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Agents in branching space-times*

Nuel Belnap

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1 Stit theory

Here I put forward some very brief and very general thoughts in connection with "stit theory" as described in FF. ① All of these thoughts are discussed at considerable length in the mentioned book. Occasionally they will refer to the following example.

Tomorrow Peter will drive his car to work. 〔1〕

What are the key elements of research strategy for Facing the future? The central aim is to use formal theory to help a little in understanding agency—human doings—as it works out in our world. We call it "stit theory." It takes its name from the prominent use of a non-truth-functional connective:

"α sees to it that _ ," which we abbreviate with "〔α stit: _〕." 〔2〕

In connection with stit, the analysis moves in several directions.

1.1 Stit theses

*收稿日期：2003－03－28
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1.1 Stit theses

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In an effort to tie the formal grammar of stit to natural language, we offer a series of so-called "theses" that concerning the intended application of the stit construction. The theses, which are stated in full in the appendix, are these:

Agentiveness of stit thesis.

Stit complement thesis.

Stit paraphrase thesis.

Imperative content thesis.

Restricted complement thesis.

Stit normal form thesis.

Together they say that in thinking about the philosophy of action, it is helpful to use stit as a normal form for expressing that an agent does something—something that natural languages express in a bewildering variety of fashions. As applied to [1], this gives

Will: [Peter *stit*: Peter drives his car to work]. [3]

Note that we have use "*Will*:" in order to express the future tense with complete explicitness. This is important in being clear about indeterminism.

1.2 Stit theory and indeterminism

Stit theory assumes *indeterminism*. Why? Stit theory is equally pre-humanist and pre-scientific. We take it as a fact is that our agentive doings and non-agentive happenings are enmeshed in a common causal order. The sense of "causal order" here should not be picked up from a determinist presupposition. The appropriate causal order is indeterministic. Nor does this involve "laws". Why should it? Bare indeterminism is what is wanted: There are initial events in our world for which more than one future is possible. It is this very, very simple concept of indeterminism that permits progress on the theory of agency. Thousands of pages written by philosophers about agency make (or seem to make) the contrary assumption of strict determinism. Hume and Kant are famous in modern philosophy for what has come to be called "compatibilism," the doctrine that agency is compatible with strict and absolute and perfect determinism. On the other hand there are also hundreds of pages (I' m making up these numbers) assuming some form of indeterministic "free will"; one has to think only of William James and his rejection of the idea of a "block universe." Rarely, however, is an indeterministic view of agency combined with a desire to let the enterprise be guided by a desire for mathematically rigorous theories. The aim of our research is to make indeterminism "intelligible."

We try to carry out this aim by proposing a simple and logically rigorous theory to help out.

1.3 The metaphysics of agency

The simplest representation of indeterminism is this: A treelike structure that opens toward the future. That is, a partial order in which there is no backward branching (see the appendix for exact statements). We call the elements of the tree "moments." A moment is an entire world-slice, so to speak, from one edge of the universe to the other. A moment is not a time; it is a concrete "super-event," caught in the causal order, with its unique past and its future of possibilities. A maximal chain of moments is called a "history," and represents one particular way in which our world might go.

In FF we argue at length against the pernicious doctrine that one of the many possible histories through a moment has a special status as "actual" or "real." We show that our understanding of assertion, prediction, promising, and the like is much improved if all histories through a moment are treated on a par, with none singled out.

One reaches the "theory of agents and choices in branching time" by adding two elements. (1) *Agent* is a set whose members are consider to be agents capable of making choices and so acting. (2) A *Choice* function, defined on all moments m and agents α, which delivers a partition of all the histories through m. The partition represents the choices open to agent α at moment m.

When there are multiple agents under consideration, one naturally assumes that the simultaneous choices of two agents α_1 and α_2 are radically independent; technically, every choice by α_1 is consistent with every choice by α_2 when the choices are simultaneous.

1.4 The semantics of indeterminism

Simple as it is, the branching-time structure does not explain itself. One has to come to terms with what prediction, and more generally the use of the future tense and other kinds of statements "about" the future, mean in connection with the tree representation of indeterminism.

This is a matter of semantics, in contrast to metaphysics. It is about how to understand certain linguistic structures under the assumption that they are used in an indeterministic situation. The key to a semantics for indeterminism, including agency as a special case, lies in combining the Prior-Thomason semantics for branching time, in which truth is relativized to histories, with the Kaplan semantics for indexical expressions.

The Prior-Thomason semantics makes the truth of this relative not only to the time of utterance, but to a history representing how the future unfolds. The semantics is entirely two-valued, since for each history, there is a definite truth value given to the example sentence. The semantics makes it possible to say precisely: When asserted, the example [1] is neither settled true nor settled false. However, it is settled true that, no matter what happens, one of the following holds: either tomorrow it will be settled true that [1] was true at the moment of utterance, or tomorrow it will be settled false that [1] was true at the moment of utterance. With slightly more brevity (but still with inevitable risk of error): The assertion is not settled one way or the other at the moment of assertion; but no matter what happens, tomorrow it will be definitely settled whether or not the assertion was true at the moment of assertion. This confusing passage involves "double time references," which are sorted out both in FF and in Belnap 2001. It may also be called "bivalence later." The subtle part is that [1] is evaluated with respect to a moment m_0 that is the moment of utterance, and with respect to each history h that passes through not m_0, but instead through some *later* moment m_1 (tomorrow) at which we are evaluating whether or not the assertion *was* true at the moment of assertion.

In this way, the theory provides a framework for normatively evaluating assertions about the future in terms of what we may call "fidelity": We say that an assertion is *vindicated* or *impugned* at some later moment depending on whether it is settled true or settled false at that later moment that the assertion was true at the moment of assertion. This apparatus is, we think, just as relevant to modelling assertions by artificial agents as it is to modelling assertions by human beings.

1.5 The semantics of agency

On the basis of the Prior-Thomason semantics that relativizes truth to momenthistories pairs, we suggest two principal semantic analyses of stit. These analyses depend essentially on an indeterminist view of agency. Both share the idea that action begins in choice, and that there is no choice and therefore no action without choices. We work out the semantics of stit in two ways. Both involve a transition from an unsettled situation to settledness due to the choice of an agent.

The *achievement stit* postulates that the relevant choice occurred at some temporal remove from its settled outcome.

The *deliberative stit* works through a concept of action based on the settled result of the choice being immediate.

1.6 Strategies

Stit theory allows a sophisticated but simple account of *strategies*, conceived of as a pattern of prescribed choices for future action. In this part of the theory, the principle of "no choice between undivided histories" in modeling agency is critical. The principle denies that it makes sense to choose today between histories that only divide tomorrow. In a nutshell: To choose a strategy is not to choose the later choices that define it.

