ldots B_m$ all be primitive conjunctions (i.e., conjunctions of atoms, where an atom is either a propositional variable or the negate of a propositional variable). Then $A \rightarrow (B_1v \ldots vB_m)$ is a theorem if and only if for some B_i, $A \rightarrow B_i$ is a theorem.

Proof. Trivially, if for some B_i, $A \rightarrow B_i$ is provable, then so is $A \rightarrow (B_1v \ldots vB_m)$. For the converse, suppose that for no B_i is $A \rightarrow B_i$ provable. Then every B_i contains some atom not contained in A. Hence the conjunctive normal form $C_1\& \ldots \&C_n$ of $B_1v \ldots vB_m$ contains a disjunction C_j, none of the atoms of which occur in A; so $A \rightarrow C_j$ is not provable. But $A \rightarrow (B_1v \ldots vB_m)$ is provable if and only if $A \rightarrow (C_1\& \ldots \&C_n)$ is provable, i.e., if and only if $A \rightarrow C_j$ is provable for each j. Hence $A \rightarrow (B_1v \ldots vB_m)$ is unprovable, as required.

Theorem. Let $p_1 \ldots p_n$ be distinct propositional letters, and let A, $B_1 \ldots B_m$, be conjunctions of the form $p_1'\&p_2' \ldots \&p_n'$ where each p_i' is either p_i or $\sim p_i$. Then $A\&{\sim}A \rightarrow (B_1v \ldots vB_m)$ is provable if and only if for each i $(1 \leq i \leq n)$ there is a B_j among $B_1 \ldots B_m$ which differs from A in at most its i-th conjunct.

Proof. Consider $A\&{\sim}A$, which has the form $p_1'\&p_2'\& \ldots \&p_n'\&(p_1''vp_2''v \ldots vp_n'')$, where p_i'' is $\sim p_i$ if p_i' is p_i, and p_i'' is p_i if p_i' is $\sim p_i$. It is seen that this expression entails $B_1v \ldots vB_m$ if and only if we have, for each i, $(p_1'\& \ldots \&p_n'\&p_i'') \rightarrow (B_1v \ldots vB_m)$. By the lemma, this entailment holds if and only if $(p_1'\& \ldots \&p_n'\&p_i'') \rightarrow B_j$, for some j. But this can hold if and only if some B_j is $p_1'\& \ldots \&p_i'\& \ldots p_n'$ or $p_1'\& \ldots \&p_i''\& \ldots p_n'$ and hence differs from A in at most the i-th conjunct.

Since every tautological Boolean disjunctive normal form satisfies the condition of the theorem, it follows then that if $A_1v \ldots vA_n$ is a tautology in Boolean disjunctive normal form, then $\sim A_1(A_1v \ldots vA_n) \rightarrow A_2v \ldots vA_n$ is provable. But of course in the general case, $\sim A(AvB) \rightarrow B$ is rejected, since (here comes the "independent proof" promised above) $\sim A(AvB) \rightarrow B$ if and only if $\sim AAv{\sim}AB \rightarrow B$, only if $\sim AA \rightarrow B$, which is absurd.

Notice that negation, which is at the bottom of all truth-functional fallacies of relevance, plays a very weak role in the theory of entailments between truth-functions: if $A_1v \ldots vA_n \rightarrow B_1\& \ldots \&B_m$ is such an entailment in normal form, then all negation signs occurring in the formula may be deleted without affecting validity. This feature of the situation reinforces our claim (which stands in fact at the core of the tradition in formal logic), that the validity of a valid entailment depends in no way on the truth or falsity of antecedent or consequent.

