Partial Differential Equations

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Contents

In Analysis
there are no theorems
only proofs

A large part of these notes are based on [**Evans, 2010**] and lectures by Heiko von der Mosel (RWTH Aachen).

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.1. Introduction and some basic notation

When studying Partial Differential Equations (PDEs) the first question that arises is: what are partial differential equations.

Let $\Omega \subset \mathbb{R}^n$ be an open set and $u : \Omega \to \mathbb{R}$ be differentiable. The partial derivatives ∂_1 is the directional derivative

$$\partial_1 u(x) \equiv \partial_{x_1} u(x) = \frac{d}{dx_1} u(x) = \frac{d}{dt}\Big|_{t=0} u(x + te_1),$$

where $e_1 = (1, 0, \dots, 0)$ is the first unit vector. The partial derivatives $\partial_2, \dots \partial_n$ are defined likewise.

Sometimes it is convenient to use multiinidces: an n-multiindex γ is a vector $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ where $\gamma_1, \dots, \gamma_n \in \{0, 1, 2, \dots, \}$. The order of a multiindex is $|\gamma|$ defined as

$$|\gamma| = \sum_{i=1}^{n} \gamma_i.$$

For a suitable often differentiable function $u : \Omega \to \mathbb{R}$ and a multiindex γ we denote with $\partial^\gamma u$ the partial derivatives

$$\partial^\gamma u(x) = \partial_{x_1}^{\gamma_1} \partial_{x_2}^{\gamma_2} \dots \partial_{x_n}^{\gamma_n} u(x).$$

For example, for $\gamma = (1, 0, 0, \dots, 0)$ we have

$$\partial^\gamma u(x) = \partial_{x_1} u,$$

i.e. a partial derivative of first order; and for $\gamma = (1, 2, 0, \dots, 0)$ we have

$$\partial^\gamma u = \partial_{122} u \equiv \partial_1 \partial_2 \partial_2 u,$$

i.e. a partial derivative of 3rd order.

The collection of all partial derivatives of k-th order of u is usually denoted by $D^k u(x) \in \mathbb{R}^{n^k}$ or (the "gradient") $\nabla^k u$. Usually these are written in matrix form, namely

$$Du(x) = (\partial_1 u(x), \partial_2 u(x), \partial_3 u(x), \dots, \partial_n u(x))$$

and

$$D^2 u(x) = (\partial_{ij} u)_{i,j=1,\dots n} \equiv \begin{pmatrix} \partial_{11} u(x) & \partial_{12} u(x) & \partial_{13} u(x) & \dots & \partial_{1n} u(x) \\ \partial_{21} u(x) & \partial_{22} u(x) & \partial_{23} u(x) & \dots & \partial_{2n} u(x) \\ \vdots & & & \vdots & \\ \partial_{n1} u(x) & \partial_{n2} u(x) & \partial_{n3} u(x) & \dots & \partial_{nn} u(x) \end{pmatrix}$$

Definition .1.1. Let $\Omega \subset \mathbb{R}^n$ an open set and $k \in \mathbb{N} \cup \{0\}$. A partial differential equation (PDE) of k-th order is an expression of the form

$$F(D^k u(x), D^{k-1} u(x), D^{k-2} u(x), \dots, Du(x), u(x), x) = 0 \quad x \in \Omega, \tag{.1.1}$$

where $u : \Omega \to \mathbb{R}$ is the unknown (also the "solution" to the PDE) and F is a given structure (i.e. map)

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \ldots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$$

- (.1.1) is called *linear* if F is linear in u: meaning if we can find for every n-multiindex γ with $|\gamma| \leq k$ a function $a_\gamma : \Omega \to \mathbb{R}$ (independent of u) such that
$$F(D^k u(x), D^{k-1}u(x), D^{k-2}u(x), \ldots, Du(x), u(x), x) = \sum_{|\gamma| \leq k} a_\gamma(x)\partial^\gamma u(x)$$
- (.1.1) is called *semilinear* if F is linear with respect to the highest order k, namely if

$$F(D^k u(x), D^{k-1}u(x), D^{k-2}u(x), \ldots, Du(x), u(x), x) = \sum_{|\gamma|=k} a_\gamma(x)\partial^\gamma u(x) + G(D^{k-1}u(x), D^{k-2}u(x), \ldots, Du(x$$

- (.1.1) is called quasilinear if F is linear with respect to the highest order k but the coefficient for the highest order may depend on the lower order derivatives of u. Namely if we have a representation of the form

$$F(D^k u(x), D^{k-1}u(x), D^{k-2}u(x), \ldots, Du(x), u(x), x) = \sum_{|\gamma|=k} a_\gamma(D^{k-1}u(x), D^{k-2}u(x), \ldots, Du(x), u(x), x)\partial^\gamma u$$

- If all the above do not apply then we call F fully nonlinear.

We have a system of partial differential equations of order k, if $u : \Omega \to \mathbb{R}^m$ is a vector and/or the structure function F is also a vector

$$F : \mathbb{R}^{m\,n^k} \times \mathbb{R}^{m\,n^{k-1}} \times \ldots \times \mathbb{R}^{m\,n} \times \mathbb{R}^m \times \Omega \to \mathbb{R}^\ell$$

for $m, \ell \geq 1$.

The goal in PDE is usually (besides modeling what PDE describes what situation) to solve PDEs, possibly subject to sidecondition (such as prescribed boundary data on $\partial\Omega$). This is rarely possible explicitely (even in the linear case) mostly the best one can hope for is address the following main questions for PDEs are

- Is there a solution to a problem (and if so: in what sense? – we will learn the distributional/weak sense and strong sense)
- Are solutions unique?
- What are properties of the solutions (e.g. does the solution depend continuously on the data of the problem)?

It is important to accept that there are PDEs without (classical) solutions and there is no general theory of PDEs. There is theory for several types of PDES.

Example .1.2 (Some basic linear equations)**.**

- Laplace equation
$$\Delta u := \sum_{i=1}^n u_{x_i x_i} = 0.$$

- Eigenvalue equation (aka Helmholtz equation)
$$\Delta u = \lambda u.$$
- Transport equation
$$\partial_t u - \sum_{i=1}^{n} b^i u_{x_i} = 0$$
- Heat equation
$$\partial_t u - \Delta u = 0$$
- Schrödinger equation
$$i\partial_t u + \Delta u = 0$$
- Wave equation
$$u_{tt} - \Delta u = 0$$

Example .1.3 (Some basic nonlinear equations)**.**

- Eikonal equation
$$|Du| = 1$$
- p-Laplace equation
$$\operatorname{div}\left(|Du|^{p-2}Du\right) \equiv \sum_{i=1}^{n} \partial_i(|Du|^{p-2}\partial_i u) = 0$$
- Minimal surface equation
$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0.$$
- Monge-Ampere
$$\det(D^2 u) = 0.$$
- Hamilton-Jacobi
$$\partial_t u + H(Du, x) = 0$$

In this course we will focus on the linear theory (the nonlinear theory is always based on ideas on the linear theory). Almost each of the linear and nonlinear equations warrants its own course, so we will focus on the basics (namely: mainly elliptic equations).

CHAPTER 1

Model equations and special solutions

I.1. Transport equation

We consider solutions $u : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ of

$$\partial_t u + b \cdot Du = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty). \tag{I.1.1}$$

The variables in $\mathbb{R}^n$ we denote by x (space) and the variable in $(0,\infty)$ by t (time).

Here $b = (b^1, \dots, b^n)$ is a constant vector, and Du is the gradient of u, so that

$$b \cdot Du(x) = \sum_{i=1}^n b^i \partial_i u$$

If we were to assume that u is sufficiently differentiable, then

$$\frac{d}{ds} u(x+sb, t+s) = b \cdot Du(x+sb, t+s) + \partial_t u(x+sb, t+s) \overset{(I.1.1)}{\equiv} 0$$

That is, u is constant in the direction $(b,1)$ in $\mathbb{R}^{n+1}$. But this means we can solve explicitely an equation of the form

$$\begin{cases} \partial_t u + b \cdot Du = 0 & \text{on } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{0\} \end{cases}$$

Namely, since (assuming enough differentiability!) u is constant on lines with slope $(b,1)$ we have

$$u(x,t) = u(x+\lambda b, t+\lambda) \quad \forall \lambda \geq -t.$$

Taking $\lambda = -t$ we then find

$$u(x,t) = u(x-tb, 0) = g(x-tb) \tag{I.1.2}$$

as the solution.

That is, if $u \in C^1(\mathbb{R}^n \times (0,\infty)) \cap C^0(\mathbb{R}^n \times [0,\infty))$ solves (I.1.1) then u is of the form (I.1.2). Also if $g \in C^1(\mathbb{R}^n)$ the u of the form (I.1.2) solves (I.1.1).

On the other hand if $g \notin C^1$ then there cannot be C^2-solutions to the transport equation! In that case one reverts to weak solutions.

(I.1.1) is called the homogeneous transport equation, since the right-hand side is zero. If we consider the inhomogeneous problem

$$\begin{cases} \partial_t u + b \cdot Du = f & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{in } \mathbb{R}^n \times \{0\} \end{cases} \tag{I.1.3}$$

for a given f we can try to do the same spiel as above:

This time we have for a sufficiently smooth solution

$$\frac{d}{ds} u(x + sb, t + s) = b \cdot Du(x + sb, t + s) + \partial_t u(x + sb, t + s) \overset{\text{(I.1.3)}}{\equiv} f(x + sb, t + s).$$

That is, we do not know that $u(x, t) - g(x - tb)$ is constantly zero, but we have by the fundamental theorem of calculus,

$$\begin{aligned} & u(x, t) - g(x - tb) \\ =& u(x, t) - u(x - tb, 0) \\ =& \int_{-t}^{0} \frac{d}{ds} u(x + sb, t + s)\, ds \\ =& \int_{-t}^{0} f(x + sb, t + s)\, ds \\ \overset{\sigma := t+s}{=}& \int_{0}^{t} f(x + (\sigma - t)b, \sigma)\, d\sigma. \end{aligned}$$

That is, as in the homogeneous case we can conclude that (under enough differentiability assumptions),

$$u(x, t) = g(x - tb) + \int_0^t f(x + (\sigma - t)b, \sigma)\, d\sigma.$$

is the unique solution to the inhomogeneous (linear) transport problem (I.1.3).

I.2. Laplace equation

Let $\Omega \subset \mathbb{R}^n$ be an open set (this will always be the case from now on). We consider the homogeneous Laplace equation

$$\Delta u = 0 \quad \text{in } \Omega \tag{I.2.1}$$

where we recall that $\Delta u = \operatorname{tr}(D^2 u) = \sum_{i=1}^n \partial_{ii} u$.

The inhomogenous equation (sometimes: Poisson equation) is, for a given function $f : \Omega \to \mathbb{R}$,

$$\Delta u = f \quad \text{in } \Omega \tag{I.2.2}$$

Definition I.2.1. A function $u \in C^2(\Omega)$ is called harmonic if u pointwise solves

$$\Delta u(x) = 0 \quad \text{in } \Omega$$

We also say, u is a solution to the homogeneous Laplace equation.

We say that u is a subsolution[1] or subharmonic if

$$\Delta u(x) \geq 0 \quad \text{in } \Omega.$$

If

$$\Delta u(x) \leq 0 \quad \text{in } \Omega$$

we say that u is a supersolution or superharmonic.

I.2.1. Fundamental Solution, Newton- and Riesz Potential. There are many trivial solutions (polynomials of order 1) of Laplace equation. But these are not very interesting. There is a special type of solution which is called *fundamental solution* (which, funny enough, is actually not a solution).

It appears when we want to compute the solution to an equation on the whole space

$$\Delta u(x) = f(x). \tag{I.2.3}$$

For this we make a brief (formal) introduction to Fourier transform:

The Fourier transform takes a map $f : \mathbb{R}^n \to \mathbb{R}$ and transforms it into $\mathcal{F}u \equiv \hat{f} : \mathbb{R}^n \to \mathbb{R}$ as follows

$$\hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\langle \xi, x\rangle} \, f(x) \, dx.$$

The *inverse Fouriertransform* $f^\vee$ is defined as

$$f^\vee(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{+i\langle \xi, x\rangle} \, f(x) \, dx.$$

It has the nice property that $(f^\wedge)^\vee = f$.

One of the important properties (which we will check in exercises) is that derivatives become polynomial factors after Fourier transform:

$$(\partial_{x_i} g)^\wedge (\xi) = -i\xi_i \hat{g}(\xi).$$

For the Laplace operator Δ this implies

$$(\Delta u)^\wedge(\xi) = -|\xi|^2 \hat{u}(\xi).$$

This means that if we look at the equation (I.2.3) and apply Fourier transform on both sides we have

$$-|\xi|^2 \hat{u}(\xi) = \hat{f}(\xi),$$

that is

$$\hat{u}(\xi) = -|\xi|^{-2} \hat{f}(\xi),$$

Inverting the Fourier transform we get an explicit formula for u in terms of the data f.

$$u(x) = -\left(|\xi|^{-2} \hat{f}(\xi)\right)^\vee (x).$$

[1] yes that notion is confusing

This is not a very nice formula, so let us simplify it. Another nice property of Fourier transform (and its inverse) is that products become convolutions. Namely

$$(g(\xi)f(\xi))^\vee(x) = \int_{\mathbb{R}^n} g^\vee(x-z)f^\vee(z)\,dz.$$

In our case, for $g(\xi) = -|\xi|^{-2}$ we get that

$$u(x) = \int_{\mathbb{R}^n} g^\vee(x-z)\,f(z)\,dz.$$

Now we need to compute $g^\vee(x-z)$, and for this we restrict our attention to the situation where the dimension is $n \geq 3$. In that case, just by the definition of the (inverse) Fourier transform we can compute that since g has homogeneity of order 2 (i.e. $g(t\xi) = t^{-2}g(\xi)$, then $g^\vee$ is homogeneous of order $2-n$. In particular

$$g^\vee(x) = |x|^{2-n}g^\vee(x/|x|).$$

Now an argument that radial functions stay radial under Fourier transforms leads us to conclude that

$$g^\vee(x) = c_1|x|^{2-n}.$$

That is, we have arrived that (by formal computations) a solution of (I.2.3) should satisfy

$$u(x) = c_1\int_{\mathbb{R}^n} |x-z|^{2-n}\,f(z)\,dz. \tag{I.2.4}$$

The constant c_1 can be computed explicitely, and we will check below that this *potential representation* of u really is true. This potential is called the Newton potential (which is a special case of so-called Riesz potentials). The kernel of the Newton potential is called the fundamental solution of the Laplace equation (which, again, is not a solution)

Definition I.2.2. The fundamental solution $\Phi(x)$ of the Laplace equation for $x \neq 0$ is given as

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x| & \text{for } n=2 \\ -\frac{1}{n(n-2)\omega_n}|x|^{2-n} & \text{for } n \geq 2 \end{cases}$$

Here ω_n is the Lebesgue measure of the unit ball $\omega_n = |B(0,1)|$.

