

Branching Space-Time Analysis of the GHZ Theorem

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Greenberger, Horne, Shimony, and Zeilinger gave a new version of the Bell theorem without using inequalities (probabilities). Mermin summarized it concisely; but Bohm and Hiley criticized Mermin's proof from contextualists' point of view. Using the branching space-time language, in this paper a proof will be given that is free of these difficulties. At the same time we will also clarify the limits of the validity of the theorem when it is taken as a proof that quantum mechanics is not compatible with a deterministic world nor with a world that permits correlated space-related events without a common cause.

1. GREENBERGER, HORNE, SHIMONY, ZEILINGER, AND MERMIN

Greenberger, Horne, Shimony, and Zeilinger (1990) developed a proof of the Bell theorems without using inequalities. Following Mermin, one can extract from their ideas not only a simple Kochen–Specker-like illustration that there are quantities that have no values independently of the measurements, but also some simple Bell–EPR-like results emphasizing locality (spatiotemporal) principles.

In the GHSZ example we consider three spin-half particles originated in a gedanken spin-conserving decay flying apart along three different straight lines in the horizontal plane. Let us denote the spin of particle i along its direction of motion by $\frac{1}{2}\hbar\sigma_z^i$, the spin along the vertical direction by $\frac{1}{2}\hbar\sigma_x^i$, and the spin along the horizontal direction orthogonal to the

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trajectory by $\frac{1}{2}h\sigma_v^i$. Assume that the quantum state of the three-particle system is

$$\Psi = \frac{1}{\sqrt{2}} (|1_z^1\rangle \otimes |1_z^2\rangle \otimes |1_z^3\rangle - |-1_z^1\rangle \otimes |-1_z^2\rangle \otimes |-1_z^3\rangle) \quad (1)$$

where $\sigma_z^i |1_z^i\rangle = |1_z^i\rangle$ and $\sigma_z^i |-1_z^i\rangle = |-1_z^i\rangle$. In this state, the possible measurement results can be $R(\sigma_x^a) = \pm 1$ and $R(\sigma_y^a) = \pm 1$ for $a = 1, 2, 3$. Consider now the following operators:

$$\begin{aligned} \Omega_1 &= \sigma_x^1 \sigma_y^2 \sigma_y^3 \\ \Omega_2 &= \sigma_y^1 \sigma_x^2 \sigma_y^3 \\ \Omega_3 &= \sigma_y^1 \sigma_y^2 \sigma_x^3 \\ \Omega_4 &= \sigma_x^1 \sigma_x^2 \sigma_x^3 \end{aligned} \quad (2)$$

The only known way to measure these quantities is to perform the measurements $\sigma_x^1, \sigma_y^2, \sigma_y^3$ and $\sigma_y^1, \sigma_x^2, \sigma_y^3$, etc., and to take the product of the corresponding results. Of course, the quantum state Ψ does not determine the measurement results $R(\sigma_x^1), R(\sigma_y^2), R(\sigma_y^3), R(\sigma_y^1), R(\sigma_x^2), R(\sigma_y^3), \dots$. But we can know in advance each $R(\Omega_j)$! The state Ψ is an eigenstate of Ω_1, Ω_2 , and Ω_3 with eigenvalue 1. Ψ is an eigenstate of Ω_4 , too, with eigenvalue -1 . Consequently, in any measurement of these quantities the results are

$$R(\Omega_1) = R(\Omega_2) = R(\Omega_3) = -R(\Omega_4) = 1 \quad (3)$$

which implies a constraint on the measurement results for the spin components of the three separated particles. This constraint provides what we shall call an *inconsistency-type correlation* among the outcomes at the three stations; that is, there are *combinations* of outcomes that are not possible, even though each *individual* outcome is possible.