It follows that we must carefully distinguish the *availability* of a strategy from *having* a strategy. In other words, the stit theory of strategies makes it imperative that one distinguish "acting *in accord with* a strategy" from "acting on a strategy." This is a special case of the following: The fundamental notions of stit and strategies may serve as a kind of foundation for intentional notions. Its benefit is that if you start that way, you easily see how difficult it is to sort out the intentional notions in any way that keeps them sorted out.

2 Branching space-times

There is much less written on branching space-times, and it is less accessible, in part because most essays on this topic were written with physical examples such as quantum mechanics in mind. ①

The fact is that histories do not really branch like a tree. Such a picture would work if there were an objective meaning for a simultaneity-slice that ran from one edge of the universe, so to speak, to the other. Such a simultaneity-slice would allow histories to split globally. But we know from Einstein that to think of such simultaneity-slices as objective is corrupt. We need another picture.

The new picture comes from the following observation. Individual histories are not really ordered like a line. Our modern reverence for classical mechanics and our related love of clock time deludes us. Not only fancy Einstein physics, but even our ordinary experience (when uncorrupted by uncritical adherence to Newton or mechanical addiction to clocks and watches) shows us that events are not strung out one after the other. Take the event of our being here now. Indeed some events lie in our future—or, better, keeping possibilities firmly in mind, we should say that some events lie in our future of possibilities. Others lie in our causal past. But there is a third category, always intuitive, and

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①The following paragraphs are extracted with little change from the as-yet unpublished essay Belnap 2002a. That essay also provides an overview of stit theory. BST theory is discussed in the following places: Belnap 1992, Szabo and Belnap 1996, Raki? c 1997, Belnap 1999, Placek 2000a, Placek 2000b, Belnap 2002b, Muller 2002, and Placek 2002.

now scientifically respectable, since we have learned in this century to be suspicious of the idea of action at a distance. In this third category are events that neither lie ahead of us as possibilities, nor do they lie behind us as determinate facts. Instead, they have a space-like relation to us. Neither later nor earlier (nor frozen into simultaneity by a mythical world-spanning clock), they are "over there".

The jargon word for this structure of events is well known: It is called a spacetime. So each history, each possible course of events, is a space-time. It needs zero training in mathematics to see that the theory of how such histories fit together into a single world will be more complicated than the theory that we pictured with a single tree. I mention only four key points underlying the theory of "branching space-times".

1. Here is the first point. You will remember that histories are closely related to the ideas of possibility and consistency. A key idea here is that what allows two events to share a history, and therefore to be consistent, is that at least one event lies in their common future. As long as there is a standpoint in Our World from which one could truly say "both of these events have happened," even if the two events are not themselves arranged one after the other, one may be confident that the two events can live together in single history. Peter' s (possible) going to the dentist in one village and Paul' s (possible) staying home in another village are consistent just in case there is some (possible) standpoint at which someone could truly say, using the past tense, that both events came to pass. Let us also turn this around: If two events are inconsistent, then no event can have both of them in its past. For example, although Peter' s choice to turn left or right is altogether local, and although we cannot picture Our World as a tree, nevertheless, there is no standpoint anywhere in Our World that has in its past both of the inconsistent events represented by Peter' s having turned left and his having turned right. These inconsistent possibilities can and must lie ahead of the point at which he makes the choice, but they cannot both lie behind anything whatsoever.

2. The second point is this. There have to be choice points, definite local events at which two histories split into radically inconsistent portions. It is presumably not true and we must not assume that when splitting occurs, it occurs in some magical worldwide way. When Peter is given the choice to go left or right on a certain occasion, that occasion is confined in space as well as in time. His little bit of free will is local, not global. And the same might be true when the choice is only metaphorical, a matter of a random outcome of some natural event such as, perhaps, the decay of a radium atom in Paris. It might be that the decay is a strictly local matter, neither influencing nor in-fluenced by contemporary happenings in, say, Manhattan. Whenever there is indeterminism, whether of choice or of chance, a good theory must give meaning to the difficult idea that the indeterminism is local, not worldwide.

3. The third point is critical to understanding Our World. When I put the point in everyday language, it sounds so obvious that you will yawn. And yet as far as I know a thoroughly controlled statement has never been made apart from the present theory of branching space-times. The key postulate for locating choice points may be put informally in the following way: Whenever we find ourselves in one history instead of in another, we may always look to the past for a choice point responsible for the splitting.

Example. Suppose that on a certain Friday in March a cat is sleeping on its mat in a certain living room in Chicago. Think of this as a particular concrete event, and let it be contingent. Take any history in which that event fails to occur, perhaps a history in which the cat spends the whole of the Friday high in a tree in Lincoln park. The theory guarantees that if you look in the causal past of the given concrete cat-on-the-mat event (the one that we are supposing occurred), you will find a definite choice point at which things could have gone either in the direction of keeping the cat-on-the-mat event possible, or in the direction of keeping the cat-in-the-tree history possible. ① You do not need to look in the future, and you also do not need to look far away at events going on "over there." The point is that examination of the causal past of the cat-on-mat event suffices. (The theory does not presume to say if the choice was up to the cat, or up to some human, or perhaps a bit of natural randomness or some combination.)

The fourth point records a recognition of an important way in which the theory should not be enstrengthened. The following is a principle that an unwary philosopher might easily be inclined to endorse.

Tempting principle. If two choice points are related in a "space-like" way, so that the second is "over there" with respect to the first, then their respective choices are entirely independent of each other. ②

Example. Aristotle tells us how two market-goers meet at the market, as an "accidental" result of their choices that morning, made separately in far-away villages. Their individual choices, we all suppose, are bound to be totally uncorrelated, that is, independent, and that is what the tempting principle says must be so.

The theory of branching histories, however, resists this temptation. And it does not do so a priori. It does so because Our World seems, as a matter of fact, to contain violations of the tempting principle. With reference once more to Einstein, modern physics seems to tell us that in fact it is possible for two utterly random choice points to be space-like related, with no hint of a line of causal connection between them, and nevertheless fail to be independent. I call this "funny business." Reichenbach has taught us that whenever we find the long arm of coincidence stretching across space, it is in our nature to look for a common cause. Funny business precisely happens when there is objective coincidence—which is to say, a failure of independence—across space, without a common cause. Since modern-day physics tells us that funny business happens, it is good that the theory of branching histories has room for funny business—and indeed has the virtue of permitting us to offer a stable (albeit conjectural) account as to the difference between (1) mere indeterminism without funny business and (2) indeterminism with funny business.

These necessarily too-brief points allude to a theory of branching histories that gives a satisfying account of how physical indeterminism can be local instead of global. It gives us an account of how choices and outcomes of natural random processes can affect only what lies in their causal future,

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①The theory will not let you exchange "event" and "history" here; precision of statement is essential.