One can explicitely check that $\Delta\Phi(x) = 0$ for $x \neq 0$ (indeed, $\Delta\Phi(x) = \delta_0$ where δ_0 is the Dirac measure at the point zero, cf. remark I.2.4).

The following statement justifies (somewhat) the notion of fundamental solution: the fundamental solution $\Phi(x)$ can be used to construct all solutions to the imhomogeneous Laplace equation:

THEOREM I.2.3. *Let u be the Newton-potential of $f \in C_c^2(\mathbb{R}^n)$, that is*

$$u(x) := \int_{\mathbb{R}^n} \Phi(x-y)\,f(y)\,dy.$$

Here $C_c^2(\mathbb{R}^n)$ are all those functions in $C^2(\mathbb{R}^n)$ such that f is constantly zero outside of some compact set.

We have

- $u \in C^2(\mathbb{R}^n)$
- $-\Delta u = f \quad in\ \mathbb{R}^n.$

Proof. First we show differentiability of u. By a substitution we may write

$$u(x) := \int_{\mathbb{R}^n} \Phi(x-y)\, f(y)\, dy = \int_{\mathbb{R}^n} \Phi(z)\, f(x-z)\, dz.$$

Now if we denote the difference quotient

$$\Delta_h^{e_i} u(x) := \frac{u(x+he_i) - u(x)}{h}$$

where e_i is the i-th unit vector, then we obtain readily

$$\Delta_h^{e_i} u(x) := \int_{\mathbb{R}^n} \Phi(x-y)\, f(y)\, dy = \int_{\mathbb{R}^n} \Phi(z)\, (\Delta_h^{e_i} f)(x-z)\, dz.$$

One checks that Φ is locally integrable (it is not globally integrable!), that is for every bounded set $\Omega \subset \mathbb{R}^n$,

$$\int_\Omega |\Phi| < \infty. \tag{I.2.5}$$

Indeed, (we show this for $n \geq 3$, the case $n = 2$ is an exercise), if $\Omega \subset \mathbb{R}^n$ is a bounded set, then it is contained in some large ball $B(0, R)$.

$$\int_\Omega |\Phi| \leq C \int_{B(0,R)} |x|^{2-n}\, dx \tag{I.2.6}$$

Using Fubini's theorem,

$$\begin{aligned}
&\int_{B(0,R)} |x|^{2-n}\, dx \\
=&\int_0^R \int_{\partial B(0,r)} |\theta|^{2-n}\, d\mathcal{H}^{n-1}(\theta)\, dr \\
=&\int_0^R r^{2-n} \int_{\partial B(0,r)} d\mathcal{H}^{n-1}(\theta)\, dr \\
=&c_n \int_0^R r^{2-n} r^{n-1} dr \\
=&c_n \int_0^R r^1 dr \\
=&c_n \frac{1}{2} R^2 < \infty.
\end{aligned}$$

This establishes (I.2.5)

On the other hand $(\Delta_h^{e_i} f)$ has still compact support for every h. In particular, by dominated convergence we can conclude that

$$\lim_{h\to 0} \Delta_h^{e_i} u(x) := \int_{\mathbb{R}^n} \Phi(x-y)\, f(y)\, dy = \int_{\mathbb{R}^n} \Phi(z)\, \lim_{h\to 0} (\Delta_h^{e_i} f)(x-z)\, dz.$$

that is

$$\partial_i u(x) := \int_{\mathbb{R}^n} \Phi(x-y)\, f(y)\, dy = \int_{\mathbb{R}^n} \Phi(z)\, (\partial_i f)(x-z)\, dz.$$

In the same way

$$\partial_{ij} u(x) := \int_{\mathbb{R}^n} \Phi(x-y)\, f(y)\, dy = \int_{\mathbb{R}^n} \Phi(z)\, (\partial_{ij} f)(x-z)\, dz.$$

Now the right-hand side of this equation is continuous (again using the compact support of f). This means that $u \in C^2(\mathbb{R}^n)$.

To obtain that $\Delta u = f$ we first use the above argument to get

$$\Delta u(x) = \int_{\mathbb{R}^n} \Phi(x-y)\, f(y)\, dy = \int_{\mathbb{R}^n} \Phi(z)\, (\Delta f)(x-z)\, dz.$$

Observe that

$$(\Delta f)(x-z) = \Delta_x (f(x-z)) = \Delta_z (f(x-z)).$$

Now we fix a small $\varepsilon > 0$ (that we later send to zero) and split the integral, we have

$$\Delta u(x) = \int_{\mathbb{R}^n} \Phi(x-y)\, f(y)\, dy = \int_{B(0,\varepsilon)} \Phi(z)\, (\Delta f)(x-z)\, dz + \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(z)\, (\Delta f)(x-z)\, dz =: I_\varepsilon + II_\varepsilon.$$

The term I_ε contains the singularity of Φ, but we observe that

$$I_\varepsilon \xrightarrow{\varepsilon\to 0} 0.$$

Indeed, this follows from the absolute continuity of the integral and since Φ is integrable on $B(0,1)$:

$$|I_\varepsilon| \le \sup_{\mathbb{R}^n} |\Delta f| \int_{B(x,\varepsilon)} |\Phi(z)| \xrightarrow{\varepsilon\to 0} 0.$$

The term II_ε does not contain any singularity of Φ which is smooth on $\mathbb{R}^n \setminus B_\varepsilon(0)$, so we can perform an integration by parts[2]

$$II_\varepsilon = \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(z)\, (\Delta f)(x-z)\, dz = \int_{\partial B(0,\varepsilon)} \Phi(z)\, \partial_\nu f(x-z)\, d\mathcal{H}^{n-1}(z) - \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \nabla\Phi(z) \cdot \nabla f(x-z)\, dz.$$

Here ν is the unit normal to the ball $\partial B(0,\varepsilon)$, i.e. $\nu = \frac{-z}{\varepsilon}$.

By the definition of Φ one computes that (using (I.2.5))

$$\left| \int_{\partial B(0,\varepsilon)} \Phi(z)\, \partial_\nu f(x-z)\, d\mathcal{H}^{n-1}(z) \right| \le \sup_{\mathbb{R}^n} |\nabla f| \int_{\partial B(0,\varepsilon)} |\Phi(z)| \xrightarrow{\varepsilon\to 0} 0.$$

2

$$\int_\Omega f\, \partial_i g = \int_{\partial\Omega} f\, g\, \nu^i - \int_\Omega \partial_i f\, g,$$

where ν is the normal of $\partial\Omega$ pointing outwards (from the point of view of Ω). ν^i is the i-th component of ν. Fun exercise: Check this rule in 1D, to see the relation what we all learned in Calc 1.

So we perform another integration by parts and have

$$\begin{aligned} II_\varepsilon =& o(1) - \int_{\partial B(0,\varepsilon)} \partial_\nu \Phi(z)\, f(x-z)\, dz + \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \underbrace{\Delta\Phi(z)}_{=0}\, f(x-z)\, dz \\ =& o(1) - \int_{\partial B(0,\varepsilon)} \partial_\nu \Phi(z)\, f(x-z)\, dz \end{aligned}$$

Here in the last step we used that $\Delta\Phi = 0$ away from the origin.

Now we observe that the unit normal on $\partial B(0,\varepsilon)$ is $\nu(z) = -\frac{z}{\varepsilon}$ and

$$D\Phi(z) = \begin{cases} -\frac{1}{2\pi}\frac{1}{|z|}\frac{z}{|z|} & n = 2, \\ -\frac{1}{n(n-2)\omega_n}(2-n)|z|^{1-n}\frac{z}{|z|} & n \geq 3. \end{cases}$$

Thus, for $|z| = \varepsilon$,

$$\partial_\nu \Phi(z) = \nu \cdot D\Phi(z) = \frac{1}{n\omega_n}\varepsilon^{1-n}$$

Thus we arrive at

$$\begin{aligned} II_\varepsilon =& o(1) - \int_{\partial B(0,\varepsilon)} \frac{1}{n\omega_n \varepsilon^{n-1}}\, f(x-z)\, d\mathcal{H}^{n-1}(z) \\ =& o(1) - \fint_{\partial B(0,\varepsilon)} f(x-z)\, d\mathcal{H}^{n-1}(z) \\ =& o(1) - f(x) + \fint_{\partial B(0,\varepsilon)} \left(f(x) - f(x-z)\right)\, d\mathcal{H}^{n-1}(z) \end{aligned}$$

Here we use the mean value notation

$$\fint_{\partial B(0,\varepsilon)} = \frac{1}{\mathcal{H}^{n-1}(\partial B(0,\varepsilon)} \int_{\partial B(0,\varepsilon)}.$$

Now one shows (exercise!) that for continuous f

$$\lim_{\varepsilon \to 0} \fint_{\partial B(0,\varepsilon)} \left(f(x) - f(x-z)\right)\, d\mathcal{H}^{n-1}(z) = 0.$$

(Indeed this is essentially Lebesgue's theorem). That is

$$II_\varepsilon = o(1) - f(x) \quad \text{as } \varepsilon \to 0$$

and thus

$$\Delta u(x) = -f(x) + o(\varepsilon),$$

and letting $\varepsilon \to 0$ we have

$$\Delta u(x) = -f(x),$$

as claimed. □

Remark I.2.4. One can argue (in a distributional sense, which we learn towards the end of the semester)

$$-\Delta\Phi = \delta_0,$$

where δ_0 denotes the Dirac measure at 0, namely the measure such that

$$\int_{\mathbb{R}^n} f(x)\, d\delta_0 = f(0) \quad \text{for all } f \in C^0(\mathbb{R}^n).$$

Observe that δ_0 is not a function, only a measure. In this sense one can justify that

$$\begin{aligned} -\Delta u(x) =& \Delta \int_{\mathbb{R}^n} \Phi(x-z) f(z) \\ =& \int_{\mathbb{R}^n} \Delta\Phi(x-z) f(z)\, dz \\ =& \int_{\mathbb{R}^n} f(z)\, d\delta_x(z) \\ =& f(x) \end{aligned}$$

I.2.2. Mean Value Property for harmonic functions. An important property (but very special to the "base Operator Δ", i.e. not that easily generalizable to more general PDEs) is the mean value property

THEOREM I.2.5 (Harmonic functions satisfy Mean Value Property). *Let $u \in C^2(\Omega)$ such that $\Delta u = 0$, then*

$$u(x) = \fint_{\partial B(x,r)} u(z)\, d\mathcal{H}^{n-1}(z) = \fint_{B(x,r)} u(y)\, dy \tag{I.2.7}$$

holds for all balls $B(x,r) \subset \Omega$.

If $\Delta u \leq 0$ then we have "$\geq$"in (I.2.7). If $\Delta u \geq 0$ then we have "$\leq$" in (I.2.7).

PROOF. Set

$$\varphi(r) := \fint_{\partial B(x,r)} u(y)\, d\mathcal{H}^{n-1}(y).$$

Observe that by substitution $z := \frac{y-x}{r}$ we have

$$\varphi(r) := \fint_{\partial B(0,1)} u(x+rz)\, d\mathcal{H}^{n-1}(z).$$

Taking the derivative in r we have

$$\varphi'(r) = \fint_{\partial B(0,1)} Du(x+rz) \cdot z\, d\mathcal{H}^{n-1}(z).$$

Transforming back we get

$$\varphi'(r) = \fint_{\partial B(x,r)} Du(y) \cdot \frac{y-x}{r}\, d\mathcal{H}^{n-1}(y).$$

Observe that $\frac{y-x}{r}$ is the outer unit normal of $\partial B(x,r)$. That is

$$\varphi'(r) = |\partial B(x,r)|^{-1} \int_{\partial B(x,r)} \partial_\nu u(y) d\mathcal{H}^{n-1}(y).$$

By Stokes or Green's theorem (aka, integration by parts)

$$\varphi'(r) = |\partial B(x,r)|^{-1} \int_{B(x,r)} \Delta u(y) dy \overset{(I.2.7)}{=} 0.$$

That is,

$$\varphi'(r) = 0 \quad \forall r \text{ if } B(x,r) \subset \Omega.$$

which implies that φ is constant, and in particular

$$\varphi(r) = \lim_{\rho \to 0} \varphi(\rho).$$

But (exercise!) for continuous functions u,

$$\lim_{\rho \to 0} \varphi(\rho) = \lim_{\rho \to 0} \fint_{\partial B(x,\rho)} u(y)\, d\mathcal{H}^{n-1}(y) = u(x),$$

we have shown that

$$u(x) = \fint_{\partial B(x,r)} u(y)\, d\mathcal{H}^{n-1}(y) \tag{I.2.8}$$

holds whenever $B(x,r) \subset \Omega$.