The measurement results $R(\Omega_j)$ are fixed in advance. That is why, according to the minimal understanding of a "value" assigned to a quantum observable, one may say that the values of $\Omega_1, \dots, \Omega_4$ are

$$V(\Omega_1) = V(\Omega_2) = V(\Omega_3) = -V(\Omega_4) = 1 \quad (4)$$

Consider now the product of these operators

$$\Omega = \Omega_1 \Omega_2 \Omega_3 \Omega_4 \quad (5)$$

Ψ is an eigenstate of Ω too with eigenvalue -1 . This fact is consistent with

$$V(\Omega_1) V(\Omega_2) V(\Omega) V(\Omega_4) = -1 \tag{6}$$

that is,

$$V(\sigma_x^1 \sigma_y^1 \sigma_z^1) V(\sigma_y^1 \sigma_x^1 \sigma_z^1) V(\sigma_y^1 \sigma_z^1 \sigma_x^1) V(\sigma_x^1 \sigma_x^1 \sigma_x^1) = -1 \tag{7}$$

And the same holds for the measurement results

$$R(\sigma_x^1 \sigma_y^1 \sigma_z^1) R(\sigma_y^1 \sigma_x^1 \sigma_z^1) R(\sigma_y^1 \sigma_z^1 \sigma_x^1) R(\sigma_x^1 \sigma_x^1 \sigma_x^1) = -1 \tag{8}$$

Until now we did not do anything beyond standard quantum mechanics. Here we come, however, to a Kochen–Specker-type argument. Let us make an additional assumption and see whether it goes through or not: Assume that we can assign values not only to the products $\sigma_x^1 \sigma_y^1 \sigma_z^1, \sigma_y^1 \sigma_x^1 \sigma_z^1, \dots$, but also to the spin-operators $\sigma_x^1, \sigma_y^1, \sigma_x^2, \sigma_y^2, \sigma_x^3, \sigma_y^3$ themselves, such that (7) can be written as

$$V(\sigma_x^1) V(\sigma_y^2) V(\sigma_y^3) V(\sigma_y^1) V(\sigma_x^2) V(\sigma_x^3) \\ \times V(\sigma_y^1) V(\sigma_y^2) V(\sigma_x^3) V(\sigma_x^1) V(\sigma_x^2) V(\sigma_x^3) = -1 \tag{9}$$

This is, however impossible, because each $V(\sigma_i^a)$ appears twice, so whatever the values $V(\sigma_i^a)$ are, the left-hand side is a positive number, instead of -1 . In this way, one finds evidence against the idea of pre-existing values for the quantities σ_i^a .

From the Kochen–Specker point of view locality considerations are absent or not so important. From the EPR point of view, one takes into account that the various measurements are distant, and finds evidence that in spite of the correlations among them, there are no “elements of reality” corresponding to the values $V(\sigma_i^a)$. One uses the term “element of reality” partly because one thinks of a presettled value of a physical quantity as corresponding to an objective feature of our world, a feature that exists independently of the measurement that reveals it. Another reason is that $V(\sigma_i^a)$ seems to satisfy the often-quoted EPR reality criterion: “*If, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.*” By virtue of the spatial separation of the three measurements, one can predict any of $V(\sigma_i^a)$, without apparent disturbance of the system, by carrying out two suitable measurements on the other two particles; see Mermin (1990a).

Since the values of the spin components cannot be prearranged in advance before the measurements, the *individual* experiments cannot merely reveal values settled in advance. In this sense one can use the GHSZ example in the Bell–EPR–common cause context, too: The nonexistence of prearranged values/measurement outcomes seems to imply the nonexistence of a *common cause* that can account for all the correlations provided by the constraint (3). In the remainder of this paper we will be discussing the GHSZ example in this context. In order to establish the correctness of the implication to no-common-cause, we use language that is more rigorous than customary. In this way our discussion will not only provide a more solid nonlocality theorem, but will avoid the vulnerable points about which the contextualists’ complaints have been made.

2. CONTEXTUALISTS’ CRITICISM

From the contextualists’ point of view the above proof is not completely acceptable. Their critique is focused in the following remarks:

- From a contextualist point of view any Kochen–Specker-type argument is vacuous. The disagreement (9), for example, proves the nonexistence of values, the existence of which has never been assumed by a contextual theory, such as the Bohm mechanics. The “beables,” if they exist, determine the result of each individual measurement *operation*. But these results are not present before the measurement operations have been completed.

