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ENTHYMEMES*

I

THERE is a classical category of arguments, discussions of which enjoyed a certain vogue toward the end of (what Bochenski has called) the Dark Ages in logic (i.e., from the Renaissance up until near the turn of the present century), namely, *enthymemes.* We take our text from Jevons 1870[1] (pp. 153–154):

A syllogism when incompletely stated is usually called an *enthymeme,* and this name is often supposed to be derived from two Greek words (εν, in, and θυμός, mind), so as to signify that some knowledge is held by the mind and is supplied in the form of a *tacit,* that is a silent or understood premise. . . . Of this nature is the following argument: 'Comets must be subject to the law of gravitation; for this is true of all bodies which move in elliptic orbits.' It is so clearly implied that comets move in elliptic orbits, that it would be tedious to state this as the minor premise in a complete syllogism of the mood Barbara, thus:

[Major premise (M)] All bodies moving in elliptic orbits are subject to the law of gravitation;

[Minor premise (m)] Comets move in elliptic orbits;

[Conclusion (C)] Therefore comets are subject to the law of gravitation.

Now everyone agrees that the syllogism as stated in full is valid and that a definition of "validity" that denied validity to the syllogism in Barbara would *ipso facto* be faulty. It is also conceded that corresponding to this form of inference there exists a certain *true* proposition,[2] namely,

If M and m, then C.

The case of the enthymematic inference and the corresponding proposition

If M, then C

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[1] Dates refer to the bibliography at the end of the paper.

[2] Or propositional form; such distinctions are irrelevant to our purposes, and will be observed only where essential (which in this paper they never are).

may, however, occasion some doubt. Asked whether the enthymematic inference is valid (or whether the proposition just above is true), we may want to answer either, "No, your premises are simply insufficient for your conclusion," or, "Yes, provided you mean to be using the obviously required premise m (which we grant that you are, we being in a tolerant mood)."

Now, whichever attitude we may wish to take, it is clear that there is a difference: in the first case there is a robust relation of logical consequence between premises and conclusion; in the second the *logical* connection is largely, if not entirely, by courtesy. And notice that no one has any difficulty in seeing the clear difference. So much, at any rate, we wish to take for granted; and we now turn to our topic, which is what to make of the situation. In particular, what sort of formal analysis of *if ... then*___ and *and* can do justice to the clearly felt distinction considered above?

For present purposes we wish to treat *and* in a purely truth-functional way: M *and* m is true if and only if M is true and also m is true.

We now ask whether the true *if ... then*___, corresponding to the syllogism, should be represented by the material "implication":

$$M \& m \supset C.$$

We have argued elsewhere at length (and somewhat peevishly) that material "implication" is simply *not* a kind of implication (Anderson and Belnap, 1961a, 1961b). We do not plan to repeat those arguments here; rather we shall offer some fresh arguments for the same thesis. Our point, incidentally, is also telling as against the view that intuitionistic "implication" is a relation of entailment, or logical deducibility.

The present argument is simply this: both views make hash of the distinction between logically valid arguments and enthymemes. For on both theories we have as a "theorem of logic" a principle according to which true premises aren't really there at all ("true premises may be suppressed"):

$$\frac{\begin{array}{l} m \\ M \& m \supset C \end{array}}{M \supset C.}$$

Suppose now that an argument is valid if and only if the corresponding material or intuitionistic "implication" statement is true. Then, since the argument from M and m to C is valid, $M \& m \supset C$ is true; and since m is true as well and, hence, suppressible, $M \supset C$ is true; hence the argument from M to C is valid, and *in precisely the same sense* as is the argument from M and m to C. But to say that the argument from M to C is thus valid is in direct contradiction to the doctrine that enthymematic arguments suppress *required* premises. "Where a necessary premiss is missing, without that premiss the inference is invalid" (Copi 1953, p. 205).

The conclusion seems inescapable that one cannot *both* hold that material and intuitionistic "implication" relations answer to the notion of a valid argument *and* try to maintain a distinction between valid arguments and enthymemes.

Now it is not unmistakably clear whether or not anyone ever seriously took material implication to explicate "valid argument," but we have no doubt on this score in respect to Lewis's theory of strict implication. We are told that the meaning of strict implication "is precisely that of ordinary inference and proof" (Lewis, 1912, p. 531; see also Lewis and Langford, 1932, p. 247).