Moreover, by Fubini's theorem

$$\begin{aligned}
\fint_{B(x,r)} u(y)\, dy &= \frac{1}{|B(x,r)|} \int_0^r \int_{\partial B(x,\rho)} u(\theta)\, d\mathcal{H}^{n-1}(\theta)\, d\rho \\
&= \frac{1}{|B(x,r)|} \int_0^r |B(x,\rho)| \fint_{\partial B(x,\rho)} u(\theta)\, d\mathcal{H}^{n-1}(\theta)\, d\rho \\
&\overset{(I.2.8)}{=} \frac{1}{|B(x,r)|} \int_0^r |\partial B(x,\rho)|\, u(x) d\rho \\
&= u(x) \frac{1}{|B(x,r)|} \int_0^r \int_{\partial B(x,\rho)} 1\, d\mathcal{H}^{n-1}(\theta) d\rho \\
&= u(x) \frac{|B(x,r)|}{|B(x,r)|} \\
&= u(x).
\end{aligned}$$

Together with (I.2.8) we have shown the claim for $\Delta u = 0$. The inequality arguments are left as an exercise. □

The converse holds as well (and there is actually a whole literature on "how many balls" one has to assume the mean value property to get harmonicity, cf. [**Llorente, 2015, Kuznetsov, 2019**])

THEOREM I.2.6 (Mean Value property implies harmonicity). *Let $u \in C^2(\Omega)$. If for all balls $B(x,r) \subset \Omega$,*

$$u(x) = \fint_{\partial B(x,r)} u(\theta)\, d\mathcal{H}^{n-1}(\theta) \tag{I.2.9}$$

then

$$\Delta u = 0 \quad \text{in } \Omega$$

PROOF. Assume the claim is false.

Then there exists some $x_0 \in \Omega$ such that $\Delta u(x_0) \neq 0$, so (by continuity of Δu) w.l.o.g. $\Delta u > 0$ in a small neighbourhood $B(x_0, R)$ of x_0.

On the other hand, setting as above

$$\varphi(r) := \fint_{\partial B(x_0,r)} u(\theta) \overset{(\text{I.2.9})}{\equiv} u(x_0)$$

we have $\varphi'(r) = 0$ for all $r > 0$ such that $B(x_0, r) \subset \Omega$. But as computed before, for $r < R$,

$$\varphi'(r) = C(r) \int_{B(x_0,r)} \Delta u \, dy > 0.$$

This $(0 = \varphi'(r) > 0)$ is a contradiction, so the claim is established. □

I.2.3. Maximum and Comparison Principles. The mean value property as above is very rigid in the sense that it holds only for very special operators such as the Laplacian. A much more general property (which for the Laplacian Δ is a direct consequence of the mean value property) are *maximum principles*, which should be seen as a "forced convexity/concavity property" for sub-/supersolutions of a large class of PDEs (2nd order elliptic, see Chapter 2 later.

In one-dimension a subsolution of Laplace's equation satisfies

$$u'' \geq 0$$

that is, subsolutions are exactly the convex C^2-functions. Convexity means that on any interval (a, b) the maximum of u is obtained at a or at b – and if the maximum is obtained in a point $c \in (a, b)$ then u is constant. The curious fact is that these properties still hold in arbitrary dimension for solutions of the Laplace equation (and later a large class of elliptic 2nd order equations), they are the so-called weak maximum principle and strong maximum principle.

Corollary I.2.7 (Strong Maximum-principle)**.** *Let $u \in C^2(\Omega)$ be subharmonic, i.e. $\Delta u \geq 0$ in Ω. If there exists $x_0 \in \Omega$ at which u attains a global maximum then u is constant in the connected component of Ω containing x_0.*

PROOF. By taking a possibly smaller Ω we can assume w.l.o.g. Ω is connected and u still attains its global maximum in $x_0 \in \Omega$.

Let

$$A := \{y \in \Omega : u(y) = u(x_0)\}.$$

We will show that $A = \Omega$ (and thus u is constant) by showing that the following three properties hold

- A is nonempty
- A is relatively closed (in Ω).
- A is open

Then A is an open and closed set in Ω, and since A is not the empty set it is all of Ω.

Clearly A is nonempty since $x_0 \in A$.

Also A is relatively closed by continuity of u: If $\Omega \ni y_k \xrightarrow{k\to\infty} y_0 \in \Omega$ then

$$u(y_0) = \lim_{k\to\infty} u(y_k) = u(x_0)$$

and thus $y_0 \in A$.

To show that A is open let $y_0 \in A$. Since Ω is open we can find a small ball $B(y_0, \rho) \subset \Omega$.

Observe that x_0 is a global maximum of u in $B(y_0, \rho)$.

The mean value property, Theorem I.2.5, and then the fact that $u(x_0) \geq u(y)$ for all y in $B(y_0, \rho)$, imply

$$u(x_0) = u(y_0) \leq \fint_{B(y_0,\rho)} u(y)\, dy \leq \fint_{B(y_0,\rho)} u(x_0)\, dy = u(x_0).$$

Since left-hand side and right-hand side coincide the inequality is actually an equality.

That is, we have

$$u(x_0) = \fint_{B(y_0,\rho)} u(y)\, dy,$$

in other words

$$\fint_{B(y_0,\rho)} u(y) - u(x_0)\, dy = 0.$$

Since $u(y) - u(x_0)$ by assumption ≤ 0 the above integral becomes

$$-\fint_{B(y_0,\rho)} |u(y) - u(x_0)|\, dy = 0.$$

that is

$$u(y) \equiv u(x_0) \quad \text{in } B(y_0, \rho),$$

that is $B(y_0, \rho) \subset A$. That is, A is open. □

Remark I.2.8. The statement of Corollary I.2.7 is false if one replaces global with local maximum (even though local maxima are locally global maxima). A counterexample is for example

$$u(x) := \begin{cases} 0 & x \leq 0 \\ x^3 & x > 0 \end{cases}$$

Then $u \in C^2(\mathbb{R})$ and

$$\Delta u = u'' \geq 0 \quad \text{in } (-1, 1)$$

Clearly u attains several local maxima, namely in $(-1,0)$ we have $u \equiv 0$, but also clearly u is not constant. The argument above in the proof of Corollary I.2.7 fails, since the point 0 is not a local maximum, and thus the set

$$A := \{x \in (-1,1) : u(x) = 0\}$$

is not open.

For the next statement we use the notation $A \subset\subset B$ ("A is compactly contained in B) which means that A is bonded and its closure $\overline{A} \subset B$. I.e. for two open sets A, B the condition $A \subset\subset B$ means in particular that ∂A has positive distance from ∂B.

Corollary I.2.9 (Weak maximum principle)**.** *Let $\Omega \subset\subset \mathbb{R}^n$ and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be subharmonic, i.e. $\Delta u \geq 0$ in Ω. Then*

$$\sup_{\Omega} u = \sup_{\partial\Omega} u,$$

i.e. "the maximal value is attained at the boundary"[3]*.*

Remark I.2.10. This statement also holds on unbounded sets Ω, one just has to define the meaning of $\sup_{\partial\Omega}$ in a suitable sense (i.e. "$\sup_{\partial\mathbb{R}^n}$" should be interpreted as $\limsup_{|x|\to\infty}$).

Proof of Corollary I.2.9. Clearly by continuity

$$\sup_{\Omega} u \geq \sup_{\partial\Omega} u.$$

To prove the converse let us argue by contradiction and assume that

$$\sup_{\Omega} u > \sup_{\partial\Omega} u. \tag{I.2.10}$$

Since u is continuous and Ω bounded this must mean that there exists a local maximum point $x_0 \in \Omega$ such that

$$u(x_0) = \sup_{\Omega} u > \sup_{\partial\Omega} u. \tag{I.2.11}$$

But in view of Corollary I.2.7 (strong maximum principle) u is then constant on the connected component of Ω containing x_0. But this implies that on the boundary of this connected component the value of u is still $u(x_0)$, which implies

$$\sup_{\partial\Omega} u \geq u(x_0).$$

But this contradicts the assumption (I.2.11). □

Remark I.2.11. A particular consequence of the strong maximum principle is the following. If for $\Omega \subset\subset \mathbb{R}^n$ we have $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying

$$\begin{cases} \Delta u \geq 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

[3]again: think of convex functions which do have this property

for some $g \in C^0(\partial\Omega)$. Then the following (equivalent) statements are true:

- If $g \leq 0$ but $g \not\equiv 0$ on $\partial\Omega$ then we have that $u < 0$ in all of Ω.
- If $g \leq 0$ then either $u \equiv 0$ or $u < 0$ everywhere in Ω.

Such a behaviour is special to the PDEs of order two. Even for

$$\Delta^2 u = \Delta(\Delta u) = 0 \quad \text{in } \Omega$$

the above statement may not hold (see e.g. [**Gazzola et al., 2010**]).

Corollary I.2.12 (Strong Comparison Principle)**.** *Let $\Omega \subset\subset \mathbb{R}^n$ open and connected. Assume that $u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy*

$$\Delta u_1 \geq \Delta u_2 \quad \text{in } \Omega.$$

If $u_1 \leq u_2$ on $\partial\Omega$, then exactly one of the following statements is true

(1) either $u_1 \equiv u_2$
(2) or $u_1(x) < u_2(x)$ for all $x \in \Omega$.

Proof. Let $w := u_1 - u_2$, then we have

$$\begin{cases} \Delta w \geq 0 & \text{in } \Omega \\ w \leq 0 & \text{in } \partial\Omega \end{cases}$$

The claim now follows from Remark I.2.11. □

The maximum principle is a great tool to get uniqueness for linear equations!

Theorem I.2.13 (Uniqueness for the Dirichlet problem). *Let $\Omega \subset\subset \mathbb{R}^n$, $f \in C^0(\Omega)$ and $g \in C^0(\partial\Omega)$ be given. Then there is at most(!) one solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of*

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Proof. Assume there are two solutions, u, v solving this equation. If we set $w := u - v$ then w is a solution to the equation

$$\begin{cases} \Delta w = 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

In view of Corollary I.2.9 we then have

$$\sup_{\Omega} w \leq \sup_{\partial\Omega} w = 0.$$

That is, $w \leq 0$ in Ω. But observe that $-w$ solves the same equation, which implies that

$$\sup_{\Omega}(-w) \leq \sup_{\partial\Omega}(-w) = 0,$$

that is $-w \leq 0$ in Ω. But this readily implies that $w \equiv 0$ in Ω, that is $v \equiv w$. □

So comparison principles are a fantastic tool for obtaining uniqueness for PDEs. Let us also note that via the so-called *Perron's method*, see e.g. [**Koike, 2004**], one can also obtain existence from such comparison principles, but we shall not investigate this further here.

I.2.4. Weak Solutions, Regularity Theory. Now we look at our first encounter with *distributional solutions*. Let $u \in L^1_{loc}(\Omega)$, that is u is a measurable function on Ω which is integrable on every compactly contained set $K \subset \Omega$, i.e.

$$\int_K |u| < \infty.$$

u certainly has no reason to be differentiable, it might not even be continuous. How on earth are we going to define

$$\Delta u = 0 \quad \text{in } \Omega?$$

The idea is that if $u \in C^2(\Omega)$ then

$$\Delta u = 0 \quad \text{in } \Omega \tag{I.2.12}$$

is equivalent to saying that

$$\int_\Omega u\, \Delta\varphi = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{I.2.13}$$

(Recall that $C_c^\infty(\Omega)$ are those smooth functions that have compact support $\operatorname{supp}\varphi \subset\subset \Omega$).

Indeed, for $\varphi \in C_c^\infty(\Omega)$ and $u \in C^2(\Omega)$ we have by integration by parts

$$\int_\Omega u\, \Delta\varphi = \int_\Omega \Delta u\, \varphi.$$

So for $u \in C^2(\Omega)$ we clearly have that (I.2.13) is equivalent to

$$\int_\Omega \Delta u\, \varphi = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{I.2.14}$$

Now if (I.2.12) holds then clearly (I.2.14) holds.

On the other hand assume that (I.2.14) holds, but (I.2.12) is false. That is assume there is $x_0 \in \Omega$ such that (w.l.o.g.)

$$\Delta u(x_0) > 0.$$

Since $u \in C^2(\Omega)$ we have $\Delta u \in C^0(\Omega)$ and thus there exists a ball $B(x_0, r) \subset\subset \Omega$ such that

$$\Delta u > 0 \quad \text{on } B(x_0, r) \tag{I.2.15}$$

Now let $\varphi \in C_c^\infty(\Omega)$ a *bump function* (or *cutoff function*), namely a function φ such that $\varphi \geq 1$ in $B(x_0, r/2)$ and $\varphi \equiv 0$ in $\Omega \backslash B(x_0, r)$, and $\varphi \geq 0$ everywhere. These bump functions really exist: they can be build by essentially scaled and glued versions of

$$\eta(x) := \begin{cases} e^{-\frac{1}{1-|x|^2}} & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

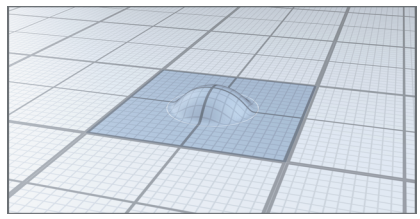

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FIGURE I.2.1. A bump function

See Figure I.2.4.

For this bump function φ we have from (I.2.15)

$$\int_\Omega \varphi \Delta u > 0$$

which contradicts (I.2.14). This proves the equivalence of (I.2.13) and (I.2.12) for C^2-functions u.

However, we notice that while (I.2.12) only makes sense for functions u that are twice differentiable, the statement (I.2.13) makes sense for all functions $u \in L^1_{loc}(\Omega)$. This warrants the following definition:

Definition I.2.14 (Weak solutions of the Laplace equation)**.** For a function $u \in L^1_{loc}(\Omega)$ we say that (I.2.12) is satisfied *in the weak sense* (or in the *distributional* sense) if

$$\int_\Omega u\, \Delta\varphi = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{I.2.13}$$

holds. The functions φ used to "test" the equation are for this very reason called *test-functions*.