②In Belnap 2002b this principle is suitable generalized and shown to be provably equivalent to an equally tempting principle saying something like "every situation that is cause-like with respect to a certain outcome event lies in the past of that event."

touching neither their past nor the vast region of space-like related events. But it does so in such a way as to allow plenty of room for individually random processes to be, as the physicists say, "entangled." Or, in the phrase I just used, the theory of branching histories helps us to come to terms with funny business.

The result is that the theory of branching histories, in addition to helping us clarify ideas of action and agency, provides low key suggestions for articulating some of the strangest phenomena uncovered by contemporary physicists. It does this by avoiding careless or fuzzy or sloppy formulations. It does this by insisting on a careful and rigorous account of what it is for indeterminism to be not immodestly global, but modestly local.

3 Agents in branching space-times

Since branching space-times is more realistic and more sophisticated than branching time as a representation of the indeterminist causal structure of our world, it is natural that one should consider agents and agency in branching space-times. The matter is hardly understood at all, which is precisely why it should be investigated. Here are some rather shallow thoughts on how some of the fundamental ideas might go.

We need to keep in mind the basic nontransitivity of "causally independent" between the actions of agents (and between their actions and other happenings). This should mimic the nontransitivity of simultaneity, and I think actually comes down to the same thing when properly understood. This consciousness of nontransitivity impacts straightaway on the fundamental idea of the independence of simultaneous actions by agents. Without this independence, there is "funny business" in the sense of EPR-like phenomena as discussed in § 2. The conceptual possibility of non-independence should be taken seriously if you are thinking about e. g. Prisoner' s Dilemma. Are you going to take their choices as independent or not? BST theory forces you to make up your mind in a clear way.

The chief difference between branching time and branching space-times lies in what "the causal order relation" relates. In the former case, moments are events taken to be "spatially" rich enough to be "occupied" by more than one agent. In the latter case, point events would seem to be so small as to admit the "presence" of at most one agent.

For this reason, in BST it seems (more or less) reasonable to represent an agent by a set of point events, whereas it makes no sense to represent an agent in branching time as a set of moments. (Agents can share moments-of-action, but not point-events-of-action.) If we do this, then " $e \in \alpha$ " can be read as "point-event e is a spatio-temporal part of α ."

This representation of an agent suggests (but certainly does not demand) that the choices open to an agent α at a point event e are exactly the same as the immediate possibilities at e in the sense of BST theory. In other words, there may well be no need for imposing a Choice function in addition to the possibilities definable from the causal ordering alone. ① Π_e is the set of immediate possibilities at e , defined in terms of undividedness at e . So when $e \in \alpha$, Π_e would represent the choices available to α at e .

Given that we are going to represent an agent α as a set of point events, what constraints make

①We should, however, leave open the possibility that some interesting ideas turn out to need Choice as a separate concept.

sense?

It seems easiest to think about if an agent in a given history looks like some portion of a "world line"; which is to say, for every agent α and history h, $\alpha \cap h$ should be a chain. I doubt that anything more is desirable, at least for "first purposes."

Since point events are so small, it seems plausible that one should never have more than one agent located at a point event. So perhaps $\alpha_1 \neq \alpha_2 \rightarrow \alpha_1 \cap \alpha_2 = \emptyset$. That requirement is probably, however, too strong; it might be better only to say that no nonvacuous point event (no point event with more than one possibility in its immediate future) can be shared by two agents. The weaker restriction would not entirely forbid that the "world lines" of two agents intersect.

There certainly is a notion of agent responsibility for immediate outcomes; the analog of the deliberative stit seems to work just the same in BST theory as it does in branching-time theory.

There should be a rich possibility for notions of agent responsibility in for example the form "α is partly responsible for O beginning to be rather than h_1". Here O is an outcome chain. $H_{\langle O\rangle}$ is the set of histories in which O begins to be (the set of histories having nonempty overlap with O). The simplest form of this partial responsibility would be when a single point event does all the work:

$\exists e[e \in \alpha \;\&\; h_1 \perp_e H_{\langle O\rangle}]$.[①]

This says that α had a choice at e such that one way keeps the outcome event O possible (at least in the immediate future) at the expense of making h_2 no longer possible, whereas another way makes O henceforth no longer possible.

It is striking that this recipe says nothing about e being in the past of O. When not, that is a (simple) case of "funny business." The theory of "agency without funny business" should be a stable theory. True or false, it doubtless corresponds to what we normally have in mind when thinking about agency and how it fits into the causal structure of our world.

A question needing research is just how to generalize this call for "no funny business" in the presence of agency.

Given the space-like choice-occasions of two agents, they should be independent.

But also, I think, a given space-like choice of an agent should be independent of all space-like related choice points, whether natural or agentive. Keeping in mind that quantum mechanics seems to say that there is funny business involving space-like related natural (non-agentive) choice points, it is hard to be sure what is best to say.

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① $H_{\langle O\rangle}$ is many histories, and there is no reason to suppose that a single point event e can be the locus for separating each of them from h_1. A more general idea is to let **I** be a set of point events that is consistent in the sense that it is a subset of some history or other (that is our definition of an "initial event.") Then $h_1 \perp_I H_{\langle O\rangle}$ means that for each history h_2 in $H_{\langle O\rangle}$, there is a point e in **O** such that $h_1 \perp_e h_2$. If all of these point events **O** represent choices of α, we should still say that α is partially responsible for the outcome **O** instead of h_1.

Check out 3-history diagram on p. 12: e_2 is not (under this definition) "partially responsible" for O_1 instead of h_3.

This subtlety should disappear (I think) if there is no funny business, since then e is forced to be in the past of O, so that the one e is bound to do all the work for $H_{\langle O\rangle}$.

A further generalization would consider a joint choice initial involving an arbitrarily complex (but consistent) set of choice points, each of which is agentive (each of which belongs to some agent). It would seem that the whole complex should be independent of any joint choice initial, no matter how complex, and no matter if agentive or natural or mixed, as long as the purely agentive joint choice is space-like related to the other complex.

The BST framework may give new ways of structuring joint agency. One might be able to have a theory of "messages" passing from one agent (at a causally earlier stage) to another agent (at a causally later stage).

Keep in mind that SLR is not a transitive relation, so that for example a choice by α_1 can well be SLR to each of two choices by α_2, those two choices being causally ordered. This may or may not complicate the idea of "state" as in Horty 2001. This is a delicate matter, which should be carefully considered.

4 Summary

I have tried to sketch some of the main ideas of stit theory, drawing on FF, and I have given a kind of overview of branching-times theory as discussed in the publications listed in note 1 on page 4. Then I have raised but definitely not answered some questions concerning how to use these materials in order to fashion a theory of agents and their choices in branching space-times.

Appendix: Lists for reference

This the appendix to Facing the future: Agents and choices in our indeterminist world (N. Belnap and M. Perloff and M. Xu, Oxford University Press, 2001); except that section references are missing. The purpose is to list in one place many of the items to which the book repeatedly refers. Cross-references to this appendix itself, either to its sections or to one of its numbered statements, are indicated by the use of one of the **boldface** forms occurring in the following display.