- The Bell–EPR or the Bell–EPR–common cause context concerns the question of what we can tell about the measurement results

$$R(\sigma_x^1), R(\sigma_y^1), \dots, R(\sigma_z^1)$$

To avoid a contradiction analogous to (9),

$$\underbrace{R(\sigma_x^1) R(\sigma_y^2) R(\sigma_z^3)}_1 \underbrace{R(\sigma_x^1) R(\sigma_x^2) R(\sigma_y^3)}_1 \times \underbrace{R(\sigma_y^1) R(\sigma_y^2) R(\sigma_z^3)}_1 \underbrace{R(\sigma_x^1) R(\sigma_x^2) R(\sigma_x^3)}_{-1} = -1 \tag{10}$$

the measurement results must be *context dependent* in the sense that, for example, $R(\sigma_x^1)$ cannot be the same in the first place, when σ_x^1 is assumed to be measured along with σ_y^2 and σ_z^3 , as it is in the second place when it

is measured with σ_y^2 and σ_y^3 (see Bohm and Hiley, 1993). So, it would be better to use a special index, to express the context in which the measurement is completed:

$$\underbrace{R_{xyx}(\sigma_x^1) R_{xyx}(\sigma_y^2) R_{xyx}(\sigma_y^3)}_1 \underbrace{R_{yxx}(\sigma_y^1) R_{yxx}(\sigma_x^2) R_{yxx}(\sigma_x^3)}_1$$

$$\times \underbrace{R_{xyx}(\sigma_y^1) R_{xyx}(\sigma_y^2) R_{xyx}(\sigma_x^3)}_1 \underbrace{R_{xxx}(\sigma_x^1) R_{xxx}(\sigma_x^2) R_{xxx}(\sigma_x^3)}_{-1} = -1 \quad (11)$$

Such an expression is not contradictory at all.

One might argue that $R_{xyx}(\sigma_x^1)$ for example, should be the same as $R_{xxx}(\sigma_x^1)$, because of the spatial separation of the three stations. There is no influence on the measurement at station 1 of the choices among the possible measurements at the other two stations. But this is irrelevant. It is a matter of fact that $R_{xyx}(\sigma_x^1)$ differs from $R_{xxx}(\sigma_x^1)$. These two measurements belong to different runs of the experiment.

We agree that both of the above remarks are relevant, but we think that this is not the last word. In the following sections we will present a reformulation of Mermin’s version of the GHSZ thought-experiment. It will avoid vague notions such as “value of a quantity,” concentrating on the Bell–EPR–common cause context only. Instead,

1. We will reformulate the problem using only some elementary objects, such as events and their causal relations.
2. In this language we will describe exactly what it could mean to say that there is no common-cause explanation of inconsistency-type correlations among separated spin measurements.
3. And we will refer to the observed correlations among the separated spin measurement results only in such a way that everything will have a correct empirical meaning.

On this purified basis, we will give an airtight proof that in a certain limited sense, the GHSZ thought-experiment does in fact imply a violation of the common-cause principle.

3. BASICS OF THE BRANCHING SPACE-TIME THEORY

Our formulation will use some of the very basic elements of the *branching space-time theory* (BST). The aim of this theory was to solve the problem: How can we combine *relativity* and *indeterminism* in a rigorous

theory? The underlying idea is that a true description of our world may require fusing Einstein spacetime with Prior/Thomason branching time. For further motivation and details of BST we refer to Belnap (1992).

From BST we rely on the following postulates and definitions on the primitives $\langle \textit{Our World}, < \rangle$, where *Our World* is the totality of (possible) point events connected to us by a succession of causal paths, and $<$ is an adaptation of the *causal ordering relation* among space-time points (the existence of a timelike or lightlike path) to the domain of possible point events. We stress that no other primitives whatsoever figure in our discussion.

Postulate. *Our World* is postulated to be a nonempty set; its members are called *point events*. $<$ is postulated a dense partial order on *Our World* such that every nonempty lower bounded chain has an infimum. A *history* is defined as a subset h of *Our World* maximal with respect to the property that if h contains two point events, then it contains an upper bound for them. It is postulated that each nonempty upper bounded chain has a supremum in each history of which it is a subset. A *choice point* for two histories is defined as a point event maximal in their intersection. Given any nonempty lower bounded chain E of point events in $h_1 - h_2$, it is postulated that there is at least one choice point e_i for h_1 and h_2 such that $e_1 < e_2$ for all $e_\epsilon \in E$ (this is the “*prior choice principle*”). An *event* is defined as a nonempty set of point events.

A good picture to have in mind is that each history is a space-time of general relativity that is free from causal degeneracy (for example, the Minkowski space-time). We do not use anything of the more detailed geometry of a space-time beyond its causal structure. So, if we wish, we can regard a history at a more abstract level as a causal space of point events satisfying the Kronheimer–Penrose (1967) axioms. The postulates above, especially the prior choice principle, constrain how these histories branch one from the other. The details are not, however, pertinent. We add the following definitions for current purposes.