And in the systems of strict implication, one *can* maintain a distinction between valid arguments and enthymemes, at least those of the sort we have considered above. In particular, if we put hooks for horseshoes, a conjoined premise may not be dropped simply on the grounds that it is true. "That is, in stating a strict implication one cannot omit a merely true premise which is one of a set of premises which together give the conclusion" (Lewis and Langford, 1932, p. 165).

So far so good.

II

Now consider the following argument:

> Copi, in the passage quoted above, is *clearly* correct in holding that valid arguments may be distinguished from enthymemes and that enthymemes as they stand (i.e., without statement of the required supporting premiss) are not valid. Therefore, if all bodies moving in elliptic orbits are subject to the law of gravitation and if comets move in elliptic orbits, then comets are subject to the law of gravitation.

The reader may find the reasoning in the foregoing paragraph somewhat intricate and difficult to follow, so we forthwith lay bare its logical form. Where A represents "enthymemes may be distinguished from valid arguments" and B represents "enthymemes as they stand are not valid," the argument has the form:

$$\frac{A \text{ and } B}{\text{If } M \text{ and } m, \text{ then } C.}$$

When one is asked whether this argument is valid, the initial reaction is likely to be "No," perhaps because the premises A and B have nothing to do with the conclusion: if M and m then C. But there have been sophisticated logicians, Lewis for example, who, on reflection, decided that the argument *is* valid, on the grounds that the consequent, if M and m then C, is necessary. But this view is debatable (notoriously), and the notion that a necessary proposition is entailed by *any* proposition has been heatedly and repeatedly attacked. That is, there have been those who (1) have held that the lack of relevance between premise and conclusion, just above, reduces to absurdity that claim that strict implication captures intuitive *entailment*, and those who (2) have defended strict implication on the grounds that the well-known paradoxes do indeed represent "facts about deducibility." We shall want ultimately to side with the first group; the ludicrous argument above cannot really be regarded as valid. But sophisticated logicians must have *something* in mind, and we shall try later to give such devils their due. Still, the argument above is no more valid as it stands than is the move from "all bodies moving in elliptic orbits are subject to the law of gravitation" to "comets must be subject to the law of gravitation," i.e., the move from M to C.

But now an analogy suggests itself. Couldn't we try to view the odd argument cited above as somehow enthymematic and by this irenic maneuver pacify both parties to the dispute? For *everyone can see* that

$$\frac{A \text{ and } B \text{ and (if } M \text{ and } m \text{ then } C)}{\text{If } M \text{ and } m, \text{ then } C}$$

is valid, and that

$$\frac{A \text{ and } B}{\text{If } M \text{ and } m, \text{ then } C}$$

arises from it by suppressing a true (indeed necessarily true) premise. And *everyone can see* that there is a difference between these two cases; else there could have been no sensible dispute in the first place.

Of course these latter enthymemes ("strict enthymemes") differ from those of section I (which, for reasons to be brought up later, we shall call "intuitionistic enthymemes"[3]) in that the suppressed premise is *necessarily* true, rather than simply *true*. But the suppression of necessary premises is, if anything, less objectionable than the suppression of contingently true ones; necessarily true premises are suppressed constantly in arguments in mathematics and in logic, simply because one does not want (ordinarily) to

[3] We have considered, and rejected, the following harrowing course: to retain "enthymeme" in its customary usage, and introduce "ananceënthymeme" for what we here call "strict enthymemes." (The term was supplied, at our insistent request, and with understandable reluctance, by a colleague in the department of classics, who asked us not to mention his name. It was Ralph L. Ward.)

bore one's audience with endless repetition of obvious logical truths (as *we* have been doing).

At any rate we shall take it, for the present, that the *sense* behind the claim that a necessary proposition is entailed by *any* old proposition is that, in the alleged entailment, a necessary premise is suppressed: to wit, the conclusion. And we offer this view as face-saver for both parties. To take another example, the statement "If A, then either A-and-B or A-and-not-B" can be regarded as *true*, all right; we have just left out an obvious, and obviously required, premise: "B or not-B."