§ 4 Stit theses Thesis 1 - Thesis 6.

§ 4 Structures of various kinds.

§ 4 BT+I+AC postulates Post. 1 - Post. 10.

§ 4 Definitions of important BT+I concepts Def. 1 - Def. 9.

§ 4 Definitions of concepts involving choice Def. 10 - Def. 14.

§ 4 Fundamental semantic concepts Def. 15 - Def. 16.

§ 4 Derivative semantic concepts Def. 17 - Def. 20.

§ 4 The grammar of the mini-language we introduce.

§ 4 A few concepts from the axiomatics of stit theory, Ax. Conc. 1, Ax. Conc. 2, and Ax. Conc. 3.

We do not list recursive semantic clauses in this appendix; instead, we refer to the proper sections of chapter ??.

§ Stit theses: Thesis 1 - **Thesis** 6. By means of various "stit theses" we try to make explicit some of our central claims. (There are also several "slogans" in § ??.)

THESIS 1 AGENTIVENESS OF STIT THESIS. (Stit thesis) [αstit: Q] is always agentive for

α. Introduced in § ??.

THESIS 2 STIT COMPLEMENT THESIS. (Stit thesis) [α stit: Q] is grammatical and meaningful for any arbitrary sentence Q. Introduced in § ??.

THESIS 3 STIT PARAPHRASE THESIS. (Stit thesis) Q is agentive for α just in case Q may usefully be paraphrased as [α stit: Q]. Therefore, up to an approximation, Q is agentive for α whenever Q $\leftrightarrow$ [α stit: Q]. Introduced in § ??.

THESIS 4 IMPERATIVE CONTENT THESIS. (Stit thesis) Regardless of its force on an occasion of use, the content of every imperative is agentive. Introduced in § ??.

THESIS 5 RESTRICTED COMPLEMENT THESIS. (Stit thesis) A variety of constructions concerned with agents and agency—including deontic statements, imperatives, and statements of intention, among others—must take agentives as their complements. Introduced in § ??.

THESIS 6 STIT NORMAL FORM THESIS. (Stit thesis) In investigations of those constructions that take agentives as complements, nothing but confusion is lost if the complements are taken to be all and only stit sentences. Introduced in § ??.

We remind the reader that we offer these theses as worthwhile heuristics, but not as pronouncements to be taken strictly. In some cases, indeed, we call explicit attention to how one or more can usefully be modified, while nevertheless continuing to advance them as excellent rough approximations.

§ **Structures.** We list the most important classes of structures that we treat. In each case, if quantifiers are not being considered, Domain may be missing. (Note. We use "BT," for example, both as a singular term naming a class of structures and as an adjective modifying "structure" or "model".)

A BT structure is any tuple (Tree, $\leqslant$, Domain) satisfying each postulate of § 4 that governs one or more of its elements. Such a structure is a branching time structure. Introduced in § ??.

A BT+AC structure is any tuple (Tree, $\leqslant$, Agent, Choice, Domain) satisfying each postulate of § 4 that governs one or more of its elements. Such a structure is an agents and choices in branching time structure. Introduced in § ??.

A BT+I structure is any tuple (Tree, $\leqslant$, Instant, Domain) satisfying each postulate of § 4 that governs one or more of its elements. Such a structure is a branching time with instants structure. Introduced in § ??.

A BT+I+AC structure is any tuple (Tree, $\leqslant$, Instant, Agent, Choice, Domain) satisfying each postulate of § 4 that governs one or more of its elements. Such a structure is an agents and choices in branching time with instants structure. Introduced in § ??.

A BT+I+AC+nbc structure is a BT+I+AC structure in which there are no busy choosers (no busy choice sequences) in the sense of Def. 14. Introduced at the beginning of chapter ??.

A BT+I+AC+bc structure is a BT+I+AC structure in which there is at least one busy chooser (at least one busy choice sequence) in the sense of Def. 14. Introduced at the beginning of chapter ??.

· § **BT+I+AC postulates: Post.** 1 - **Post.** 10.

The postulates for agents and choices in branching time were introduced informally in chapter ?? and studied at length in chapter ??. These postulates play two roles. (i) They serve to define the kinds of structures of § 4, considered abstractly. (ii) When a structure is considered as anidealized representation of our world, the postulates count as part of a broadly substantive theory. To emphasize this second aspect of our project, this book sometimes uses "Our World" in place of "Tree." Note that several postulates are cast in terms of phrases that are defined in § 4.

POST. 1 NONTRIVIALITY. (BT+I+AC postulate) Tree is a nonempty set: Tree $\neq \emptyset$. Discussed in § ??.

POST. 2 CAUSAL ORDER. (BT+I+AC postulate) Tree is partially ordered by $\leqslant$:

Reflexivity. $(m \leqslant m)$.

Transitivity. $(m_1 \leqslant m_2 \,\&\, m_2 \leqslant m_3) \rightarrow m_1 \leqslant m_3$.

Antisymmetry. $(m_1 \leqslant m_2 \,\&\, m_2 \leqslant m_1) \rightarrow m_1 = m_2$.

Discussed in § ??.

POST. 3 NO BACKWARD BRANCHING. (BT+I+AC postulate) Incomparable moments in Tree never have a common upper bound; or contrapositively, if two moments have a common upper bound, then they are comparable: $(m_1 \leqslant m_3 \,\&\, m_2 \leqslant m_3) \rightarrow (m_1 \leqslant m_2 \text{ or } m_2 \leqslant m_1)$. Discussed in § ??.

POST. 4 HISTORICAL CONNECTION. (BT+I+AC postulate) Every two moments have a lower bound: $\forall m_1 \forall m_2 \exists m_0 [m_0 \leqslant m_1 \,\&\, m_0 \leqslant m_2]$. In other words, every two histories intersect. Discussed in § ??.

POST. 5 Instant AND INSTANTS. (BT+I+AC postulate)

1. Partition. Instant is a partition of Tree into equivalence classes; that is, Instant is a set of nonempty sets of moments such that each moment in Tree belongs to exactly one member of Instant.

2. Unique intersection. Each instant intersects each history in a unique moment; that is, for each instant i and history h, $i \cap h$ has exactly one member.

3. Order preservation. Instants never distort historical order: Given two instants i_1 and i_2 and two histories h and h′, if the moment at which i_1 intersects h precedes, or is the same as, or comes after the moment at which i_2 intersects h, then the same relation holds between the moment at which i_1 intersects h′ and the moment at which i_2 intersects h′.

Discussed in § ??.

POST. 6 AGENTS. (BT+I+AC postulate) Agent is a nonempty set. Discussed in § ??.