To distinguish the notion of solution we used before, we say that if $\Delta u = 0$ in a differentiable function sense tjem u is a *strong solution* or *classical solution*.

We also have shown above the following statement

Proposition I.2.15. *Let $u \in C^2(\Omega)$. Then the following two statements are equivalent:*

(1) u is a weak solution to the Laplace equation $\Delta u = 0$ in Ω
(2) u is a classical solution of $\Delta u = 0$ in Ω.

Weyl proved that this equivalence holds for $u \in L^1_{loc}$ (i.e. with no a priori differentiablity at all) – this is our first result of *regularity theory*: showing that weak solutions which are a priori only integrable are actually differentiable. Observe: the reason this works here is that we have a homogeneous equation $\Delta u = 0$, and that Δ is a constant-coefficient linear elliptic operator (and one can spend much more time for proving similar results for more

general linear elliptic operators). Having said that, in some sense, the regularity theory for elliptic equations is always somewhat based on the following Theorem, Theorem I.2.16 (albeit in a hidden way).

THEOREM I.2.16 (Weyl's Lemma). *Let $u \in L^1_{loc}(\Omega)$ for $\Omega \subset \mathbb{R}^n$ open. If u is a weak solution of Laplace equation, i.e.*

$$\int_\Omega u\,\Delta\varphi = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{I.2.13}$$

then $u \in C^\infty(\Omega)$ and Δu in the classical sense.

Observe that this theorem (rightfully) does not say anything about u on $\partial\Omega$, this is a purely interior result!

The proof of Theorem I.2.16 exhibits the structure that many proofs in PDE have. First on obtains some *a priori estimates* (namely under the assumption that everything is smooth we find good estimates). Then we show that these estimates hold also for rough solutions by an approximation argument.

The a priori estimates for the Laplace equations are called the Cauchy estimates. These are truly amazing: They say that if we solve the Laplace equation we can estimate *all derivatives, in pretty much any norm* simply by the L^1-norm of the function.

Lemma I.2.17 (Cauchy estimates)**.** *Let $u \in C^\infty(\Omega)$ be harmonic, $\Delta u = 0$ in Ω. Then we have for any ball $B(x_0, r) \subset \Omega$ and for any multiindex γ of order k,*

$$|\partial^\gamma u(x_0)| \le \frac{C_k}{r^{n+k}}\,\|u\|_{L^1(B(x_0,r))}.$$

In particular we have for any $\Omega_2 \subset\subset \Omega$ that

$$\sup_{\Omega_2} |D^k u| \le C(\operatorname{dist}(\Omega_2, \Omega), k)\|u\|_{L^1(\Omega)}$$

PROOF OF THE CAUCHY ESTIMATES, LEMMA I.2.17. For $k = 0$ we argue with the mean value property for harmonic functions, Theorem I.2.6. We have for any ρ such that $B(x_0, \rho) \subset \Omega$ and any $x \in B(x_0, \rho/2)$,

$$|u(x)| = \left|\fint_{B(x,\rho/2)} u(z)\,dz\right| \le \frac{C}{\rho^n}\int_{B(x,\rho/2)} |u(z)|\,dz \le \frac{C}{\rho^n}\int_{B(x_0,\rho)} |u(z)|\,dz.$$

That is, we have obtained that for if $\Delta u = 0$ on $B(x_0, \rho)$ then

$$\sup_{B(x_0,\rho/2)} |u| \le \frac{C}{\rho^n}\|u\|_{L^1(B(x_0,\rho))}. \tag{I.2.16}$$

This proves in particular the case $k = 0$ (taking $\rho =: r$).

For the case $k = 1$ we use a technique called "*differentiating the equation*" (and in more general situations where this is used in a discretized version we will study later is due to Nirenberg, cf. Section III.3.2). Observe that $\Delta u = 0$ in Ω implies

$$\Delta \partial_i u = \partial_i \Delta u = 0 \quad \text{in } \Omega$$

So if we set $v := \partial_i u$ we have that $\Delta v = 0$ in Ω. For $x \in B(x_0, \rho/4)$, again from the mean value property for harmonic functions, Theorem I.2.6, we get with an additional integration by parts

$$\begin{aligned}
|\partial_i u(x)| = &\left| \fint_{B(x,\rho/4)} \partial_i u(z)\, dz \right| = \frac{C}{\rho^n} \left| \int_{\partial B(x,\rho/4)} u(\theta)\, \nu^i d\mathcal{H}^{n-1}(\theta) \right| \\
\leq &\frac{C}{\rho^n} \rho^{n-1} \sup_{B(x,\rho/4)} |u| \\
\leq &\frac{C}{\rho^n} \rho^{n-1} \sup_{B(x_0,\rho/2)} |u|
\end{aligned}$$

Now in view of the estimates in the step $k = 0$, namely (I.2.16), we arrive at

$$\sup_{B(x_0,\rho/4)} |\nabla u(x)| \leq \frac{C}{\rho^{n+1}} \|u\|_{L^1(B(x_0,\rho))}.$$

Differentiating the equation again, we find by induction that (the constant changes in each appearance!)

$$|\nabla^k u(x_0)| \leq \sup_{B(x_0,4^{-k}\rho)} |\nabla^k u(x)| \leq \frac{C}{\rho^{n+1}} \|\nabla^{k-1} u\|_{L^1(B(x_0,4^{1-k}\rho))} \leq \ldots \leq \frac{C}{\rho^{n+k}} \|u\|_{L^1(B(x_0,\rho)}.$$

If we want to show the estimate on $\Omega_2 \subset\subset \Omega$ we now pick $\rho < \operatorname{dist}(\Omega_2, \partial\Omega)$ and obtain the claim. □

Proof of Weyl's Lemma: Theorem I.2.16. We use a mollification argument, i.e. we approximate u with smooth functions u_ε that also solve (in the classical sense) the Laplace equation.

Let $\eta \in C_c^\infty(B(0,1))$ be another bump function, this time with the condition $\eta(x) = \eta(-x)$, i.e. η is even, $\eta \geq 0$ everywhere, and normalized such that

$$\int_{\mathbb{R}^n} \eta = 1.$$

We rescale η by a factor $\varepsilon > 0$ and set

$$\eta_\varepsilon(x) := \varepsilon^{-n} \eta(x/\varepsilon).$$

Then the convolution[4] is defined as

$$u_\varepsilon(x) := \eta_\varepsilon * u(x) := \int_{\mathbb{R}^n} \eta_\varepsilon(y-x)\, u(y)\, dy$$

Clearly this is not well-defined for all x, if $u \in L^1_{loc}(\Omega)$ only. But it is defined for all $x \in \Omega$ such that $\operatorname{dist}(x, \partial\Omega) > \varepsilon$, since $\operatorname{supp} \eta_\varepsilon(\cdot - x) \subset B(x, \varepsilon)$.

But observe that derivatives on u_ε hit only the kernel η_ε (which is smooth) (there is a dominated convergence to be used to show that, and for this we need L^1_{loc}!)

$$\partial^\gamma u_\varepsilon(x) := \eta_\varepsilon * u(x) := \int_{\mathbb{R}^n} \partial^\gamma \eta_\varepsilon(y-x)\, u(y)\, dy$$

That is $u_\varepsilon \in C^\infty(\Omega_{-\varepsilon})$ where

$$\Omega_{-\varepsilon} = \{x \in \Omega,\ \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$$

The fun part (which we used above already) is that convolutions behave well with differential operators, namely we will show now that $\Delta u_\varepsilon = 0$ in $\Omega_{-\varepsilon}$:

For this let $\psi \in C^\infty_c(\Omega_{-\varepsilon})$ a testfunction, then we have

$$\int_{\Omega_{-\varepsilon}} u^\varepsilon(x)\, \Delta\psi(x)\, dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(y)\eta_\varepsilon(x-y)\, \Delta\psi(x)\, dy\, dx = \int_{\mathbb{R}^n} u(y) \int_{\mathbb{R}^n} \eta_\varepsilon(x-y)\, \Delta\psi(x)\, dx\, dy$$

Now, by integration by parts (for any fixed $y \in \mathbb{R}^n$)

$$\int_{\mathbb{R}^n} \eta(x-y)\, \Delta\psi(x)\, dx = \int_{\mathbb{R}^n} \Delta_x \eta_\varepsilon(x-y)\, \psi(x)\, dx = \int_{\mathbb{R}^n} \Delta_y \eta_\varepsilon(x-y)\, \psi(x)\, dx = \Delta_y \int_{\mathbb{R}^n} \eta_\varepsilon(x-y)\, \psi(x)\, dx$$

So if we set

$$\varphi(y) := \eta_\varepsilon * \psi(y) \equiv \int_{\mathbb{R}^n} \eta_\varepsilon(x-y)\, \psi(x)\, dx$$

then we have by the support condition on ψ that $\varphi \in C^\infty_c(\Omega)$, and thus

$$\int_{\Omega_{-\varepsilon}} u^\varepsilon(x)\, \Delta\psi(x)\, dx = \int_{\mathbb{R}^n} u(y)\, \Delta\varphi(y)\, dy \overset{(I.2.13)}{=} 0.$$

This argument works for any $\psi \in C^\infty_c(\Omega_{-\varepsilon})$, that is u^ε is weakly harmonic in $\Omega_{-\varepsilon}$. But since $u_\varepsilon \in C^\infty(\Omega_{-\varepsilon})$ this implies in view of Proposition I.2.15 that in the strong sense

$$\Delta u_\varepsilon = 0 \quad \text{in } \Omega_{-\varepsilon}.$$

So now u_ε is a smooth solution to Laplace's equation, so we use the a priori estimates of Lemma I.2.17.

Fix $\Omega_2 \subset\subset \Omega$. Between Ω_2 and Ω we can squeeze two more set Ω_3, and Ω_4,

$$\Omega_2 \subset\subset \Omega_3 \subset\subset \Omega_4 \subset\subset \Omega.$$

For any ε small enough, namely

$$\varepsilon < \operatorname{dist}(\Omega_3, \partial\Omega_4) \quad \text{and} \quad \varepsilon < \operatorname{dist}(\Omega_3, \partial\Omega_4)$$

[4] we have seen this operation for the Fourier Transform argument above after (I.2.3), there we used a nonsmooth kernel $|\cdot|^{2-n}$ for the convolution

we have that $\Delta u^\varepsilon = 0$ in Ω_3, so by the Cauchy estimates, Lemma I.2.17, we have for any $k \in \mathbb{N}$

$$\sup_{\Omega_2} |\nabla^k u_\varepsilon| \leq C(k, \Omega_2, \Omega_3) \, \|u_\varepsilon\|_{L^1(\Omega_3)}.$$

Now we estimate, by Fubini,

$$\|u_\varepsilon\|_{L^1(\Omega_3)} \leq \int_{\Omega_3} \int_{\mathbb{R}^n} |\eta_\varepsilon(x-y)| \, |u(y)| \, dy \, dx = \int_{\mathbb{R}^n} |u(y)| \int_{\Omega_3} |\eta_\varepsilon(x-y)| dx \, dy$$

Since ε is small enough we have that

$$\operatorname{supp} \left(\int_{\Omega_3} |\eta_\varepsilon(x - \cdot)| dx \right) \subset \Omega_4.$$

So we get

$$\|u_\varepsilon\|_{L^1(\Omega_3)} \leq \|u\|_{L^1(\Omega_4)} \sup_{y \in \mathbb{R}^n} \int_{\Omega_3} |\eta_\varepsilon(x-y)| dx \leq \|u\|_{L^1(\Omega_4)} \int_{\mathbb{R}^n} |\eta_\varepsilon(z)| dz.$$

Now we use the definition of η_ε to compute via substitution[5]

$$\int_{\mathbb{R}^n} |\eta_\varepsilon(z)| dz = \varepsilon^{-n} \int_{\mathbb{R}^n} |\eta(z/\varepsilon)| dz = \varepsilon^{-n} \int_{\mathbb{R}^n} |\eta(z/\varepsilon)| dz = \int_{\mathbb{R}^n} |\eta(\tilde{z})| d\tilde{z} = 1.$$

The last equality is due to the normalization of η, $\int \eta = 1$.

That is, we have shown that for any $k \in \mathbb{N} \cup \{0\}$

$$\sup_{\Omega_2} |\nabla^k u_\varepsilon| \leq C(k, \Omega_2, \Omega_3) \, \|u\|_{L^1(\Omega_4)},$$

and the right-hand side is finite since $u \in L^1_{loc}(\Omega)$ and $\Omega_4 \subset\subset \Omega$.

This estimate holds for any $\varepsilon > 0$, so u_ε and all its derivative are uniformly equicontinuous (in ε). By Arzela-Ascoli (and a diagonal argument in k) we find a converging subsequence $\varepsilon \to 0$ and a function $u_0 \in C^\infty(\Omega_2)$ such that for any $k \in \mathbb{N} \cup \{0\}$.

$$|\nabla^k u_\varepsilon(x) - \nabla^k u_0(x)| \xrightarrow{\varepsilon \to 0} 0 \quad \text{locally uniformly in } \Omega_2.$$

We claim that $u = u_0$ in almost every point (since u is an L^1_{loc}-function it is actually a the class of maps equal up to almost every point, u_0 is a continuous representative of the class u). Indeed, by the normalization $\int \eta = 1$ which implies $\int \eta_\varepsilon = 1$ we have

$$|u_\varepsilon(x) - u(x)| = \left| \int \eta_\varepsilon(y-x) \, (u(y) - u(x)) \, dy \right| \leq C(\eta) \fint_{B(x,\varepsilon)} |u(y) - u(x)| \, dy.$$

So, by the Lebesgue differentiation theorem, we have for almost every $x \in \Omega_2$,

$$\lim_{\varepsilon \to 0} |u_\varepsilon(x) - u(x)| = 0,$$

that is

$$u_0 = u \quad \text{a.e. in } \Omega_2.$$

Thus $u \in C^\infty(\Omega_2)$, and $\Delta u = 0$ in classical sense in Ω_2.