Definition 3.1. An *initial event* I is a nonempty upper bounded chain. An *outcome event* O is a nonempty lower bounded chain. A *stable event* is both initial and outcome, and is contained in every history it overlaps.

Initial events represent situations that can have any one of a variety of outcomes, which are represented by outcome events. Both initial and outcome events have definite loci in *Our World*; but the two sorts of events have different structures.

Initial events and outcome events, because of their respective structures, can fit together into “spreads.” Roughly speaking, a spread—which we are going to define just below—consists in a single initial event, and a variety of individually possible but mutually inconsistent outcomes of that initial. A spread with its outcomes is one way of representing an “experiment” and its possible outcomes. It also sometimes makes sense to think of a spread as a choice-situation for an experimenter, with the outcomes representing the available choices. The meaning of a spread is, however, more general; it can describe any kind of indeterministic situation without implying either the presence or the absence of any human activity at the “preparation” of the initial event or “observation” of the outcomes. Thus, the following formulation draws no distinction between the “a measurement process” and any other process going on in Our World:

Definition 3.2. A *spread* σ is an ordered pair the first member of which is an initial event I and the second of which is a nonempty set Ω of outcome events; we, write $\sigma = I \rightarrow \Omega$. Each spread must satisfy the following three conditions:

- (i) I causally precedes each $O \in \Omega$.
- (ii) Every history containing I overlaps some $O \in \Omega$.
- (iii) No history overlaps two distinct members of Ω .

Thus given any historical course in which the initial event I comes to a close, exactly one of the outcome events in Ω commences. In all cases spreads have two features: (i) they have a definite locus in Our World; and (ii) their internal structure has a definite causal structure, since their mutually inconsistent “outcomes” are located after their initials in the causal ordering.

We next define an “ n -spread” with its “outcome vectors” in order to give us a convenient way of simultaneously considering a number of experiments and their possible results.

Definition 3.3. An *n -spread* is a finite sequence of spreads:

$$\Sigma = \{\sigma_i = I_i \rightarrow \Omega_i\}_{i=1,2,\dots,n}$$

The collection of the initial events $\{I_1, I_2, \dots, I_n\}$ is called the set of initials of Σ . An outcome vector \mathbf{O} is a vector whose i th term is one of the outcomes of the i th spread of Σ , $\mathbf{O} \in \Omega_1 \times \Omega_2 \times \dots \times \Omega_n$.

The next definition uses the expressive power of BST to give an exact account of what it means to say that a set of events is “consistent.”

Definition 3.4. A set of initial and outcome events is said to be *consistent* if there is a history that contains each of its initial events and overlaps each of its outcome events. An outcome vector is said to be consistent if its terms form a consistent set of outcome events.

Observe that there is an important difference in how initial events and outcome events enter into consistency relations: In the history in question, initial events must finish, while outcome events must begin. Given the difference in their structures, that is to be expected.

We also distinguish several senses of “consistency” of an n -spread Σ .

Definition 3.5. Σ is *minimally consistent* if its set of initials is consistent, and *maximally consistent* if all of its outcome vectors are consistent. And Σ is *1-consistent* if its set of initials is consistent with each single outcome of each of its spreads in the following sense: for each outcome event $O \in \Omega_j$, where $1 \leq i \leq n$, the set of events $\{I_1, I_2, \dots, I_n, O\}$ is consistent.

Lastly, in contrast to the language available in most discussions of Belllike phenomena, the language of BST permits us to say with absolute rigor what it is for a set of experimental situations to be spatially separated.

Definition 3.6. Σ is *space-like* iff it is minimally consistent, and no initial of one of its spreads causally precedes any outcome of a distinct spread.

The following sums up what will be used explicitly.

Fact 1. Let σ be a spread, let Σ be an n -spread, and let E be an initial or outcome event of Σ .

- E is consistent with the initial of σ if and only if E is consistent with some outcome of σ .
- E is consistent with the set of initials of an n -spread if and only if E is consistent with some outcome vector of Σ .
- If E is consistent with a set of initials and outcomes, then E is consistent with each element of the set.
- If Σ is maximally consistent then it is 1-consistent. If it is 1-consistent then it is minimally consistent.