POST. 7 CHOICE PARTITION. (BT+I+AC postulate) Choice is a function defined on agents and moments. Its value for agent α and moment m is written as $Choice^{\alpha}_{m}$. For each agent α and moment m, $Choice^{\alpha}_{m}$ is a partition into equivalence classes of the set $H_{(m)}$ of all histories to which m belongs. Discussed in § ??.

POST. 8 NO CHOICE BETWEEN UNDIVIDED HISTORIES. (BT+I+AC postulate) If two histories are undivided at m, then no possible choice for any agent at m distinguishes between the two histories. That is, one of two histories undivided at m belongs to a certain choice possible for α at m if and only if the other belongs to exactly the same possible choice. In symbols from Def. 4 and Def. 12: $h_1 \equiv_{m_0} h_2 \rightarrow h_1 \equiv^{\alpha}_{m_0} h_2$. Discussed in § ??.

POST. 9 INDEPENDENCE OF AGENTS. (BT+I+AC postulate) If there are multiple agents: For each moment and for each way of selecting one possible choice for each agent, α, from among α's set of choices at that moment, the intersection of all the possible choices selected must contain at least one history. In symbols: for each $m \in$ Tree, and for each function f_m on Agent such that $f_m(\alpha) \in Choice^{\alpha}_m$ for all $\alpha \in$ Agent, $\bigcap \{f_m(\alpha): \alpha \in \text{Agent}\} \neq \emptyset$. Discussed in § ??.

POST. 10 RICHNESS OF Domain. (BT+I+AC postulate) The domain of quantification, Domain, must include Tree, Hist, Instant, and Agent as subsets. Discussed in § ??.

That concludes the list of postulates for BT+I+AC theory, that is, the theory of agents and choices in branching time with instants.

§ [. Branching-time-with-instants definitions: Def. 1 - Def. 9] Branching-time-with-instants definitions: Def. 1 - Def. 9 We list key definitions of BT+I+AC concepts. Most are intended as revelatory. In each case we suppose that we are given a BT+I+AC structure, ⟨Tree, $\leqslant$, Instant, Agent, Choice⟩.

DEF. 1 MOMENTS. (Definition) A moment is defined as a member of Tree. We let m and w range over moments; and we let M range over sets of moments. Discussed in § ??.

DEF. 2 CAUSAL ORDER. (Definition)

$m_1 \leqslant m_2$ is the strict partial order of Tree associated with the partial order $\leqslant$; that is, $m_1 \leqslant m_2$ iff $(m_1 \leqslant m_2 \ \& \ m_1 \neq m_2)$. We refer to either of these as the causal ordering of Tree.

We read $\leqslant$ and its converse with the plain words earlier/later, below/above, lower/upper, backward/forward, and so on, and insert proper when we intend $<$ or its converse. When m_1 is properly earlier than m_2, we also say, in a fashion much more revealing of our intentions, that m_1 is in the (causal) past of m_2; and m_2 is in the (causal) future of possibilities of m_1.

Discussed in § ??.

DEF. 3 CHAINS AND HISTORIES. (Definition)

Moments m_1 and m_2 are comparable if $\leqslant$ goes one way or the other: $(m_1 \leqslant m_2)$ or $(m_2 \leqslant m_1)$.

A chain, c, in Tree is a subset of Tree such that every pair of its members is comparable: $c \subseteq \text{Tree} \ \& \ \forall m_1 \forall m_2 [m_1, m_2 \in c \rightarrow m_1$ and m_2 are comparable]. We let c range over chains in Tree.

A history, h, of Tree is a maximal chain in Tree: h is a history of Tree iff h is a chain in Tree, and no proper superset of h is itself a chain in Tree. We let h range over all histories.

Hist is the set of all histories of Tree. We let H range over subsets of Hist.

Discussed in § ??.

DEF. 4 MOMENTS AND HISTORIES. (Definition)

1. $H_{(m)}$ is the set of histories in which m lies (or the set of histories "passing through" m): $h \in H_{(m)}$ iff $m \in h$.

2. m/h is a moment-history pair provided m is a moment and $h \in H_{(m)}$ (which is to say, $m \in h \ \& \ h \in \text{Hist}$). The notation "m/h" is an alternative way of naming the ordered pair ⟨m, h⟩, except that the use of "m/h" always presupposes that $m \in h$.

3. Moment-Hist is the set of all moment-history pairs m/h.

4. $h_1 \equiv_{m_0} h_2$ iff $m_0 \in h_1 \cap h_2$ and there is an m_1 such that $m_0 < m_1$, and $m_1 \in h_1 \cap h_2$ (unless there is no m_1 such that $m_0 \leqslant m_1$). We say that h_1 and h_2 are undivided at m_0. We adapt undividedness-at to pasts as well: Two histories extending a past, p, are undivided at p iff they share a

moment properly later than p, so that they appear as a single line as p comes to a close.

5. $h_1 \perp_{m_0} h_2$ iff $h_1 \neq h_2$ and m_0 is the least upper bound of $h_1 \cap h_2$. We say that h_1 and h_2 split at m_0.

6. Tree is deterministic at a moment, m [or at a past, p] iff every pair of histories through m [or p] is undivided at m [or p].

7. A set, H, of histories is an immediate possibility at a moment, m_0 iff H is a subset of $H_{(m_0)}$ that is closed under undividedness atm_0: $(h_1 \in H$ and $h_1 \equiv_{m_0} h_2) \rightarrow h_2 \in H$.

8. $H_{(M)} = \{h: (M \cap h) \neq \emptyset\}$, so that $H_{(M)}$ is the set of histories that pass through at least one member of the set of moments, M. For suitable M, this is the set of histories in which M comes to be.

9. $H_{[M]} = \{h: M \subseteq h\}$, so that $H_{[M]}$ is the set of histories entirely containing M. For suitable M, this is the set of histories in which M passes away.

Discussed in § ?? and § ??.

DEF. 5 BOUNDS. (Definition)

A moment, m, is a [proper] lower bound of a set of moments, M, iff m is [properly] earlier than every member of M; and similarly for [proper] upper bounds of chains.

We let $m < M$ iff m is a proper lower bound of M, and similarly in other cases.

We let $M_1 < M_2$ iff every member of M_1 is earlier than every member of M_2, and similarly in other cases.

Greatest lower bounds and least upper bounds of sets of moments are as usual in the theory of partial orders.

Discussed in § ??.

DEF. 6 CUTS, PASTS, AND FUTURES. (Definition)

A historical cut for a history, h, is a pair $\langle p, f\rangle$ of sets of moments such that (i) neither p nor f is empty, (ii) $p < f$, and (iii) $(p \cup f) = h$.

p is the past history of f iff f is a future history of p iff $\langle p, f\rangle$ is a historical cut.

M is the causal past of a future history, f, iff M is the set of all proper lower bounds of f.

We often say just past because given no backward branching, a causal past is the same as a historical past.