[5] observe for $\tilde{z} = z/\varepsilon$ we have in n space dimensions $d\tilde{z} = \varepsilon^{-n} dz$

Since this holds for any $\Omega_2 \subset \Omega$ we have shown

$u \in C^\infty(\Omega)$, and $\Delta u = 0$ in classical sense in Ω. $\square$

Corollary I.2.18 (Liouville)**.** *Let $u \in C^2(\mathbb{R}^n)$ and $\Delta u = 0$ in all of $\mathbb{R}^n$. If u is a bounded function then $u \equiv const$.*

Proof. Fix $x_0 \in \mathbb{R}^n$. In view of Lemma I.2.17 we have for such a function u, for any radius $r > 0$,

$$|Du(x_0)| \leq \frac{C}{r^{n+1}} \|u\|_{L^1(B(x_0,r))}$$

If u is bounded,

$$\|u\|_{L^1(B(x_0,r))} \leq C\, r^n \sup_{\mathbb{R}^n} |u| < \infty$$

and thus

$$|Du(x_0)| \leq Cr^{-1} \sup_{\mathbb{R}^n} |u|.$$

This holds for any $r > 0$, so if we let $r \to \infty$, we get

$$|Du(x_0)| = 0,$$

which holds for any $x_0 \in \mathbb{R}^n$. That is, $Du \equiv 0$, and by the fundamental theorem of calculus this means u is a constant. $\square$

I.2.5. Harnack Principle. Above we learned, e.g. in Corollary I.2.7 of the strong maximum principle. Another type of maximum principle is the Harnack inequality.

Theorem I.2.19. *Let $\Omega \subset \mathbb{R}^n$ open. For any open, connected, and bounded $U \subset\subset \Omega$ there exists a constant $C = C(U, \Omega)$ such that for any solution $u \in C^2(\Omega)$ with $u \geq 0$ and such that*

$$\Delta u = 0 \quad \text{in } \Omega$$

we have

$$\sup_U u \leq C \inf_U u$$

Proof. The proof is based on the mean value formula, Theorem I.2.5, namely for any $x \in U$ and any $r < \operatorname{dist}(U, \partial\Omega)$ we have

$$u(x) = \fint_{B(x,r)} u(z)\, dz$$

Let now $R := \frac{1}{4}\operatorname{dist}(U, \partial\Omega)$. For any $x_0 \in U$ and any $x \in B(x_0, R)$ we have (here we use $u \geq 0$ and that $B(x, R) \subset B(y, 2R)$ for $x, y \in B(x_0, R)$)

$$u(x) = \fint_{B(x,R)} u(z)\, dz \leq 2^n \fint_{B(y,2R)} u(z)\, dz = 2^n\, u(y).$$

Again, this holds for any $x, y \in B(x_0, R)$. Taking the supremum for $x \in B(x_0, R)$ and the infimimum on $y \in B(x_0, R)$ we get

$$\sup_{B(x_0,R)} u \leq 2^n \inf_{B(x_0,R)} u. \tag{I.2.17}$$

That is we have the Harnack principle on any Ball $B(x_0, R)$. Since U is bounded, open and compactly contained in Ω we can now cover all of U by finitely many balls $(B_\ell)_{\ell=1}^N$ which lie inside Ω centered at points in U and of radius R. The supremum of u on Ω can be located in some ball B_{i_1}, in the sense that

$$\sup_U u \leq \sup_{B_{i_0}} u, \tag{I.2.18}$$

and the infimum is attained in some ball B_{i_2} in the sense of

$$\inf_U u \geq \inf_{B_{i_1}} u. \tag{I.2.19}$$

(Observe that B_{i_0}, B_{i_1} may contain points in $\Omega \backslash U$. Since U is connected, there is a chain of balls $(B_{\ell_i})_{i=1}^K$, $K \leq N$, such that $B_{\ell_i} \cap B_{\ell_{i+1}} \neq \emptyset$ and such that $\ell_1 = i_1$ and $\ell_K = i_2$. Observe that this implies in particular,

$$\inf_{B_{\ell_i}} u \leq \sup_{B_{\ell_{i+1}}} u. \tag{I.2.20}$$

Then we can conclude via the following chain of estimates

$$\sup_U u \overset{(I.2.18)}{\leq} \sup_{B_{i_0}} u = \sup_{B_{\ell_1}} u \overset{(I.2.17)}{\leq} 2^n \inf_{B_{\ell_1}} u \overset{(I.2.20)}{\leq} 2^n \sup_{B_{\ell_2}} u \leq \ldots \leq K 2^n \inf_{B_{\ell_K}} u \overset{(I.2.19)}{\leq} K 2^n \inf_U u.$$

□

I.2.6. Green Functions. Our next goal are *Green's functions*. In some way Green functions are a restriction of the fundamental solution to domains $\Omega \subset \mathbb{R}^n$ factoring in also boundary data.

Recall that for the fundamental solution Φ we showed in Theorem I.2.3 that for the *Newton potential*

$$u(x) := \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy \tag{I.2.21}$$

we have $\Delta u = f$. It is an interesting observation that (for reasonable f) we have

$$\lim_{|x| \to \infty} u(x) = 0.$$

That is the Newton potential approach solves an equation of

$$\begin{cases} \Delta u = f & \text{in } \mathbb{R}^n \\ u = 0 & \text{on the boundary, i.e. for } |x| \to \infty. \end{cases}$$

The *Greens function* is a way to restrict this construction to domains Ω. Instead of the Fundamental solution $\Phi(x-y)$ we get the Green kernel $G(x,y)$. Instead of the Newton potential we consider

$$u(x) = \int_\Omega G(x,y)\, f(y)\, dy$$

and hope that this object solves

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The Greens function G (which depends on Ω) can be computed explicitely only for very specific Ω (balls, half-spaces) – which is somewhat related to the fact that there is not necessarily a reasonable Fourier transform for generic sets Ω.

But one can abstractly show that the Green functions exists for reasonable sets Ω. The idea is as follows: We know that the Newton potential as in (I.2.21) solves the right equation $\Delta u = f$, but it does not satisfy $u = 0$ on $\partial\Omega$. So let us try to correct the Newton potential and choose the Ansatz

$$u(x) := \int_\Omega \Phi(x-y)\, f(y)\, dy - \int_\Omega H(x,y)\, f(y)\, dy$$

By our computatins for Theorem I.2.3 we have that then for $x \in \Omega$

$$\Delta u(x) := f(x) - \int_\Omega \Delta_x H(x,y)\, f(y)\, dy,$$

so it would be nice if

$$\Delta_x H(x,y) = 0 \quad \forall\, x, y \in \Omega.$$

Moreover we would like that $u(x) = 0$ on $\partial\Omega$, which would be satisfied if

$$\Phi(x-y) = H(x,y) \quad \forall x \in \partial\Omega, y \in \Omega.$$

That is, for each fixed $y \in \Omega$ we should try to find a function $H(\cdot, y)$ that solves

$$\begin{cases} \Delta_x H(\cdot, y) = 0 & \text{in } \Omega, \\ H(\cdot, y) = \Phi(\cdot - y) & \text{on } \partial\Omega. \end{cases} \tag{I.2.22}$$

Observe that for fixed $y \in \Omega$ the boundary condition $\Phi(\cdot - y) \in C^\infty(\partial\Omega)$ is a smooth function, since for $y \in \Omega$ we clearly have

$$\inf_{x \in \partial\Omega} |x - y| > 0.$$

That is, there is a good chance to solve this equation (I.2.22) (and from Theorem I.2.13 we know that there is *at most one* solution).

Definition I.2.20 (Green function)**.** For given Ω, if there exists H as in (I.2.22) then we call

$$G(x,y) := \Phi(x-y) - H(x,y)$$

the Green function on Ω.

One can show that G is symmetric, i.e. that

$$G(x,y) = G(y,x) \quad \forall\, x \neq y \in \Omega \tag{I.2.23}$$

While the Green function are usually not explicit, some properties and estimates can be shown, and there is an extensive research literature on the subject, e.g. see [**Littman et al., 1963**]. The Green function is also specially important from the point of view of stochastic processes, see e.g. [**Chen, 1999**].

We will only investigate the most basic property:

THEOREM I.2.21. *Let $\Omega \subset\subset \mathbb{R}^n$, $\partial\Omega \in C^1$ $f \in C^0(\Omega)$ and $g \in C^0(\partial\Omega)$. Assume that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution to*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{I.2.24}$$

Then if G is the Green function for Ω from Definition I.2.20 we have for any $x \in \Omega$,

$$u(x) = \int_\Omega G(x,y)\, f(y)\, dy - \int_{\partial\Omega} g(\theta)\partial_{\nu(\theta)} G(x,\theta)\, d\mathcal{H}^{n-1}(\theta).$$

PROOF. Recall the Gauss-Green formula[6] on (smooth enough) domains A,

$$\int_A u(y)\Delta v(y) - \Delta u(y)\, v(y)\, dy = \int_{\partial A} u(\theta)\partial_\nu v(\theta) - \partial_\nu u(\theta)\, v(\theta)\, d\mathcal{H}^{n-1}(\theta). \tag{I.2.25}$$

We apply this to formula to $A = \Omega \backslash B(x,\varepsilon)$ and $v(y) := G(x,y)$. Observe that by symmetry of G, (I.2.23),

$$\Delta_y G(x,y) = \Delta_x G(x,y) = 0 \quad x \neq y,$$

so, also in view of (I.2.24), (I.2.25) becomes

$$-\int_A G(x,y)\, f(y)\, dy = \int_{\partial A} u(\theta)\partial_\nu G(x,\theta) - \partial_\nu u(\theta)\, G(x,\theta)\, d\mathcal{H}^{n-1}(\theta). \tag{I.2.26}$$

Now we argue as in the proof of Theorem I.2.3. Observe that H is a smooth function.

We have

$$\begin{aligned} &\int_{\partial A} u(\theta)\partial_\nu G(x,\theta) d\mathcal{H}^{n-1}(\theta) \\ &= \int_{\partial\Omega} g(\theta)\partial_\nu G(x,\theta) - \int_{\partial B(x,\varepsilon)} u(\theta)\partial_\nu \Phi(x-\theta)\, d\mathcal{H}^{n-1}(\theta) + \int_{\partial B(x,\varepsilon)} u(\theta)\partial_\nu H(x-\theta)\, d\mathcal{H}^{n-1}(\theta) \\ &\xrightarrow{\varepsilon\to 0} \int_{\partial\Omega} g(\theta)\partial_\nu G(x,\theta) - u(x) + 0. \end{aligned}$$

[6] this is a special case of the integration by parts formula

and

$$\begin{aligned}
&\int_{\partial A} \partial_\nu u(\theta) G(x,\theta) d\mathcal{H}^{n-1}(\theta) \\
=&\int_{\partial\Omega} \partial_\nu u(\theta) G(x,\theta) - \int_{\partial B(x,\varepsilon)} \partial_\nu u(\theta) G(x,\theta)\, d\mathcal{H}^{n-1}(\theta) \\
=&0 - \int_{\partial B(x,\varepsilon)} \partial_\nu u(\theta) G(x,\theta)\, d\mathcal{H}^{n-1}(\theta) \\
\xrightarrow{\varepsilon\to 0}&0.
\end{aligned}$$

This proves the claim. □

In special situations one can actually construct explicit Green's function. Let us firstly consider the Half-space

$$\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : \ x_n > 0\}.$$

So we need to find a solution to the equation

$$\begin{cases} \Delta_x H(\cdot, y) = 0 & \text{in } \mathbb{R}^n_+, \\ H(\cdot, y) = \Phi(\cdot - y) & \text{on } \mathbb{R}^{n-1} \times \{0\} \equiv \partial\mathbb{R}^n_+. \end{cases}$$

Since H at the boundary has to coincide with Φ it is likely that H should be somewhat of the form of Φ – only the singularity has to be getten rid of – the idea is a reflection:

$$H(x,y) := \Phi(x - y^*)$$

where

$$y^* = (y_1, \dots, y_n)^* = (y_1, \dots, y_{n-1}, -y_n).$$

It is a good exercise to check that

(1) H is symmetric, $H(x,y) = H(y,x)$
(2) H is smooth in $\mathbb{R}^n_+ \times \mathbb{R}^n_+$ (since $x^* = y$ implies $x_n = -y_n$, so x and y cannot both lie in the upper half-space if this happens)
(3) The reflection does not change the PDE, namely $\Delta_x H = 0$ for $x, y \in \mathbb{R}^n_+$.
(4) Indeed $H(x,y) = \Phi(x - y)$ for $x \in \mathbb{R}^{n-1} \times \{0\}$ and $y \in \mathbb{R}^n_+$.

So we set

$$G(x,y) := \Phi(x - y) - \Phi(x - y^*) = \Phi(x - y) - \Phi(x^* - y)$$

When we now use the representation formula, as in Theorem I.2.21, then we need to compute $\partial_{\nu(y)} G(x,y)$ for $y \in \mathbb{R}^{n-1} \times \{0\}$. Observe that the outwards unit normal $\nu(y) = (0, \dots, 0, -1)$, so we compute

$$\partial_{\nu(y)} G(x,y) = -\partial_{y_n}\Phi(x - y) + \partial_{y_n}\Phi(x^* - y) = c_n \frac{x_n - y_n}{|x - y|^n} - c_n \frac{x_n + y_n}{|x - y|^n} = \tilde{c}_n \frac{x_n}{|x - y|^n}.$$

If we write the variables in $\mathbb{R}^n_+$ as $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$ and $x_n > 0$, then as in Theorem I.2.21 we indeed obtain, e.g., if

$$u(x) := c_n \int_{\mathbb{R}^{n-1}} \frac{x_n}{(|x'-y'|^2 + |x_n - y_n|^2)^{\frac{n}{2}}} \, g(y') \, dy'$$

then u satisfies indeed (for "reasonable" g)

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ \lim_{x_n \to 0} u(x) = g(x') \\ \lim_{x_n \to \infty} u(x) = 0. \end{cases}$$

The formula for u is called the *Poisson formula* on the Half-space $\mathbb{R}^n_+$, also the *harmonic extension of g from $\mathbb{R}^{n-1}$ to $\mathbb{R}^n_+$*.