Using the definitions just given, BST suggests the following partial (and pre-probabilistic) analysis of the concept of the “common cause.” The idea is that the question of a common cause does not arise unless you are given a spacelike n -spread Σ that is 1-consistent, so that the joint coming to a close of its initials is consistent with each outcome of each of its

spreads. (One does not even look for a common cause if there is a violation of 1-consistency; if, that is, an otherwise possible outcome of one the experiments is prevented merely by initializing the other experiments.) But you are also given an outcome vector \mathbf{C} that is inconsistent—in spite of the fact that each term of \mathbf{C} is consistent with the set of initials of Σ . It is this combination that raises a question. And when the question does arise, one looks for a causal locus in the common past of the outcomes of spreads in Σ . What one looks for can be understood in terms of Reichenbach's (1956) concept of "screening off," here adapted to the language of consistency: The inconsistency of \mathbf{C} represents a "correlation" among the terms of the n -spread, in spite of their spacelike separation, and screening off makes the "correlation" disappear in the context of each outcome of the common cause σ . As a partial analysis, we collect the necessary conditions for a spread to be a common cause for an inconsistent outcome vector. If our analysis were not restricted to inconsistency-type correlations, we would perhaps need to enrich or refine these conditions, but for present purposes the following definition works out:

Definition 3.7. Let σ be a spread, Σ a spacelike n -spread that is 1-consistent, and \mathbf{C} an inconsistent outcome vector of Σ . σ is a common cause for \mathbf{C} in Σ only if

- CC1. (*Causal priority*) The initial of σ is causally prior to every outcome of every spread in Σ .
- CC2. (*Consistency*) Each outcome of σ is consistent with the set of initials of Σ .
- CC3. (*Screening off of inconsistency*) σ "screens off" \mathbf{C} . That is, each outcome O of σ is inconsistent with some one term of \mathbf{C} .

The expression "common cause for the inconsistent outcome vector \mathbf{C} " sounds a little awkward in ordinary language; we use it as short for something like "common causal locus for \mathbf{C} " or "common-cause explanation for \mathbf{C} ." Whichever, it clearly and rigorously refers to an actual place in *Our World* such that what happens there "explains away" the spacelike correlation represented by \mathbf{C} .

4. BST FORMULATION OF THE GHSZ–MERMIN THEOREM

We now turn to redescribing Mermin's version of the GHSZ theorem in the language of BST, dropping all language of particles, state, and systems. Nor do we even use much of the language of BST itself, confining

ourselves to the notions introduced in the previous section. We shall, however, help ourselves to heuristic interpretations, with the caution that the hard information is contained in the explicit numbered stipulations below.

At each of the three “stations” 1, 2, or 3 there can be one of the two “measurements” of σ_x or σ_y , each of which can have one of two “outcomes” – or +. We interpret this situation as follows. There are, to begin with, three widely separated “pre-preparation” initial events I_1, I_2 and I_3 , with the interpretations, e.g.,

I_1 : Station 1 has been pre-prepared (so to speak) so that it will be set to measure either on the x axis or on the y axis (but not both).

Next, the pre-preparation initial I_1 (for example) has two possible outcomes x_1 and y_1 , with the interpretation, e.g.,

x_1 : Station 1 is prepared to measure on the x axis.

Now x_1 , for example, is not only an outcome event, but also an initial event, with its own two outcomes, x_1^- and x_1^+ , with the interpretation, e.g.,

x_1^- : At station 1 the outcome was spin “–” on the x axis.

The picture to have is that the three stations are widely separated, but that at each station the events come in rapid succession. Keep in mind that our theoretical language-implies little or nothing in the system/state vocabulary, much less in the language of classical physics, much less in the language of quantum mechanics. What happened, happened; that’s it. We record in the following “stipulations” the minimal story that quantum mechanics, no doubt, tells.

Stipulation 1.

- We use labels $\{1, 2, 3\}$ for the three “stations,” labels $\{x, y\}$ for the two “measurement-types,” and labels $\{-, +\}$ for the two “outcome types.” (The three quoted phrases are to be taken only as heuristic. Theoretically the labels we use are just labels; the suggested concepts do not figure as part of theory.) We use i and j as ranging over $\{1, 2, 3\}$.
- We shall consider three initial events I_i , six stable events x_i, y_i , and twelve outcome events $x_i^-, x_i^+, y_i^-, y_i^+$.
- These events fit together into spreads as follows. For $i=1, 2, 3$, σ_i, σ_i^x , and σ_i^y are spreads, where $\sigma_i = I_i \rightarrow \{x_i, y_i\}$ and $\sigma_i^x = x_i \rightarrow \{x_i^-, x_i^+\}$ and $\sigma_i^y = y_i \rightarrow \{y_i^-, y_i^+\}$. It follows from the stability of x_i and y_i that for $i=1, 2, 3$, $\sigma_i^* = I_i \rightarrow \{x_i^-, x_i^+, y_i^-, y_i^+\}$ is a spread.