M is a future of possibilities, or a causal future of a past, p, iff M is the set of all proper upper bounds of p.

Given indeterminism, a future history must be distinguished from a future of possibilities, so that in this book we never say merely "future".

We say that $\langle p, M\rangle$ is a causal cut iff p is a past, and M is the future of possibilities of p.

All of these usages may straightforwardly be adapted to speak of pasts and futures of single moments.

Discussed in § ??.

DEF. 7 SEMI-LATTICE CONDITION. (Definition) The semi-lattice condition says that for every two (distinct) histories there is a moment at which they split (Def. 4 (5)). Discussed in § ??.

DEF. 8 INITIAL AND OUTCOME EVENTS, AND TRANSITIONS. (Definition)

I is an initial event iff I is a nonempty and upperbounded chain.

O is an outcome event iff O is a nonempty and lower-bounded chain.

O is an immediate outcome of I iff O is a (possible) outcome of I, and if furthermore no moment lies properly between I and O.

⟨I, O⟩ is an [immediate] transition iff O is an [immediate] outcome of I.

⟨I, O⟩ is a contingent transition iff ⟨I, O⟩ is a transition, and if some history is dropped in passing from the completion of I to the beginning of O: $H_{[I]} - H_{\langle O \rangle} \neq \emptyset$.

Discussed in § ??.

DEF. 9 INSTANTS. (Definition)

The members of Instant are called instants. i ranges over instants.

$i_{(m)}$ is the uniquely determined instant to which the moment m belongs, the instant at which m "occurs".

$m_{(i,h)}$ is the moment in which instant i cuts across (intersects with) history h; that is, $i \cap h = \{m_{(i,h)}\}$.

Order preservation can conveniently be stated in the symbols just introduced: $m_{(i_1,h_1)} < m_{(i_2,h_1)}$ implies $m_{(i_1,h_2)} < m_{(i_2,h_2)}$.

Fact: $m_{(i_{(m_0)},h_0)}$, a function of m_0 and h_0, is the moment on history h_0 that occurs at the same instant as does m_0: $i_{(m_{(i_{(m_0)},h_0)})} = i_{(m_0)}$

$i \,|\!> m = \{m_0 : m < m_0 \,\&\, m_0 \in i\}$. We say that $i \,|\!> m$ is the horizon from moment m at instant i.

Where i_1 and i_2 are instants, we may induce a linear time order (not a causal order!) by defining $i_1 \leqslant i_2$ iff $m_1 \leqslant m_2$ for some moment m_1 in i_1 and some moment m_2 in i_2. Instants can also be temporally (not causally) compared with moments, m: $i_1 < m$ iff $m_1 < m$ for some moment m_1 in i_1; and $m < i_2$ iff $m < m_2$ for some moment m_2 in i_2.

Discussed in § ??.

Agent-choice definitions: Def. 10 - **Def.** 14.

DEF. 10 AGENTS. (Definition) To be an agent is to be a member of Agent. We let α and β range over agents, and sometimes over terms denoting agents. Γ ranges over sets of agents. Discussed in § ??.

DEF. 11 CHOICE NOTATION. (Definition)

1. Choice represents all the choice information for the entire Tree.

2. $Choice^{\alpha}_{m}$ gives all the choice information for the agent α and the moment m; $Choice^{\alpha}_{m}$ should be thought of as a set of possible choices, and we call it "the set of choices possible for α at m".

3. $Choice^{\alpha}_{m}$ (h) is defined only when h passes through m, and is then the unique possible choice (a set of histories) for α at m to which h belongs. The notation is justified by the fact that according to Post. 7, each member of $H_{(m)}$ picks out a unique member of $Choice^{\alpha}_{m}$ to which it belongs.

4. $Choice^{\alpha}_{m}$ (m_1) is defined only when m_1 is in the proper future of the moment m of choice. Pick any history, h, containing m_1, a history that will a fortiori contain m. Then $Choice^{\alpha}_{m}$ (m_1) is defined as $Choice^{\alpha}_{m}$ (h). This definition is justified by no choice between undivided histories, Post. 8.

5. $\underline{Choice^{\alpha}_{m}}$ (m_1) is defined only when instants are present, and when m_1 is properly future

to m. Recall that $i_{(m_1)}$ is the instant on which m_1 lies. Then $\underline{Choice}^{\alpha}_{m}$ (m_1) is defined as the set of all moments on the instant $i_{(m_1)}$ that also lie on some history in $Choice^{\alpha}_{m}$ (m_1). In symbols, $\underline{Choice}^{\alpha}_{m}(m_1) = i_{(m_1)} \cap \bigcup Choice^{\alpha}_{m}(m_1)$. ①

Discussed in § ??.

DEF. 12 CHOICE EQUIVALENCE. (Definition)

1. $h_1 \equiv^{\alpha}_{m} h_2$ iff $Choice^{\alpha}_{m}(h_1) = Choice^{\alpha}_{m}(h_2)$, and we say that h_1 and h_2 are choice equivalent for α at m.

2. $h_1 \perp^{\alpha}_{m} h_2$ iff $m \in h_1 \cap h_2$ and $Choice^{\alpha}_{m}(h_1) \neq Choice^{\alpha}_{m}(h_2)$. We say that h_1 and h_2 are choice separated for α at m.

3. $m_1 \equiv^{\alpha}_{m} m_2$ is defined only when instants are present, and when $m < m_1$, m_2. Then $m_1 \equiv^{\alpha}_{m} m_2$ iff $\underline{Choice}^{\alpha}_{m}(m_1) = \underline{Choice}^{\alpha}_{m}(m_2)$. We say that m_1 is choice equivalent to m_2 for α at m. Be warned that in contrast to $\underline{Choice}$, there is no mnemonic in either the phrase "m_1 is choice equivalent to m_2 for α at m", nor in the symbols "$m_1 \equiv^{\alpha}_{m} m_2$", that forces the recognition that m_1 and m_2 belong to the same instant. (Other studies—but not this book—use the same notation for a concept defined via Choice instead of $\underline{Choice}$.)

Discussed in § ??.

DEF. 13 INSEPARABILITY/SEPARABILITY. (Definition)

$h_1 \equiv^{\alpha}_{M} h_2$, read "$h_1$ is inseparable from h_2 for α in M", iff $\forall m\,[m \in M \cap h_1 \cap h_2 \rightarrow h_1 \equiv^{\alpha}_{m} h_2]$.

$m_1 \equiv^{\alpha}_{M} m_2$, read "$m_1$ is inseparable from m_2 for α in M", iff $\forall m\,[(m \in M \,\&\, m < m_1 \,\&\, m < m_2) \rightarrow m_1 \equiv^{\alpha}_{m} m_2]$.