The other situation where we can compute the Green's function is the ball. For simplicity let us consider $\Omega = B(0,1)$, the unit ball centered at zero. Again the first goal is to find $H(x,y)$ that corrects the fundamental solution. In the case of the half-space $\mathbb{R}^n_+$ we set $H(x,y) = \Phi(x - \tilde{y})$, i.e. we reflected the y-variable in a way that did not interfere with the PDE but removed the singularity (and coincided with $\Phi(x-y)$ on the boundary.

So lets do the same for the ball. The canonical operation that reflects points from the ball $B(0,1)$ outside and vice versa is called the inversion at a sphere,

$$y^* := \frac{y}{|y|^2} : B(0,1) \to B(0,1)^c.$$

(Although it is not explicitely used here, it is good to know: the inversion at the sphere is a conformal transform, i.e. it preserves angles). So a first attempt would be to set

$$H(x,y) := \Phi\left(\left|x - \frac{y}{|y|^2}\right|\right),$$

which takes care of the singularity of Φ (for $y, x \in B(0,1)$ we have $|x - \frac{y}{|y|^2}| \neq 0$, and does not disturb the PDE for $G(x,y)$. However such a $G(x,y)$ is not equal to $\Phi(x-y)$ for $|x| = 1$. So we need to adapt G to the boundary data. Observe that for $|x| = 1$,

$$\begin{aligned} &|y|^2 \left|x - \frac{y}{|y|^2}\right|^2 \\ =&|y|^2 \left(|x|^2 + \frac{1}{|y|^2} - 2\langle x, \frac{y}{|y|^2}\rangle\right) \\ =&\left(|y|^2|x|^2 + 1 - 2\langle x, y\rangle\right) \\ \overset{|x|=1}{=}&\left(|y|^2 + |x|^2 - 2\langle x, y\rangle\right) \\ =&|x-y|^2. \end{aligned}$$

That is why we set

$$G_{B(0,1)}(x,y) := \Phi\left(|y|\left|x - \frac{y}{|y|^2}\right|\right),$$

which satisfies all the requested properties.

From this we obtain (without proof)

THEOREM I.2.22 (Poisson's formula for the ball). *Assume $g \in C^0(\partial B(0,r))$. Define*

$$u(x) := c_n \int_{\partial B(0,r)} \frac{1}{r}\frac{r^2 - |x|^2}{|x-\theta|^n} g(\theta)\, d\mathcal{H}^{n-1}(\theta)$$

Then

(1) $u \in C^\infty(B(0,r))$
(2) $\Delta u = 0$ *in* $B(0,r)$
(3)

$$\lim_{B(0,r)\ni x\to x_0} u(x) = g(x_0) \quad \textit{for any } x_0 \in \partial B(0,r)$$

I.2.7. Methods from Calculus of Variations – Energy Methods. As we have seen, comparison principles is a strong tool for uniqueness (and also existence). These arguments also work in some situations of nonlinear pdes, where the theory of distributional solutions does not work, but the theory of Viscosity solutions can be applied, see [**Koike, 2004**].

On the other hand, the comparison methods are (currently) restricted to first or second-order equations, and to scalar equations. For systems or higher-order PDEs they seem not to be that helpful.

In this section we have a short look on energy methods, which is a basic tool of distributional theory. They do not rely on any comparison principle, and they are often used for higher-order differential equations and systems. On the other hand for some fully nonlinear equations ("non-variational" equations, equtions "not in divergence form") they cannot be well applied.

The ideas should be reminiscent of the arguments we employed for the weak solutions in Theorem I.2.16.

Assume that we have

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{I.2.27}$$

We have seen before Theorem I.2.16 that this equation is related to the integral equation

$$\int_\Omega Du \cdot D\varphi + f\varphi = 0 \quad \forall \varphi \in C_c^\infty(\Omega).$$

The interesting point is that this expression is a Frechet-Derivative of a function acting on the map u in direction φ.

Indeed one can characterize solutions as minimizers of an energy functional. This is sometimes called the *Dirichlet principle*.

THEOREM I.2.23 (Energy Minimizers are solutions and vice versa). *Assume $f \in C^0(\overline{\Omega})$.*

Denote the class of permissible functions

$$X := \Big\{ u \in C^2(\overline{\Omega}), \quad u = 0 \quad \text{on } \partial\Omega \Big\}$$

and define the energy

$$\mathcal{E}(u) := \int_\Omega \frac{1}{2}|Du|^2 + fu = 0.$$

Let $u \in X$ be a minimizer of E in X, i.e.

$$\mathcal{E}(u) \leq E(v) \quad \forall v \in X.$$

Then u solves (I.2.27).

Conversely, if $u \in X$ solves (I.2.27), then u is a minimizer of $\mathcal{E}$ in the set X.

PROOF. We compute what is called the *Euler-Lagrange-equations* of $\mathcal{E}$: Let $\varphi \in C_c^\infty(\Omega)$, then certainly $u + t\varphi \in X$ for all $t \in \mathbb{R}$. That is the minimizing property says that the function

$$E(t) := \mathcal{E}(u + t\varphi)$$

has a minimum in $t = 0$. By Fermats theorem (one checks easily that E is differentiable in t)

$$\frac{d}{dt}\Big|_{t=0} E(t) \equiv E'(0) = 0.$$

Now observe that

$$\frac{d}{dt}\Big|_{t=0} |D(u + t\varphi)|^2 = 2\langle Du, D\varphi\rangle$$

and

$$\frac{d}{dt}\Big|_{t=0} f\,(u + t\varphi) = f\,\varphi.$$

Thus, we arrive at

$$0 = \frac{d}{dt}\Big|_{t=0} E(t) = \int_\Omega Du \cdot D\varphi + f\varphi = 0.$$

That is, u is a *weak solution* of (I.2.27). But $u \in C^2(\overline{\Omega})$, so we argue similar to the proof of Proposition I.2.15:

By an integration by parts (for $\varphi \in C_c^\infty(\Omega)$ there are no boundary terms), we thus have

$$0 = \int_\Omega Du \cdot D\varphi + f\varphi = 0 = -\int_\Omega (\Delta u - f)\varphi.$$

Since $\Delta u - f$ is continuous, and the last estimate holds for any smooth $\varphi \in C_c^\infty(\Omega)$ we get that (as for Proposition I.2.15, or otherwise by the fundamental lemma of calculus of variations, Lemma I.2.24,

$$\Delta u - f = 0.$$

That is the first claim is proven: minimizers are solutions.

For the converse assume u solves (I.2.27). Let w be any other map in X. Then we have

$$\int_\Omega (\Delta u - f)(u - w) = 0.$$

Observe that u and w have the same boundary value 0 on $\partial\Omega$. Thus, when we perform the following integration by parts we do not find boundary terms,

$$0 = -\int_\Omega \nabla u \cdot \nabla(u - w) + f(u - w) = 0. \tag{I.2.28}$$

Now we compute (using Young's inequality or Cauchy-Schwarz $2ab \leq a^2 + b^2$)

$$\begin{aligned}\int_\Omega |\nabla u|^2 + fu &\overset{(I.2.28)}{=} \int_\Omega \nabla u \cdot \nabla w + fw \\ &\leq \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla w|^2 + fw \\ &= \frac{1}{2}\int_\Omega |\nabla u|^2 + E(w)\end{aligned}$$

Subtracting $\frac{1}{2}\int_\Omega |\nabla u|^2$ from both sides in the estimate above we obtain

$$E(u) \leq E(w).$$

That is, we have shown: if u solves the equation, then u is a minimizer. □

Above we have used the following statement.

Lemma I.2.24 (Fundamental Lemma of the Calculus of Variations)**.** *Let $\Omega \subset \mathbb{R}^n$ be any open set and assume $f \in L^1_{loc}(\Omega)$, i.e. for any $\Omega' \subset\subset \Omega$ we have*

$$\int_{\Omega'} |f| < \infty.$$

(1) If

$$\int_\Omega f(x)\,\varphi(x) \geq 0 \quad \textit{for all } \varphi \in C_c^\infty(\Omega) \textit{ that are nonnegative, } \varphi \geq 0,$$

then

$$f \geq 0 \quad \textit{almost everywhere in } \Omega.$$

(2) If

$$\int_\Omega f(x)\,\varphi(x) = 0 \quad \textit{for all } \varphi \in C_c^\infty(\Omega) \textit{ that are nonnegative, } \varphi \geq 0,$$

then

$$f \equiv 0 \quad \textit{almost everywhere in } \Omega.$$

The proof is left as an exercise, it is a combination of convolution arguments as in Theorem I.2.16 and the argument used for Proposition I.2.15.

THEOREM I.2.25 (Uniqueness). *Assume* $f \in C^0(\overline{\Omega}) \cap L^1(\Omega)$

Denote the class of permissible functions

$$X := \Big\{ u \in C^2(\overline{\Omega}), \quad u = 0 \quad \text{on } \partial\Omega \Big\}$$

Then there is at most one solution $u \in X$ *to* (I.2.27)

PROOF. Assume $u, w \in X$ are two solutions, then

$$\Delta(u - w) = 0.$$

Multiplying by $u - w$ and integrating by parts (observe that there are no boundary terms since $u = w$ on $\partial\Omega$, we obtain

$$\int_\Omega |\nabla(u - w)|^2 = 0.$$

But this implies that $\nabla u - w \equiv 0$, so $u - w \equiv const$. Since $u = w$ on the boundary that constant is zero, and $u \equiv w$. □

These methods can be extended, e.g. for higher order differential equations (where no maximum principle holds), e.g.

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases}$$

See exercises.

CHAPTER 2

Second order linear elliptic equations, and maximum principles

From now on, we will use the Einstein summation convention, which is summarized as "we sum over repeated indices".

For example, for vectors $a = (a^1, \dots, a^n)$, $b = (b^1, \dots, b^n)$, we write

$$\langle a, b \rangle = a^i b^i := \sum_{i=1}^{n} a^i b^i.$$

This notation is used a lot in coordinate and tensor computation in physics (e.g. relativity, hence the name). In contrast to Physics (and Geometry) we will not care about "raised" and "lowered" indices. I.e.

$$a^i b^i = a_i b_i = a^i b_i = a_i b^i \equiv \sum_{i=1}^{n} a^i b^i$$

We will also use this for matrices, namely if $A = (a_{ij})_{i,j=1}^n$ and $b = (b^1, \dots, b^n)$ is a vector

$$a_{ij} b^j = \sum_{j=1}^{n} a_{ij} b^j \equiv (Ab)_i.$$

Observe that in particular we implicitely always assume we know what dimension n we thing about. This notation needs some time to get accostumed to, but it is very beneficial for computations.

II.1. Linear Elliptic equations

Second order elliptic equationd are a class of equations that in some sense are *governed* by the Laplacian operator.

Definition II.1.1 (Linear elliptic equations)**.** (1) ("non-divergence form") linear second order operators are defined to be operators of the form

$$L := a_{ij}\partial_{ij} + b_i \partial_i + c$$

for coefficents $a_{ij}, b_i, c : \Omega \to \mathbb{R}$. They act as follows on functions $u \in C^2(\Omega)$

$$Lu(x) := a_{ij}(x)\partial_{ij} u(x) + b_i(x)\partial_i u(x) + c(x)\, u(x).$$

L is called a constant coefficient operator, if the coefficients a_{ij}, b_i and c are all constant.

(2) ("divergence form") linear second order operators are defined to be operators of the form
$$L := \partial_i \left(a_{ij}\partial_j\right) + b_i\partial_i + c$$
for coefficents $a_{ij}, b_i, c : \Omega \to \mathbb{R}$. They act as follows on functions $u \in C^2(\Omega)$
$$Lu(x) := \partial_i \left(a_{ij}(x)\partial_j u(x)\right) + b_i(x)\partial_i u(x) + c(x)\, u(x).$$

(3) Clearly, divergence on non-divergence form are very similar if a_{ij} is smooth enough, but they are not if that is not the case (and in general

(4) (divergence-form or non-divergence form) operators L are called elliptic (also often called uniformly elliptic and bounded) if there exists an *ellipticity constants* $\Lambda > 0$ such that
$$\xi^T A\xi \equiv \xi^i a_{ij}\xi^j \geq \frac{1}{\Lambda}$$
and
$$\sup_\Omega |a_{ij}|, |b_i|, |c| < \infty.$$

For simplicity, although this is not strictly necessary we will below always assume A is symmetric.

Example II.1.2.

- The operator Δ is clearly elliptic in the above sense, with
$$a_{ij} = \delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$$
- Operators like $\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ are not (uniformly elliptic), since $|\nabla u| = 0$ cannot be excluded. These operators are called degenerate elliptic.

Definition II.1.3. $u \in C^2(\Omega)$ is called a subsolution of $-Lu = f$ for an elliptic operator L, if
$$-Lu \leq 0 \quad \text{in } \Omega$$
and a supersolution if
$$-Lu \geq 0 \quad \text{in } \Omega.$$
$u \in C^2(\Omega)$ is called a solution if it is both sub- and supersolution.

In the following we will restrict ourselves to elliptic non-divergence operators!

II.2. Maximum principles

The first result is a generalization of the weak maximum principle for Δ, Corollary I.2.9.