But if we are asked whether strict implication can be taken as an account of entailment, or logical deducibility, the answer must again, as in the case of material and intuitionistic "implication," be "No."[4] For as Lewis himself points out, strict implication is enthymematic in the sense we have just described; necessarily true premises may be suppressed. "The omission of a premise which is *a priori* or logically undeniable does not affect the validity of deduction" (Lewis and Langford, 1932, p. 165). How *could* this be so? Can we make *no* distinction between valid and invalid arguments when, as may happen, all the propositions in the argument are necessary? We believe, rather, that the situation is as the citation above from Copi may be misinterpreted as saying: "Where a necessary premise is missing, without that premise the inference is invalid." Necessary premises are just as necessary as premises that are not necessary.

III

In the first two sections we have thrown the word "entailment" about in a way that suggested we knew what it meant. Well, we claim to. And, in particular, we claim that the formal system E of entailment accurately captures the intuitive concept of entailment, or logical deducibility. The literature on the system E is burgeoning,[5] and neither time nor space permit anything like a full account of what is known about it. But we here list briefly a few of its more important features, by way of preparing for a formal account of enthymemes in the next section.

Roughly, the system E is just like Lewis's S4, except that paradoxical assertions are missing. Such principles as $A \,\&\, \bar{A} \rightarrow B$, $A \rightarrow B \vee \bar{B}$, and $NA \rightarrow . B \rightarrow A$ [where NA means A is necessary, defined as $(A \rightarrow A) \rightarrow A$], are not provable in E (where the arrow

[4] We note that exactly the same argument can be made against the usual semantical definition of "validity" for the first-order functional calculus—if, that is, the horseshoe is thought of as an *if . . . then* _ _ _ relation.

[5] See items by Anderson, Belnap, and Wallace, in the bibliography.

is the symbol for entailment). But any *sensible* kind of inference is allowed. The first two properties listed below show some ways in which E is weaker than strict implication; the third through sixth show that, in spite of weaknesses, the system is strong.

1. As a minimal requirement for relevance of A to B (in propositional logic), it seems reasonable to demand that if $A \rightarrow B$ is accepted, then A and B should share a propositional variable. E satisfies this condition (Belnap 1960b).

2. Entailments are *sui generis* in the following sense: only entailments entail entailments. More sharply: if $A \rightarrow . B \rightarrow C$ is provable in E, then A contains an arrow (Ackermann 1956). Moreover, it is never the case that the *denial* of an entailment entails an entailment; i.e., no theorems have the form $\overline{A \rightarrow B} \rightarrow . C \rightarrow D$.

So the arrow is weaker than either the horseshoe or the hook. On the other hand, a decent theory of entailment, while avoiding fallacies of modality and of relevance, should not throw *all* the babies out with the bathwater. A satisfactory theory will isolate the disease of raw irrelevance, keeping everything of the standard theories that can possibly be kept:

3. E contains the full two-valued propositional calculus (Ackermann 1956), except of course that $\bar{A} \vee B$ is not to be thought of as an implication relation.

4. Every truth-functional compound coentails a disjunctive normal form and a conjunctive normal form, which means that, for conjunction and disjunction, the usual associative, commutative, and distributive rules hold and that we have rules of double negation and De Morgan's laws. More generally, if $A_1 \vee \cdots \vee A_m$ is a disjunctive normal form of a truth-functional A and if $B_1 \& \cdots \& B_n$ is a conjunctive normal form of a truth-functional B, then A entails B if and only if each A_i entails each B_j; i.e., if and only if each A_i and each B_j share a (negated or unnegated) variable (Belnap, 1959; Anderson and Belnap, 1961b).

5. Generalization to quantifiers is easy, and the extensional fragment of the system (already known in the literature as "the first-order functional calculus") is complete, as can be seen by a rather trivial proof (after Schütte 1953).

6. The semantics of first-degree entailments (i.e., entailments $A \rightarrow B$, where A and B contain only truth functions and quantifiers) is completely understood (Anderson and Belnap, 1961c), though there are many open questions about the semantics of more complicated formulas.

Such are, in rough outline, some of the principal features of the system E of entailment, which embodies the intuitive concept of *entailment*, or *logical consequence* (or so we claim). Further discussion of it will have to await another time and place; we turn now to a reconsideration of enthymemes within the context of the system E.