$h_1 \perp^{\alpha}_{M} h_2$, read "$h_1$ is separable from h_2 for α in M", iff $\exists m\,[m \in M \cap h_1 \cap h_2 \,\&\, h_1 \perp^{\alpha}_{m} h_2]$.

$m_1 \perp^{\alpha}_{M} m_2$, read "$m_1$ is separable from m_2 for α in M", iff $\exists m\,[m \in M \,\&\, m < m_1 \,\&\, m < m_2 \,\&\, m_1 \perp^{\alpha}_{m} m_2]$.

Discussed in § ??.

DEF. 14 ADDITIONAL CHOICE CONCEPTS. (Definition)

A moment m is a choice point for α iff there is more than one possible choice for α at m.

A possible choice for α at m is vacuous or trivial iff it is the only possible choice for α at m; and is otherwise nonvacuous. There can only be a vacuous choice for α at m when $Choice^{\alpha}_{m} = \{H_{(m)}\}$, in which case m itself is said to be a trivial choice point for α.

A chain c is a busy choice sequence for α iff (i) c is both lower and upper bounded in Tree, and (ii) c is an infinite chain of (nontrivial) choice points for α.

α is a busy chooser iff some c in Tree is a busy choice sequence for α.

Discussed in § ?? and § ??.

That concludes the list of major definitions of the BT+I+AC theory of agents and choices in branching time with instants.

Basic semantic definitions: **Def.** 15 - **Def.** 16. Sometimes we use K as ranging over the kinds of struc-

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①The underlining on Choice is intended as a mnemonic calling to mind the horizontal picture of an instant.

tures listed in §4. Rather than rehearse basic semantic concepts for each K, we run through the following system of definitions only for the case in which K = BT+I+AC, leaving other cases to be understood by analogy.

DEF. 15 INTERPRETATION AND MODEL. (Definition) Let G = ⟨Tree, ≤, Instant, Agent, Choice, Domain⟩ be a BT+I+AC structure.

J is an G-interpretation for a BT+I+AC structure G = ⟨Tree, ≤, Instant, Agent, Choice, Domain⟩ iff J is a function defined on propositional variables p, individual constants u (some of which are agent terms α and one of which, t, is to denote "the non-existing object"), operator letters f, and predicate letters F, such that J assigns to each propositional variable a function from Moment-Hist into {T, F}; assigns to each individual constant (including t) a member of Domain; assigns to each agent term a member of Agent; assigns to each n-ary operator letter a function from Moment-Hist into functions from $Domain^n$ into Domain; and assigns to each n-ary predicate letter a function from Moment-Hist into functions from $Domain^n$ into {T, F}. ①

M is a BT+I+AC model (based) on G iff M is a pair ⟨G, J⟩, where G is a BT+I+AC structure, and where J is an G-interpretation.

Discussed in § ?? and § ??.

DEF. 16 POINT, DENOTATION, AND TRUTH. (Definition)

A BT+I+AC point is a tuple ⟨M, m_c, a, m/h⟩, such that M is a BT+I+AC model, m_c ∈ Tree, is a function from the individual variables into Domain, m ∈ h, and h ∈ Hist. Henceforth we assume that ⟨M, m_c, a, m/h⟩ is a BT+I+AC point. We speak of the various parameters in ⟨M, m_c, a, m/h⟩ in the following way: M is the BT+I+AC model, m_c is the moment of use, is the assignment (of values to the variables), m is the moment of evaluation, and h is the history of evaluation.

We let π be the point ⟨M, m_c, a, m/h⟩, and adopt the following convention: In any context in which we write "π," we will understand the expressions Tree, ≤, Instant, Agent, Choice, Domain, G, J, M, m_c, a, m, and h just as if we had written "⟨M, m_c, a, m/h⟩".

Other structures and missing parameters. Points are defined in the same way for the other structures listed in §4. The assignment parameter, a, however, may be missing if quantification is not at issue, and the moment-of-context parameter, m_c, may be missing if the context of use is not under discussion.

We say that a BT+I+AC point, ⟨M, m_c, a, m/h⟩, is context initialized iff the moment of use, m_c, is identical to the moment of evaluation, m. That is, context-initialized BT+I+AC points have always the form ⟨M, m_c, a, m/h⟩.

Semantic value. For any categorematic expression E, be it term or sentence, $Val_{M,m_c,a,m/h}$ (E) is "the semantic value of E at the point ⟨M, m_c, a, m/h⟩". $Val_{M,m_c,a,m/h}$ (E) is defined recursively by clauses given in § ?? and § ??.

Denotation. Where t is any term, $Val_{M,m_c,a,m/h}$ (t) ∈ Domain. We read $Val_{M,m_c,a,m/h}$ (t) as "the denotation of t at the point ⟨M, m_c, a, m/h⟩".

Truth. $Val_{M,m_c,a,m/h}$ (A) is "the truth value of A at ⟨M, m_c, a, m/h⟩". Where A is any

①Some chapters use the equivalent procedure of assigning a subset of Moment-Hist to propositional variables, and more generally using a subset of Moment-Hist wherever we have used a function from Moment-Hist into {T, F}.

sentence, $Val_{M,m_c,a,m/h}(A) \in \{T, F\}$.

Alternative notation for truth: M, m_c, a, m/h $\models$ A iff $Val_{M,m_c,a,m/h}(A) = T$. Either is read "A is true at point ⟨M, m_c, a, m/h⟩".

Discussed in § ?? and § ??.

§ [. Derivative semantic definitions: Def. 17 - Def. 20] Derivative semantic definitions:

Def. 17 - Def. 20 A number of semantic concepts are standardly defined by quantifying over various parameters. As in Def. 16, we sometimes drop the assignment and momentof-context parameters when not relevant.

DEF. 17 SETTLED TRUTH. (Definition)

A is settled true at a moment m with respect to M, m_c, and a iff M, m_c, a, m/h $\models$ A for all h $\in H_{(m)}$. We may drop h, writing M, m_c, a, m $\models$ A.

A is settled true throughout a set of moments, M, with respect to M, m_c, and a iff M, m_c, a, m $\models$ A for all m $\in$ M. In addition to dropping h, we may replace m by M, writing M, m_c, a, m $\models$ A.

We most often write just M, M $\models$ A and M, m $\models$ A since we use these concepts most often when assignment and context are not relevant.

Discussed in § ??.

DEF. 18 EQUIVALENCE. (Definition)

Expressions E_1 and E_2, either both terms or both sentences, are (semantically) equivalent iff for all BT+I+AC points ⟨M, m_c, a, m/h⟩, $Val_{M,m_c,a,m/h}(E_1) = Val_{M,m_c,a,m/h}(E_2)$ (very same semantic value at all BT+I+AC points). We write $E_1 \cong E_2$.