THEOREM II.2.1 (Weak maximum principle for $c = 0$). *Let $\Omega \subset\subset \mathbb{R}^n$, $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be an L-subsolution, i.e.*

$$-Lu \leq 0 \quad \text{in } \Omega \tag{II.2.1}$$

If L is (non-divergence form) linear elliptic operator with $c \equiv 0$, then

$$\sup_{\Omega} u = \sup_{\partial\Omega} u.$$

If instead of (II.2.1) we have

$$-Lu \geq 0 \quad \text{in } \Omega$$

then

$$\inf_{\Omega} u = \inf_{\partial\Omega} u.$$

PROOF. First we assume instead of (II.2.1)

$$-Lu > 0 \quad \text{in } \Omega \tag{II.2.2}$$

Clearly, by continuity of u in $\overline{\Omega}$,

$$\sup_{\Omega} u \geq \sup_{\partial\Omega} u$$

If we had

$$\sup_{\Omega} u > \sup_{\partial\Omega} u,$$

then we would find the global (and thus a local) maximum $x_0 \in \Omega$, at which we have $Du(x_0) = 0$ and $D^2u(x_0) \leq 0$. But this implies (recall $c \equiv 0$)

$$Lu(x_0) = a_{ij}(x_0)\partial_{ij}u(x_0) + b_i(x_0)\underbrace{\partial_i u(x_0)}_{=0}$$

Since $a_{ij}(x_0)$ is elliptic, and $\partial_{ij}u(x_0) \geq 0$ we have

$$a_{ij}(x_0)\partial_{ij}u(x_0) \geq 0.$$

(This is a general Linear Algebra fact, if A, B are symmetric, nonnegative matrices, then their Hilbert-Schmidt Scalar product $A : B := a_{ij}b_{ij} \geq 0$.) That is, we have

$$Lu(x_0) \geq 0$$

which is a contradiction to (II.2.2).

We conclude that under the assumption (II.2.2) we have

$$\sup_{\Omega} u = \sup_{\partial\Omega} u.$$

In order to weak the assumption to (II.2.2) we consider, for some $\gamma > 0$, $v_\gamma(x) := e^{\gamma x_1}$, where x_1 is the first component of $x = (x_1, \ldots, x_n)$. Observe that

$$Lv_\gamma(x) = \left(a_{11}(x)\gamma^2 + b_1(x)\gamma\right) e^{\gamma x_1}$$

Since L is elliptic we have $a_{11} \geq \frac{1}{\Lambda}$ and $b_1 \geq -\Lambda$, so

$$Lv_\gamma(x) = a_{11}(x)\gamma^2 + b_1(x)\gamma \geq e^{\gamma x_1}\gamma\left(\frac{1}{\Lambda}\gamma - \Lambda\right).$$

If we choose $\gamma = 3\Lambda$ we thus find

$$Lv_\gamma(x) > 0 \quad \text{in } \Omega.$$

Consequently, under the assumption (II.2.1) we have for any $\varepsilon > 0$, for $w_\varepsilon := u + \varepsilon v_\gamma$,

$$Lw_\varepsilon(x) > 0 \quad \text{in } \Omega.$$

and thus by the first step

$$\sup_\Omega w_\varepsilon = \sup_{\partial\Omega} w_\varepsilon$$

Since $w_\varepsilon = u + \varepsilon v_\gamma$ and v_γ is continuous (and Ω is bounded) we have

$$\left|\sup_\Omega u - \sup_{\partial\Omega} u\right| \leq C(\Omega)\varepsilon.$$

Letting $\varepsilon \to 0$ we obtain the claim.

The inf claim follows by taking $-u$ instead of u. □

Also in the case $c \not\equiv 0$ a type of weak maximum principle holds (essentially mimmicking the above argument):

THEOREM II.2.2 (Weak maximum principle for $c \leq 0$). *Let $\Omega \subset\subset \mathbb{R}^n$, $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ solve*

$$-Lu \leq 0.$$

If $c \leq 0$ in Ω we have

$$\sup_\Omega u \leq \sup_{\partial\Omega} u_+,$$

where u_+, u_- denotes the positive part of u, namely

$$u_+ = \max\{0, u\}$$

If on the other hand

$$-Lu \geq 0.$$

and $c \leq 0$ in Ω we have

$$\inf_\Omega u \geq \inf_{\partial\Omega}(-u_-),$$

where u_+, u_- denotes the positive part of u, namely

$$u_+ = \max\{0, u\}, \quad u_- = -\min\{0, u\}$$

In particular, if $Lu = 0$ then

$$\sup_\Omega |u| \leq \sup_{\partial\Omega} |u|$$

PROOF. Let us assume $-Lu \leq 0$. First we observe that if

$$\sup_\Omega u \leq 0$$

then there is nothing to show, since we have $u_+ \geq 0$ by definition and thus

$$\sup_\Omega u \leq 0 \leq \sup_{\partial\Omega} u_+.$$

So w.l.o.g. we may assume that $\sup_\Omega u > 0$. Set

$$\Omega_+ := \{x \in \Omega : \quad u(x) > 0\}.$$

Since u is continuous $\Omega_+ = u^{-1}((0,\infty))$ is an nonempty, open set.

Define the elliptic operator L_0 by

$$L_0 u := Lu - cu = a_{ij}\partial_{ij}u + b_i\partial_i u.$$

Since $-Lu \leq 0$ we have $-L_0 u \leq cu \leq 0$ in Ω_+ — since by assumption $c \leq 0$. So we have, using also Theorem II.2.1,

$$\sup_\Omega u \overset{u\leq 0:\,\Omega\setminus\Omega_+}{\leq} \sup_{\Omega_+} u \overset{II.2.1}{=} \sup_{\partial\Omega_+} u = \sup_{\partial\Omega_+} u_+ \leq \sup_{\partial\Omega} u_+.$$

In the last step we used that $\partial\Omega_+ \subset \overline{\Omega}$ can be split into two parts: the part $\partial\Omega_+ \subset \Omega$ (on this part we have $u = u_+ = 0$), and the part $\partial\Omega_+ \subset \partial\Omega$ where $u_+ \geq 0$.

This settles the claim for $-Lu \leq 0$, the claim for $-Lu \geq 0$ is a similar argument.

For the last case assume that $-Lu = 0$. By the arguments before we have then (observe that $|u| = u_+ + u_-$).

$$\sup_\Omega u \leq \sup_{\partial\Omega} u_+ \leq \sup_{\partial\Omega} |u|.$$

and

$$\inf_\Omega u \geq \inf_{\partial\Omega}(-u_-),$$

which can be rewritten as

$$-\inf_\Omega u \leq -\inf_{\partial\Omega}(-u_-) = \sup_{\partial\Omega}(u_-) \leq \sup_{\partial\Omega} |u|.$$

Now at least one of the following cases holds:

$$\sup_\Omega |u| = \sup_\Omega u, \quad \text{or} \quad \sup_\Omega |u| = -\inf_\Omega u$$

but in both cases the estimates above imply

$$\sup_\Omega |u| \leq \sup_{\partial\Omega} |u|$$

□

Example II.2.3 (Counterexample for $c \geq 0$)**.** Consider

$$Lu = \Delta u + 5u$$

for $\Omega = (-1,1) \times (-1,1)$. Then

$$u = (1 - x^2) + (1 - y^2) + 1$$

satisfies

$$-Lu = -1 \leq 0.$$

However,

$$\sup_{\Omega} u \geq u(0) = 3,$$

and

$$\sup_{\partial\Omega} u = 1.$$

As it was the case for the Δ-operator, Theorem I.2.13, the weak maximum principle implies uniqueness results.

Corollary II.2.4 (Uniqueness for the Dirichlet problem)**.** *Let L be as above a non-divergence form linear elliptic operator, $\Omega \subset\subset \mathbb{R}^n$ with smooth boundary, $c \leq 0$, $f \in C^0(\Omega)$, $g \in C^0(\partial\Omega)$. Then there exists at most one solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of the Dirichlet boundary problem*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Proof. exercise. □

Corollary II.2.5 (Comparison principle)**.** *Let L be a linear elliptic differential operator (non-divergence form), and assume that $c \leq 0$ in $\Omega \subset\subset \mathbb{R}^n$. Let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy $-Lu \leq -Lv$ in Ω. Then $u \leq v$ on $\partial\Omega$ implies $u \leq v$ in Ω.*

Proof. exercise □

Corollary II.2.6 (Continuous dependence on data)**.** *Let L be a linear elliptic differential operator (non-divergence form), and assume that $c \leq 0$ in $\Omega \subset\subset \mathbb{R}^n$.*

Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$\begin{cases} -Lu = f & \text{in } \Omega \\ u = g & \text{in } \Omega \end{cases}$$

where $f \in C^0(\overline{\Omega})$ and $g \in C^0(\partial\Omega)$.

Then for some constant $C = C(b, \Lambda)$ we have

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |g| + C(b, \Lambda) \sup_{\Omega} |f|.$$

PROOF. exercise □

Our next goal is the the strong maximum principle, for this we use the following result by Hopf:

Lemma II.2.7 (Hopf Boundary point Lemma)**.** *Let $B \subset \mathbb{R}^n$ be a ball, and let L be as above. Let $u \in C^2(B) \cap C^0(\overline{B})$ and assume that for $x_0 \in \partial B$ we have*

- *$u(x) < u(x_0)$ for all $x \in B$*
- *$-Lu \leq 0$ in B.*
- *One of the following*
  (1) $c \equiv 0$
  (2) $c \leq 0$ and $u(x_0) \geq 0$
  (3) $u(x_0) = 0$
  Then for ν the outwards facing normal of B at x_0 (i.e. if $B = B(y_0, \rho)$ then for $\nu = \frac{x_0 - y_0}{\rho}$
  $$\partial_\nu u(x_0) > 0,$$
  if that derivative exists.

PROOF. W.l.o.g. we may assume

$$B = B(0, R), \quad c \leq 0, \quad u(x_0) = 0, \quad u < 0 \quad \text{in } B(0, R): \tag{II.2.3}$$

Indeed, the condition $B = B(0, R)$ can be assumed simply by shifting. As for the other conditions set (recall that $c_+ = \max\{c, 0\}$)

$$\tilde{L} := L - c_+.$$

and

$$\tilde{u} := u - u(x_0).$$

Then in B,

$$-\tilde{L}\tilde{u} = -(L - c_+)(u - u(x_0)) = -Lu + c_+ u + cu(x_0) - c_+ u(x_0) \leq c_+ (u - u(x_0)) + c\, u(x_0)$$

If $c \equiv 0$ then we readily have $-\tilde{L}\tilde{u} \leq 0$.

If $c \leq 0$ we have $c_+ \equiv 0$, and again obtain $-\tilde{L}\tilde{u} \leq 0$.

If $u(x_0) = 0$ then $c_+ u \leq 0$, since $u \leq u(x_0) = 0$ by assumption.

Since $c - c_+ \leq 0$ we observe that $\tilde{L}$ is an operator that satisfies the missing conditions in (II.2.3). Thus, indeed, (II.2.3) can be assumed w.l.o.g.

So assume (II.2.3) from now on.

Set for some $\alpha > 0$

$$v_\alpha(x) := e^{-\alpha|x|^2} - e^{-\alpha R^2}.$$

Clearly $0 \le v_\alpha \le 1$ in $B = B(0,R)$. Moreover

$$v_\alpha \equiv 0 \quad \text{on } \partial B(0,R).$$

For $\rho \in (0,R)$ denote by $A(\rho,R)$ the annulus $B(0,R)\backslash B(0,\rho)$. We will show next

(II.2.4) For any $\rho \in (0,R)$ there exists $\alpha > 0$ such that $\quad -Lv_\alpha < 0 \quad$ in $A(\rho,R)$

For this we first compute

$$\partial_i v_\alpha(x) = -2\alpha x_i e^{-\alpha|x|^2}. \tag{II.2.5}$$

Next we compute

$$\partial_{ij} v_\alpha(x) = \left(-2\alpha\delta_{ij} + 4\alpha^2 x_i x_j\right) e^{-\alpha|x|^2}$$

so (using the ellipticity conditions, $a_{ij}x_ix_j \ge |x|^2$, and $|a|, |b|, |c| \le \Lambda$,

$$\begin{aligned} -Lv(x) &= -a_{ij}\partial_{ij}v - b_i\partial_i v - cv \\ &= -a_{ij}\left(-2\alpha\delta_{ij} + 4\alpha^2 x_i x_j\right) e^{-\alpha|x|^2} - b_i\left(-2\alpha x_i e^{-\alpha|x|^2}\right) - ce^{-\alpha|x|^2} + \underbrace{ce^{-\alpha R^2}}_{\le 0} \\ &\le \left(2\alpha\Lambda - 4\alpha^2\Lambda|x|^2 + 2\alpha\Lambda|x| + \Lambda\right) e^{-\alpha|x|^2}. \end{aligned}$$

That is, for $x \in A(\rho,R)$,

$$-Lv(x) \le \underbrace{\left(-4\alpha^2\Lambda\rho^2 + 2\alpha\Lambda + 2\alpha\Lambda R + \Lambda\right)}_{\le 0 \text{ for } \alpha \gg 1} e^{-\alpha|x|^2}$$

If we take α large, the (negative) α^2-term dominates, that is for $\alpha \gg 1$ (depending on $\rho > 0$), Λ and R we have (II.2.4).

Next, we consider the equation for $u + \varepsilon v$, which in view of (II.2.4) becomes

$$-L(u + \varepsilon v) < 0 \quad \text{in } A(\rho,R).$$

The weak maximum principle, Theorem II.2.2, implies

$$\sup_{A(\rho,R)} u + \varepsilon v \le \sup_{\partial A(\rho,R)} (u + \varepsilon v)_+. \tag{II.2.6}$$

The boundary $\partial A(\rho,R)$ is the union of $\partial B(0,R)$ where $v \equiv 0$ and since u is continuous and $u < 0$ in $B(0,R)$ we have $u \le 0$ on $\partial B(0,R)$. That is $(u + \varepsilon v)_+ = 0$ on $\partial B(0,R)$.