IV

Now what has been required, classically, of "valid" enthymemes? Two things: (1) that it be obvious what the suppressed premise is; (2) that the suppressed premise be obviously true. Degrees of obviousness are of course psychological matters, which we dare not touch. But when the psychological requirements of obviousness are stripped away, we are left with a consideration that logic should be able to handle: the suppressed premise must be *true*.

This leads us to suggest that if we are right in maintaining that E correctly captures the intuitive notion of a valid argument, then (allowing ourselves propositional quantifiers) the following would be a reasonable definition of an enthymematic *if ... then___* (which we symbolize by $\rightarrow^*$):

$$A \rightarrow^* B =_{df} (\exists r)[r \;\&\; (r \;\&\; A \rightarrow B)].$$

That is, we want to say that the *if ... then___* proposition corresponding to any enthymematic argument is true if and only if there is a (suppressed) premise which is true and which would convert the enthymeme into a valid argument. And when no more than *contingent* truth of the suppressed premise is required, we have enthymemes of the sort previously referred to as "intuitionistic." We now justify that terminology.

That intuitionistic "implication" is a relation of logical implication, or deducibility, has not been maintained even by some of its strongest proponents. Heyting 1956, for example, writes:

The *implication* $p \rightarrow q$ can be asserted, if and only if we possess a construction r, which, joined to any construction proving p (supposing that the latter be effected), would automatically effect a construction proving q. In other words, a proof of p, together with r, would form a proof of q (p. 98).[6]

One could hardly ask for a clearer or more explicit statement to the effect that intuitionistic "implication" is *enthymematic*; i.e., when $p \rightarrow q$ is intuitionistically true, there is a true premiss r, *which may be required* for the construction of q from p, but which is omitted in the antecedent p.

Now apparently Heyting has in mind a distinction between the sense in which p intuitionistically "implies" q (in the presence of r)

[6] As another example we mention an interesting paper of Curry 1959; he says "The absoluteness of absolute [i.e., intuitionistic] implication does not depend on any claim to its being a definition of logical consequence. It does not pretend to be anything of the sort" (p. 20).

and some other sense of *if . . . then*_ _ _ in which *if p and r then q* is true. Nothing is said about the latter relation, nor indeed can the intuitionistic formalism reflect the intuitive distinction between "implies" and "would automatically effect a construction." But if we are correct in claiming that the definition above (of $\rightarrow^*$) captures the notion of an enthymematic *if . . . then*_ _ _ and if Heyting is correct in saying that intuitionistic "implication" is enthymematic then we should expect that intuitionistic "implication" and our enthymematic "implication" should exactly coincide.

And this turns out to be the case. It develops that if we add propositional quantifiers to the positive fragment of E (i.e., if we consider that fragment EP^+ got from E by omitting those axioms involving the negation sign, and adding propositional quantifiers) and if we further define

$$A \supset B =_{df} (\exists r)[r \& (r \& A \rightarrow B)],$$

then the set of all theorems of EP^+ that contain only $\supset$, &, and $\vee$ *coincides exactly* with the positive fragment of intuitionistic logic as formalized by Heyting. That is, "A intuitionistically 'implies' B" *means* that there is some true proposition r such that the conjunction of r and A *entails* B.[7]

So we may justifiably (and in conformity with Heyting's own intuitive discussion) think of the intuitionistic horseshoe as expressing a notion of enthymematic (intuitionistically enthymematic) "implication": the antecedent doesn't *really* guarantee the truth of the consequent—but there *are* some true propositions which, together with the antecedent, could really be shown to have the consequent as a logical consequence.

There has been abroad for a long time a notion to the effect that entailment is a "simply psychological" matter[8]—and that the only difference between (say) the law of transitivity and the "law" $A \supset . B \supset A$ is that one has a nice, warm psychological feeling in the presence of the former and a somewhat sick psychological feeling in the presence of the latter. We think that the result just stated justifies a charge of psychologism in the other direction. The *only* reason anyone could ever have had for thinking that we could justifiably omit required premises that happened to be true is that

[7] See Myhill, 1953, which suggested this idea to us.