Expressions E_1 and E_2, both terms or both sentences, are in-context equivalent iff for all context-initialized BT+I+AC points, ⟨M, m_c, a, m/h⟩, $Val_{M,m_c,a,m/h}(E_1) = Val_{M,m_c,a,m/h}(E_2)$ (the very same value at all context-initialized points). We write $E_1 \cong^{in\text{-}ctx} E_2$.

Discussed in § ??.

DEF. 19 IMPLICATION. (Definition)

A set Γ of sentences implies a sentence A iff for all BT+I+AC points ⟨M, m_c, a, m/h⟩, if M, m_c, a, m/h $\models A_1$ for every member A_1 of Γ, then M, m_c, a, m/h $\models$ A (truth preservation at all points). We write Γ $\models$ A.

A set Γ of sentences in-context implies a sentence A iff for all context-initialized BT+I+AC points ⟨M, m_c, a, m/h⟩, if M, m_c, a, m/h $\models A_1$ for every member A_1 of Γ, then M, m_c, a, m/h $\models$ A (truth preservation at all context-initialized points). We write $\Gamma^{in\text{-}ctx}$ A.

Discussed in § ??.

DEF. 20 VALIDITY CONCEPTS. (Definition)

For M a model, A is valid in M (or M-valid) iff M, m_c, a, m/h $\models$ A for every $m_c \in$ Tree and a over Domain and m $\in$ Tree. We write M $\models$ A.

For G a structure, A is valid in G (or G-valid) iff M $\models$ A for every G-model M. We write G $\models$ A. When G is understood, especially when G is taken as a representation of Our World, we say that A is valid.

When K is a class of structures such as those listed in § 4, A is valid in K (or K-valid) iff G $\models$ A for every structure G in K.

Each of these validity concepts has also an incontext version defined by restricting points to context-initialized points. When symbols are wanted, we write "$\models^{in\text{-}ctx}$" instead of "$\models$". Thus, A is (i) in-context M-valid iff M $\models^{in\text{-}ctx}$ A, (ii) in-context G-valid iff G $\models^{in\text{-}ctx}$ A, and (iii) incontext K-valid iff K $\models^{in\text{-}ctx}$ A.

Discussed in § ??.

§ **Grammar.** Part of our project involves speaking of a mini-language that we offer as illuminating. Here we indicate our ways of speaking of its grammar, and, within the general semantic framework outlined in § 4 and § 4, we point to chapter ?? for the semantics of each feature of the language.

Base clauses. Typically any one of our discussions draws on only some of the following items, which we introduce by simultaneously describing how we speak of the language that is our target. See § ?? for their semantics.

p (and sometimes q) ranges over propositional variables.

u ranges over individual constants, including two sorts of special terms: (i) α ranges over agent terms (and frequently over the agents themselves), and (ii) t is a term that artificially denotes "the non-existing object", to be available as a throwaway value of definite descriptions when existence or uniqueness fails.

x_j ranges over individual variables.

f ranges over operator letters.

F ranges over predicate letters.

Terms and sentences.

t ranges over terms of any kind. f (t_1, ⋯ , t_n) is a term.

A ranges over sentences. We also sometimes use B, C, D, P, and especially Q as ranging over sentences. F (t_1, ⋯, t_n) is a sentence.

For recursive semantic clauses, see § ??

Truth functions and identity. $\sim$, &, $\vee$, $\supset$, $\equiv$, T (truth), and $\perp$ (falsehood) are the usual truthfunctional connectives, and = is the usual identity predicate. See § ??.

Stit-free functors.

$\forall x_j$ A and ιx_j (A). See § ??.

Sett: A, Poss: A, and Can: A. See § ??.

Prior tenses Was: A, Will: A, Was-always: A, and Will-always: A, and temporal operators At-$inst_t$: A and At-mom_t: A. See § ??.

Stit functors.

The achievement stit, [α astit: A]. See § ?? (witness by moments) and § ?? (witness by chains).

The Brown stit, [α bstit: Q], is mentioned in § ?? and discussed in Horty 2001.

The deliberative stit, [α dstit: A]. See § ??.

The Chellas stit, [α cstit: A]. See § ??.

The strict joint stit, [T sstit: Q]. See § ??.

The transition stit, [α tstit: $m \overset{e}{\Rightarrow} A$]. See § ??.

Plain stit. [α stit: A]. Used both for the achievement stit when there is no ambiguity, and for the general stit idea.

§ [. Axiomatics concepts: Ax. Conc. 1, Ax. Conc. 2, and Ax. Conc. 3] Axiomatics concepts: Ax. Conc. 1, Ax. Conc. 2, and Ax. Conc. 3

AX. CONC. 1 REFREF EQUIVALENCE. (Axiomatics concept) The refref equivalence, also called just refref, is the following:

$$[\alpha \text{ stit}: \sim [\alpha \text{ stit}: \sim [\alpha \text{ stit}: A]]] \equiv [\alpha \text{ stit}: A] \text{ (refref)}$$

See § ??, § ??, chapter ??, and chapter ??.

AX. CONC. 2 STIT DEONTIC EQUIVALENCES. (Axiomatics concept) The stit deontic equivalences are the following.

Frbn: [α stit: Q] ↔ Oblg: [α stit: ∼ [α stit: Q]]

Perm: [α stit: Q] ↔ Oblg: [α stit: ∼ [α stit: Q]]

Oblg: [α stit: Q] ↔ Frbn: [α stit: ∼ [α stit: Q]]

Oblg: [α stit: Q] ↔ Oblg: [α stit: ∼ [α stit: ∼ [α stit: Q]]].

See § ?? and § ??.

AX. CONC. 3 AXIOM SYSTEMS. (Axiomatics concept) We study or refer to the following axiom systems.

La1 is the Logic for the basic achievement stit with 1 agent, without the refref equivalence (hence with the possibility of busy choice sequences). The language of La1 contains truth functions together with- stit sentences for just a single agent, α. See chapter ??.

La1+rr is the Logic for the achievement stit with 1 agent and the refref equivalence (hence with no busy choice sequences). The language of La1+rr is the same as the language of La1. See chapter ??.

Ld1 is the Logic for deliberative stit with 1 agent. In addition to truth functions and stit sentences, the language of Ld1 includes Sett:. See chapter ??.

Ldm is the Logic for the deliberative stit with many agents. In addition to truth functions and stit sentences, the language of Ldm includes identity, and hence can express multiplicity of agents. See chapter ??.

Ldm_n (for $n \geqslant 1$) is the Logic for the deliberative stit with many agents, where at each moment each agent is limited to at most n+1 choices. The language of each Ldm_n is the same as the language of Ldm. Also Ldm_0 is defined as Ldm. See chapter ??.

SA is the chapter ?? combination of Stit theory with Andersonian devices; or the Sanction with a logic of Agency. SA_0 is the fragment of SA described in § ??.

KD is the standard system of deontic logic from FØ llesdal and Hilpinen 1971.

S4 and S5 are the standard modal logics of C. I. Lewis.

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