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A SIMPLE TREATMENT OF TRUTH FUNCTIONS

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In this note we present an axiomatization of the classical two-valued propositional calculus, for which proofs of decidability, consistency, completeness, and independence, are almost trivial (given an understanding of truth tables).

Notation and definitions. 1. As primitives we use $^-$ for negation, $\vee$ for disjunction, and we assume that object language variables are specified. Well-formed formulas (wffs) are as usual, and other truth-functional connectives may be introduced by definition. Metavariables A, B, C, ..., A_i, ..., range over wffs.

2. B is a *primitive disjunction* if it has the form $A_1 \vee A_2 \vee \ldots \vee A_n$ ($n \geqq 1$; parentheses restored *ad lib.*), where each A_i is either a variable or a negate of such.

3. (*a*) A is a *disjunctive part of* A. (*b*) If $B \vee C$ is a *disjunctive part of* A, then B is a *disjunctive part of* A, and so is C.

4. $\phi(A)$ is a wff of which A is a disjunctive part, and $\phi(B)$ is the result of replacing A in $\phi(A)$ by B.

Axioms. If A and $\bar{A}$ both occur in a primitive disjunction B, then B is an axiom.

Rules. 1. From $\phi(A)$ to infer $\phi(\bar{\bar{A}})$.

2. From $\phi(\bar{A})$ and $\phi(\bar{B})$ to infer $\phi(\overline{A \vee B})$.

$\bar{\bar{A}}$ and $\overline{A \vee B}$ will be called *terminal formulas* of the respective conclusions.

Decision procedure. A decision procedure is immediate, since provability of $\phi(\bar{\bar{A}})$ reduces to provability of $\phi(A)$, and provability of $\phi(\overline{A \vee B})$ reduces to provability of $\phi(\bar{A})$ and $\phi(\bar{B})$. It follows that provability of any wff reduces to provability of primitive disjunctions — and we can see by inspection whether or not a primitive disjunction is an axiom.

Consistency and completeness. Elementary truth table considerations show that primitive disjunctions are tautologies if and only if they are axioms, and that premisses of rules are tautologies if and only if their conclusions are. Hence A is provable if and only if A is a tautology.

Independence. No axiom can be the conclusion of a rule, and nothing of the form $\phi(\bar{\bar{A}})$ $\{\phi(\overline{A \vee B})\}$ is provable without rule 1 {rule 2}.

Comments. We remark that the decision procedure also provides a uniform proof procedure involving no variables other than those of the expression to be tested, and that since application of rules is permutable in the sense of Curry [4], terminal formulas may be chosen arbitrarily, the choice uniquely determining the premisses.

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The following is prompted by a comment of the referee.

There are two senses in which we may say that a system S has a "derived rule" *from* $A_1, \ldots, A_n$ *to infer* B: (i) if there is a proof in S of B from hypotheses $A_1, \ldots, A_n$, using only primitive rules, and (ii) if there is an effective way of finding a proof in S of B, given proofs of $A_1, \ldots, A_n$. (The latter is the sense of Church [3], p. 83; and the distinction between (i) and (ii) is also drawn in Curry and Feys [5], pp. 50–51.) The referee points out that in sense (i), the rule of detachment *from* $\bar{A} \vee B$ *and* A *to infer* B is not derivable in our formulation of the propositional calculus,[1] and the same is indeed true for any truth-preserving rule which leads from longer premisses to shorter conclusions. But of course in sense (ii) any such rule *is* derivable.

Our interest in the present formulation lies not only in its simplicity, but also in the fact that the primitive rules (unlike detachment) are *entailments* in the sense of E ([2]). It follows that E contains the full two-valued calculus, in spite of the fact that detachment is not derivable for E, even in sense (ii). (For an alternative argument see Ackermann ([1])).

As a further application, our formulation may be used to show that Parry's system of *analytische Implikation* [6] also contains the two-valued calculus (granted that a rule of adjunction, obviously required, is added to his system).

REFERENCES

[1] WILHELM ACKERMANN, *Begründung einer strengen Implikation*, this JOURNAL, vol. 21 (1956), pp. 113–128.

[2] ALAN ROSS ANDERSON and NUEL D. BELNAP, JR., *A modification of Ackermann's 'rigorous implication'*, (abstract), this JOURNAL, vol. 23 (1958), pp. 457–458.

[3] ALONZO CHURCH, **Introduction to mathematical logic,** vol. 1, Princeton, 1956.

[4] HASKELL B. CURRY, *The permutability of rules in the classical inferential calculus*, this JOURNAL, vol. 17 (1952), pp. 245–248.

[5] HASKELL B. CURRY and ROBERT FEYS, **Combinatory logic,** Amsterdam, 1958.

[6] W. T. PARRY, *Ein Axiomensystem fur eine neue Art von Implikation (analytische Implikation)*, **Ergebnisse eines mathematischen Kolloquiums,** vol. 4 (1933), pp. 5–6.

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[1] (*Added December 14, 1959.*) The distinction between the senses in which detachment is and is not derivable amounts to a distinction between completeness and Post completeness. The referee points out that the first explicit application of the distinction is made by Henry Hiż, **American Mathematical Society Notices** vol. 5 (1958) abstract no. 538–34.