[8] See again Curry, 1959: ". . . it goes without saying that absolute implication is an objective concept in precisely the same sense that material implication is. It does not depend on any subjective feelings of 'entailment' " (p. 21). Similarly, "These methods [those of McKinsey] are capable of taking some of the mysticism out of the idea of necessity" (p. 23). We contend that the only mysticism in the idea of logical necessity is such as has been *put* there, largely by Quine, Goodman, White, and their followers.

we all knew that they were true, and the fact just didn't need to be mentioned (in the social context of the argument). At least it is hard to see any other reason why anyone would think (harking back to an earlier example) that the move from M to C was justified at all.

If, moreover, we define intuitionistic negation as follows:

$$\neg A =_{df} A \supset (p)p,$$

then the set of theorems of EP$^+$ containing only $\supset$, $\neg$, &, and $\vee$ *coincides exactly* with the full system of intuitionistic propositional calculus. That is, "A is intuitionistically false" *means* that A intuitionistically "implies" that every proposition is true.

We take these facts as evidence that when people have talked of the horseshoe as a relation of "implication" or "entailment" or "deducibility" or any of a host of equivalents, they have *really* had in mind (at best) a relation of (intuitionistically) *enthymematic* "implication," "entailment," "deducibility," etc.

Turning now from intuitionistic enthymemes to strict enthymemes, we observe that the appropriate definition (in EP$^+$) of a strictly enthymematic *if* ... *then*_ _ _ would be

$$A \prec B =_{df} (\exists r)[Nr \,\&\, (r \,\&\, A \rightarrow B)] \qquad \text{(where N is necessity).}$$

And it turns out that the set of theorems of EP$^+$ that contain only $\prec$, &, and $\vee$ *exactly coincides* with the positive fragment of Lewis's system S4 of strict implication.[9] That is, "A strictly 'implies' B" (in the sense of the positive part of S4) *means* that there is some *necessarily* true proposition r such that the conjunction of r and A *entails* B. Again, the intuitive considerations of our second section are borne out: if we consider only *strict* enthymemes, in which the discarded premise must be necessary, then we get precisely the (positive) system advocated by the advocates of S4.

V

We claim in this paper to have established (or commented on the establishability of) the following (obvious) truths:

1. A distinction can be drawn between valid arguments and enthymemes, and between enthymemes of two sorts: strict and intuitionistic. This is so trivial a point that it hardly seems worth arguing for. It *does* need arguing for, but that is only because currently accepted analyses of *implication* are so goofy.

2. Neither the horseshoe nor the hook will do as an analysis of the notion of a *valid argument* (most elementary and advanced texts to the contrary notwithstanding), since neither theory can distinguish between valid arguments and strict enthymemes.

[9] In a formulation where the primitives are $\prec$, &, and $\vee$.

3. The formal system E of entailment *does* suffice as an explication of the intuitive notion of entailment, or formal deducibility, and *does* account for the differences among valid arguments, strict enthymemes, and intuitionistic enthymemes.

4. What are we to think about enthymematic arguments? A priori, one might say that the notion of an enthymematic argument is sociological (depending on what we can expect our hearers to know) or psychological (depending on what *we* claim to know) and therefore is *a*logical: it is not a part of logic to try to take account of who knows what; and anyway our intuitions concerning enthymemes are altogether too unstable to support any kind of mathematical or logical theory. We on the contrary maintain that there is nothing intrinsically *bad* about sociology and psychology, even from a logician's point of view. Moreover, one can give a precise interpretation to the (admittedly, at the outset, somewhat *vague*) notion of an enthymematic argument. Thus:

5. If we are sufficiently careless in stating our arguments so that we allow ourselves to suppress any true premise, then we arrive *precisely* at the intuitionistic theory of "implication."

6. If we are somewhat more careful, and allow ourselves to suppress only necessarily true premises, then we arrive *precisely* at strict "implication" (in the sense of the positive fragment of S4).

7. If we are *very* careful, and always put down all the premises we need (i.e., if we argue *logically*), then we arrive *precisely* at the formal system E of logical implication (without quotes, this time), or *entailment.*

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MODALITIES AND EXTENDED SYSTEMS *

MODAL words are used in many different situations, and it is not clear which of them various modal logics are trying to catch and systematize. Often modal words are used ontologically, to state that some beings or some facts are considered necessary, other facts being contingent. Seemingly, not everything that happens happens with the same degree of urgency; 3 divides 21 by

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