And as final evidence for our contention we make the following observations:

The truth of A-or-B, with truth-functional "or," is not a sufficient condition for the truth of "If it were not the case that A, then it would be the case that B." Example: It is true that either Napoleon was born in Corsica or else the number of the beast is perfect (with truth-functional "or"); but it does not follow that had Napoleon *not* been born in Corsica, 666 would equal the sum of its factors. On the other hand the intensional varieties of "or" which *do* support the disjunctive syllogism are such as to support corresponding (possibly counterfactual) subjunctive conditionals. When one says "that is either *Drosophilia melanogaster* or *D. virilis,* I'm not sure which," and on finding that it wasn't *D. melanogaster,* concludes that it was *D. virilis,* no fallacy is being committed. But this is precisely because "or" in this context *means* "if it isn't one, then it is the other." Of course there is no question here of a relation of logical *entailment* (which has been our principal interest); evidently some other sense of "if . . . then . . ." is involved. But it should be equally clear that it is not *simply* the truth-functional "or" either, from the fact that a speaker would naturally feel that if what he said was true, then if it *hadn't* been *D. virilis,* it *would* have been *D. melanogaster.* And in the sense of "or" involved, it does not follow from the fact that it is *D. virilis* that it is either *D. melanogaster* or *D. virilis*—no more than it follows solely from the fact that it was *D. virilis,* that if it hadn't been, it would have been *D. melanogaster.*

The logical differences we have been discussing are subtle, and we think it is difficult or impossible to give conclusive evidence favoring the distinctions among the various senses of "or" we have been considering. But whether or not the reader is in sympathy with our views, it might still be of interest to find a case (if such exists) where a person other than a logician making jokes *seriously* holds a proposition A-or-B, in a sense warranting inference of B with the additional premise not-A, but is unwilling to admit *any* subjunctive conclusion from A-or-B. If no such examples exist, then we will feel we have made our case (and if examples do exist, we reserve the right to try to find something funny about them).

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NOTES

[1] Anderson and Belnap (5). The axioms for the pure calculus E_I of entailment are $A \rightarrow A \rightarrow B \rightarrow B$, $A \rightarrow B \rightarrow .B \rightarrow C \rightarrow .A \rightarrow C$, and $(A \rightarrow .A \rightarrow B) \rightarrow .A \rightarrow B$, with *modus ponens* as

sole rule. Further information may be found in Anderson (2), Anderson and Belnap (3, 4), Anderson, Belnap, and Wallace (6), and Belnap (7, 8, 9, 10).

In the present paper we use → for entailment, v for disjunction, & for conjunction, and ~ for negation. Where convenient we also write A&B as AB, and we use ⊃ for material "implication." Parentheses are omitted under the conventions of Church, *Introduction to Mathematical Logic*: outermost parentheses are omitted; a dot may replace a left-hand parenthesis, the mate of which is to be restored at the end of the parenthetical part in which the dot occurs (otherwise at the end of the formula); otherwise parentheses are to be restored by association to the left.

[2] Parry (20) presents a notion of *analytische Implikation*, also offered as an explication of "contained in the subject." Let us say that A is *analytically relevant* to B if every variable occurring in B also occurs in A. Then Parry understands the matter in such a way that for A to "contain" B, A must be analytically relevant to B (and he shows his system has this property: A→B is provable only if all the variables in B are also in A). But there is surely a sense in which AvB is "contained" in A; viz., the sense in which the concept Sibling (which is most naturally defined as Brother-or-Sister) is contained in the concept Brother. Certainly "All brothers are siblings" would have been regarded as analytic by Kant (16).

Another system having the same property is that of Hintikka (15). He writes: "Formulae which are tautologically equivalent by the propositional calculus are equivalent provided that they contain occurrences of exactly the same free variables, and so are expressions obtained from them by replacing one or more free individual variables by bound ones." He then writes A↔B when this metalogical equivalence holds, and he lets A→B abbreviate A↔(A&B). Hintikka nowhere suggests that his → is to be understood as entailment, but we take the opportunity of pointing out that his condition is neither necessary nor sufficient for tautological entailmenthood: ~B&A&~A→B satisfies his condition, and A→AvB fails.

[3] We have not defined a unique normal form, but it is readily shown that if one normal form of A→B is a tautological entailment, then all are, since they differ only in the order of conjuncts and disjuncts.

[4] We are indebted to Neil Gallagher for pointing out an error in an earlier attempt at interpreting the matrix.

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Infinite Analysis

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In this article I will examine those analyses which assert that a sentence of ordinary language is equivalent to an infinite conjunction of other sentences. An example of this would be the usual phenomenalist analysis where sentences about material objects are said to be equivalent to an infinite conjunction of statements about sense experiences. Conversely, the behaviorist