- These spreads fit together into 3-spreads as follows: $\Sigma_{123} = \{\sigma_1 \sigma_2 \sigma_3\}$, $\Sigma_{123}^* = \{\sigma_1^* \sigma_2^* \sigma_3^*\}$. We will also consider the following 3-spreads:

$$\begin{aligned}
 \Sigma_{xxx} &= \{\sigma_1^x \sigma_2^x \sigma_3^x\} \\
 \Sigma_{xyx} &= \{\sigma_1^x \sigma_2^y \sigma_3^x\} \\
 \Sigma_{xxy} &= \{\sigma_1^x \sigma_2^x \sigma_3^y\} \\
 \Sigma_{xyy} &= \{\sigma_1^x \sigma_2^y \sigma_3^y\}
 \end{aligned}
 \tag{12}$$

That σ_i and the σ_i^x, σ_i^y are spreads already gives us the following spatiotemporal information: for $i = 1, 2, 3$, we have $I_i < x_i, y_i < x_i, x_i^+, y_i, y_i^+$. The following stipulation is additional.

Stipulation 2. The 3-spread Σ_{123}^* is spacelike. This means that the stations are sufficiently far apart in a spacelike sense, and the events at each station are sufficiently close in a timelike sense, so as to guarantee that no measurement outcome at one station has the initial of a distinct station in its causal past. A fortiori, Σ_{123} and all the 3-spreads in (12) are spacelike.

Stipulation 3. The following stipulations are based on quantum mechanics (see Section 1).

- An outcome vector of the 3-spread Σ_{123}^* (and hence an outcome vector of a 3-spread in (12)) is consistent if and only if (i) there is a *mixture* of x and y and an *even* number of minuses, or (ii) there is *no mixture* of x and y and an *odd* number of minuses.
- Here are four examples, all used below. $(x_1^+ x_2^- x_3^+)$, which is an outcome vector of Σ_{xxx} , is consistent (no mixture of x and y , odd $-$). Each of $(x_1^+ x_2^- x_3^+)$, $(x_1^+ y_2^+ y_3^-)$, and $(x_1^+ y_2^- x_3^+)$ (an outcome vector of respectively Σ_{xyx} , Σ_{xyy} , and Σ_{xxy}) is inconsistent (mixed x and y , odd number of minuses).
- Furthermore, Σ_{123} is maximally consistent: no setting of measurement type at one station can be prohibited by any selection of measurement types at the other stations. The same fact can be restated by saying that all of the 3-spreads in question are minimally consistent. The 3-spreads in (12) are none of them, however, maximally consistent. Instead these 3-spreads each has a property intermediate between minimal consistency and maximal consistency: Each is 1-consistent, which is to say of each that the joint realization of its measurement settings (initials) permits the later occurrence of any its spreads (though of course not jointly!).

An inconsistent outcome vector of a 1-consistent spacelike n -spread is what sends us in search of a common cause. And maybe there is one. We do not prove that there isn't. Instead, we prove something weaker.

Theorem 4.1. There is no single spread σ that is a common cause for every inconsistent outcome vector of each of the four 3-spreads Σ_{xxx} , Σ_{xyx} , Σ_{xxy} , and Σ_{xyy} .

This theorem rules out, or seems to rule out, that the "gedanken spin-conserving decay," if analyzed as an initial having a variety of possible outcomes, could serve as the single common-cause explanation of all the measurement correlations.

Proof. Suppose for *reductio* that σ is a common cause of each inconsistent outcome vector of each of the 3-spreads Σ_{xxx} , Σ_{xyx} , Σ_{xxy} , and Σ_{xyy} . Therefore, even without a full analysis of the concept of a common cause, it is a part of our supposal that CC2 and CC3 hold for σ and each of Σ_{xxx} , Σ_{xyx} , Σ_{xxy} , and Σ_{xyy} .

We begin by choosing an outcome O of σ . CC2 for Σ_{xxx} implies that O is consistent with the set of initials $\{x_1^+x_2^+x_3^+\}$ of Σ_{xxx} . This implies by Fact 1 that O is consistent with at least one consistent outcome vector of Σ_{xxx} . At this point we ought to pick an *arbitrary* such consistent outcome vector, but instead we ask you to settle for a *persuasive example*: $(x_1^+x_2^+x_3^+)$, known to be consistent by Stipulation 3. So since O is consistent with the outcome vector $(x_1^+x_2^+x_3^+)$, by Fact 1

O is consistent individually with each of x_1^+ , x_2^+ , and x_3^+

Now consider that $(x_1^+x_2^+y_3^-)$ must by Stipulation 3 be inconsistent. So by CC3 for Σ_{xyx} , O must be inconsistent with one of those outcomes individually. But O is consistent with each of the first two, so it must be that

O is inconsistent with y_3^-

CC2 for Σ_{xxy} implies that O is consistent with the initials $\{x_1^+x_2^+y_3^+\}$ of Σ_{xxy} . By Fact 1 O is therefore consistent with the initial y_3^+ of σ_3^+ . The outcomes of σ_3^+ are y_3^+ and y_3^- . By Fact 1 O must be consistent with one of these two outcomes. Since O is inconsistent with the former, it must be that

O is consistent with y_3^+

Next consider the outcome vector $\{x_1^+, y_2^+, y_3^+\}$ of Σ_{vtt} . By Stipulation 3 it is inconsistent, and so by CC3 for Σ_{vtt} , together with previously established consistencies, it follows that

O is inconsistent with y_2^+

Lastly, consider $\{x_1^+, y_2^-, x_3^+\}$, which by Stipulation 3 is inconsistent. By CC3 for Σ_{vtt} , together with previously established consistencies, we have that

O is inconsistent with y_2^-

But the inconsistency of O with both y_2^+ and y_2^- implies, by Stipulation 3 and Fact 1, the inconsistency of O with y_2 . This contradicts the implication of CC2 for Σ_{vtt} , by way of Fact 1, that O is consistent with y_2 , and concludes the *reductio*.

That of course was a proof by example. To generalize, recast the argument by replacing the definite labels with variables. ■

5. LIMITATIONS

There seem to be three ways Our World might be:

Level I. Deterministic, that is, there is only one history.

Level II. Indeterministic, but without any “strange” correlations between spatially separated happenings. In other words, each inconsistent outcome vector of each spacelike n -spread has a common-cause explanation.

Level III. Indeterministic, and with “strange” correlations between spatially separated happenings.

The strength of our result is that it establishes the relation between a Bell-like phenomenon and a no-common-cause-like phenomenon with absolute rigor, relying on the causal ordering in Our World as sole primitive. As we announced at the outset, however, there are limitations.

- A frequent interpretation of the GHSZ story as well as the other Bell-like theorems is that they show that certain phenomena predicted by quantum mechanics (but describable pre-theoretically, without quantum mechanics) are *incompatible with determinism*. One of us has elsewhere questioned the legitimacy of this interpretation as arising from insufficient care in applying the usual

formalism (Szabó, 1995a, b). Certainly our proof cannot bear such an interpretation, because it begins by *stipulating* the existence of certain spreads, which are explicitly indeterministic phenomena. We may in fact be permitted to doubt that other proofs fare better in this respect. Although these proofs are not sufficiently rigorous to be sure, they seem to share with our proof the *hypothesis* that indeterministic phenomena occur.

- Our proof of Theorem 4.1 makes no use of the causal-priority condition CCI of Definition 3.7. More work is needed in understanding the interplay of the idea of causal priority with other concepts. We need especially to advance our understanding of the proposition that the values of certain measurements cannot be arranged in advance.
- Bell-like theorems are often interpreted as showing that certain phenomena predicted by quantum mechanics are inconsistent with the principle of the common cause; or, in other words, that these phenomena involve spacelike correlations without a common cause. We are in this paper explicit that Theorem 4.1 does not say, simply, that there are correlations without a common cause. Instead, it says what it can prove: There are certain sets of correlations such that one cannot find a single common-cause locus for all of them. In other words, although Theorem 4.1 does not strictly imply that Our World belongs to Level III, it moves us sharply in that direction. So even though our result succeeds in ruling out that a single “gedanken decay” can account for all the correlations, its limited nature suggests the interest in finding a better theorem.

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