On $\partial B(0,\rho)$, since $u < 0$ on $B(0,R)$ we have $\sup_{\partial B(0,\rho)} u < 0$, and consequently, since $v \le 1$ we have for all $0 < \varepsilon < \varepsilon_0 := -\sup_{\partial B(0,\rho)} u$

$$u + \varepsilon v < 0 \quad \text{on } \partial B(0,\rho)$$

That is (II.2.6) implies

$$u + \varepsilon v \le 0 \quad \text{in } A(\rho,R). \tag{II.2.7}$$

Now fix $\rho \in (0,R)$, choose ε, α so that the above is true.

Denote $\nu := \frac{x_0}{|x_0|}$ the outwards unit normal to ∂B at $x_0 \in \partial B$. Observe that for all small $0 < t \ll 1$ (depending on ρ) we have $x_0 - t\nu \in A(\rho,R)$.

Recall that by assumption $u(x_0) = 0$, then (II.2.7) implies for any small $t > 0$,

$$u(x_0 - t\nu) + \varepsilon v(x_0 - t\nu) \overset{\text{(II.2.7)}}{\leq} 0 = u(x_0) + \varepsilon v(x_0).$$

This leads to (again: for all $0 < t \ll 1$)

$$\frac{u(x_0 - t\nu) - u(x_0)}{t} \leq -\varepsilon \frac{v(x_0 - t\nu) - v(x_0)}{t}$$

Letting $t \to 0^+$ on both sides we obtain

$$-\partial_\nu u(x_0) \leq \varepsilon \partial_\nu v(x_0). \tag{II.2.8}$$

Observe that (II.2.5) implies

$$\partial_\nu v(x_0) = \partial_i v(x_0) \frac{(x_0)_i}{R} = -2\alpha \frac{|x_0|^2}{R} e^{-\alpha R^2} < 0$$

That is (II.2.8) implies

$$-\partial_\nu u(x_0) < 0$$

which implies the claim. □

The Hopf Lemma, Lemma II.2.7 implies the strong maximum principle.

Corollary II.2.8 (Strong maximum principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open and connected set, (but Ω may be unbounded). Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy*

$$-Lu \leq 0 \quad \text{in } \Omega.$$

If $c \equiv 0$ or if $c \leq 0$ and $\sup_\Omega u \geq 0$ then we have the following.

If there exists $x_0 \in \Omega$ such that

$$u(x_0) = \sup_\Omega u$$

then $u \equiv u(x_0)$ in Ω.

Proof. Assume the claim is false. Via the modification as in the proof of Lemma II.2.7, we may assume w.l.o.g. $u \leq 0$ in Ω and $u(x_0) = 0$ for some $x_0 \in \Omega$, but $u \not\equiv 0$.

Let

$$\Omega_- := \{x \in \Omega : \quad u(x) < 0\}.$$

Observe that Ω_- is open (u is continous) and $\Omega_- \neq \emptyset$ (because $u \leq 0$ and $u \not\equiv 0$).

Observe that the boundary of Ω_- does cannot be contained in $\partial\Omega$, i.e.

$$\partial\Omega_- \cap \Omega \neq \emptyset.$$

Indeed this follows from connectedness: Let $\gamma \subset \Omega$ be a continuous path from x_0 to a point in Ω_-. Then there has to be a point on γ where γ leaves Ω_-. This point lies in $\partial\Omega_-$ and in Ω.

This means we can find a point $x_1 \in \Omega_-$ which is close to $\partial\Omega_-$ but not close to $\partial\Omega$, i.e.

$$x_1 \in \Omega_-, \quad \rho := \operatorname{dist}(x_1, \partial\Omega_-) < 10\operatorname{dist}(x_1, \partial\Omega).$$

By definition of the distance

$$B(x_1, \rho) \subset \Omega_-, \quad \overline{B(x_1, \rho)} \backslash \Omega_- \neq \emptyset.$$

Let $x_2 \in \partial B(x_1, \rho) \backslash \Omega_-$. Since by construction $x_2 \in \partial\Omega_- \cap \Omega$ we have $u(x_2) = 0$ by continuity. Moreover $u < 0$ in $B(x_1, \rho) \subset \Omega_-$.

Since everything takes place well within Ω, the conditions of the Hopf Lemma, Lemma II.2.7, are satisfied and thus for ν the outwards facing normal at x_2 to $\partial B(x_1, \rho)$

$$\partial_\nu u(x_2) > 0.$$

But on the other hand $x_2 \in \Omega$ is a local maximum for u, so $Du(x_2) = 0$, which is a contradiction. The claim is then proven. □

A consequence of the Hopf Lemma, Lemma II.2.7, and the strong maximum principle, Corollary II.2.8, is the uniqueness for the Neumann problem.

Corollary II.2.9 (Uniqueness for Neumann-boundary problem)**.** *Let $\Omega \subset\subset \mathbb{R}^n$ be open and connected. Moreover we assume a boundary regularity of $\partial\Omega$, the interior sphere condition*[1]*:*

Assume that for any $x_0 \in \partial\Omega$ there exists a ball $B \subset \Omega$ such that $x_0 \in \overline{B}$.

Then the following holds for any elliptic operator as above with $c \equiv 0$: For any given $f \in C^0(\Omega)$ and any $g \in C^0(\partial\Omega)$ there is at most one solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of the Neumann boundary problem

$$\begin{cases} -Lu = f & \text{in } \Omega \\ \partial_\nu u = g & \text{on } \partial\Omega, \end{cases}$$

up to constant functions. That means, the difference of two solutions u, v is constant, $u - v \equiv c$.

Proof. The difference of two solutions u, v, $w := u - v$ satisfies[2]

$$\begin{cases} -Lw \leq 0 & \text{in } \Omega \\ \partial_\nu w = 0 & \text{on } \partial\Omega, \end{cases}$$

Firstly, assume that there exists $x_0 \in \Omega$ such that $\sup_\Omega w = w(x_0)$. Then, by the strong maximum principle, Corollary II.2.8, we have $w \equiv w(x_0)$ and the claim is proven. If this is not the case, then there must be $x_0 \in \partial\Omega$ with $w(x_0) > w(x)$ for all $x \in \Omega$. If we take a ball from the interior sphere condition of $\partial\Omega$ at x_0 then on this ball B we can apply Hopf

[1]This condition does not allow for outwards facing cusps. One can show that every set Ω whose boundary $\partial\Omega$ is a sufficiently smooth manifold satisfies the interior sphere condition

[2]actually we have $=$ in the equation below, but the argument works for $\leq$ as well

Lemma, Lemma II.2.7, which leads to $\partial_\nu w(x_0) > 0$, which is ruled out by the Neumann boundary assumption $\partial_\nu w = 0$. $\square$

CHAPTER 3

Sobolev Spaces

III.1. Basic concepts from Functional Analysis

We will always consider real vector spaces with a norm $(X, \|\cdot\|)$, where the norm needs to satisfy

- for all $x \in X$: $\|x\|_X = 0 \Leftrightarrow x = 0$
- $\|\lambda x\|_X = |\lambda| \, \|x\|_X$ for all $x \in X$ and $\lambda \in R$.
- $\|x + y\|_X \leq \|x\|_X + \|y\|_X$ for all $x, y \in X$.

A normed vector space is a metric space via the metric $d_X(x, y) := \|x - y\|$.

A normed vector space $(X, \|\cdot\|)$ is complete if every Cauchy sequence has a limit in X. We then say X is a Banach space (with the norm $\|\cdot\|$).

Sometimes normed spaces have an interior product (aka scalar product), $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ which satisfy

- $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in X$.
- $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$ for all $x, y, z \in X$ and $\lambda, \mu \in \mathbb{R}$.

If X has such an interior product, then then it is called pre-Hilbert. If X is moreover complete (i.e. a Banach space) then it is called a Hilbert space.

Between Banach spaces X, Y we define

$$L(X, Y) := \{T : X \to Y : \quad T \text{ continuous and linear}\},$$

which is a Banach space with the norm

$$\|T\|_{L(X,Y)} := \sup_{\|x\|_X \leq 1} \|Ty\|_Y.$$

The dual space $X^* = L(X, \mathbb{R})$ is the space of continuous linear functionals on X (called the dual space of X).

Example III.1.1 (Example 1: Hölder spaces)**.** The space $C^{k,\gamma}(\overline{\Omega})$ for $k \in \mathbb{N}_0$ and $\gamma \in [0, 1]$ is defined as the set of all functions $f \in C^0(\overline{\Omega})$ such that $f \in C^k(\Omega)$ and the k-the

derivatives of f, $D^k f$ are Hölder continuous

$$[D^k f]_{C^\gamma(\Omega)} := \sup_{x,y\in\Omega} \frac{|D^k f(x) - D^k f(y)|}{|x-y|^\gamma} < \infty$$

For $\gamma = 1$ we say this is Lipschitz continuity.

One norm is

$$\|f\|_{C^{k,\gamma}(\Omega)} := \|f\|_{L^\infty(\Omega)} + [D^k f]_{C^\gamma(\Omega)}.$$

where

$$\|f\|_{L^\infty(\Omega)} \equiv \|f\|_{C^0(\Omega)} := \sup_\Omega |f|.$$

Example III.1.2 (Example 2: Lebesgue spaces)**.** For $p \in [1,\infty)$ the class $f \in L^p(\Omega)$ is given as

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ measurable s.t.} \|f\|_{L^p(\Omega)} < \infty \right\} / \sim,$$

where

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p \right)^{\frac{1}{p}}$$

and we factor module the equivalence relation $\sim$ where

$$f \sim g \quad \text{if } f - g = 0 \text{ almost everywhere.}$$

For $p = \infty$ we set

$$\|f\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_\Omega |f| = \inf \{\Lambda > 0 : \mathcal{L}^n(|f| > \Lambda) = 0\}.$$

For $p = 2$ L^p has a scalar product

$$\langle f, g\rangle_{L^2(\Omega)} := \int_\Omega fg.$$

Observe that the set of functions $C_c^\infty(\Omega)$ has no proper norm. Often one resorts to using the Schwartz class of functions which is at least nicely metrizable.

We will work a lot with L^p-spaces, so let us state some of the basic properties:

Lemma III.1.3 (Hölder inequality)**.** *For $1 \le p, q, r \le \infty$, if $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ we have*

$$\|fg\|_{L^r(\Omega)} \le \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}$$

whenever $f \in L^p(\Omega)$, $g \in L^q(\Omega)$.

In particular we have

$$\int_\Omega fg \le \|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)}$$

where we denote (from now on) $p' = \frac{p}{p-1}$ the *Hölder dual.*

Moreover, $f \in L^p(\Omega)$ implies $f \in L^q_{loc}(\Omega)$ for any $1 \le q \le p$. (Set $g := \chi_\Omega \in L^{q'}_{loc}$ and use Hölder)

Dual spaces of L^p are characterized, namely:

THEOREM III.1.4 (Riesz Representation theorem). *Let $1 < p < \infty$. Assume that*

$$T \in (L^p(\Omega))^*,$$

i.e. $T : L^p(\Omega) \to \mathbb{R}$ is linear, and bounded:

$$|Tf| \leq \|f\|_{L^p(\Omega)} \quad \text{for all } f \in L^p(\Omega)$$

Then for $q = p' = \frac{p}{p-1}$ we for some $g \in L^q(\Omega)$

$$Tf = \int_\Omega fg.$$

THEOREM III.1.5 (Density of smooth functions in $L^p(\Omega)$). *For $1 \leq p < \infty$ smooth, compactly supported functions are dense in $L^p(\Omega)$, that is for every $f \in L^p(\Omega)$ there exists $f_k \in C_c^\infty(\Omega)$ such that*

$$\|f - f_k\|_{L^p(\Omega)} \xrightarrow{k\to\infty} 0.$$

In particular, in view of Hahn-Banach theorem, we can weaken a little bit the condition of Theorem III.1.4.

Corollary III.1.6 (Riesz Representation theorem)**.** *Let $1 < p < \infty$. Assume that $T : C_c^\infty(\Omega) \to \mathbb{R}$ is linear, and bounded in $L^p(\Omega)$:*

$$|Tf| \leq \|f\|_{L^p(\Omega)} \quad \text{for all } f \in C_c^\infty(\Omega).$$

Then for $q = p' = \frac{p}{p-1}$ we for some $g \in L^q(\Omega)$

$$Tf = \int_\Omega fg \quad \text{for all } f \in C_c^\infty(\Omega.$$

In particular, Tf can be extended to a bounded, linear operator on all of $L^p(\Omega)$.

A nice consequence for $p \in (1, \infty)$:

$$\|f\|_{L^p(\Omega)} = \sup_{\varphi \in C_c^\infty(\Omega),\, \|\varphi\|_{L^{p'}(\Omega)} \leq 1} \int_\Omega f\,\varphi$$

III.2. Philosophy of Distributions and Sobolev spaces

Let $f \in C^0(\Omega)$. From the first time we learned about functions we thought about them as a collection of points $x \in \Omega$ with there respective value $f(x)$. I.e.

$$f \in C^0(\Omega) \quad \leftrightarrow \quad \{(x, f(x)) \quad x \in \Omega\}$$

In some sense we test/sample f at every test-point x to obtain $f(x)$ and a representation for f.

The idea of distributions[1]is that we sample functions not at test-points $x \in \Omega$ but at test-functions $\varphi \in C_c^\infty(\Omega)$, i.e. we identify

$$f \in C^0(\Omega) \quad \leftrightarrow \quad \{(\varphi, f[\varphi]) \quad \varphi \in C_c^\infty(\Omega)\} \tag{III.2.1}$$

where $f[\varphi]$ measures the action of f on φ (from the functional analytic perspective we identify f with an operator of a dual space of a function space containing C_c^∞, and this is called the *distributional* interpretation of functions),

$$f[\varphi] := \int_\Omega f\varphi.$$

For continuous functions we know that the relation (III.2.1) is one-to-one on continuous functions, this was used for Weyl's lemma, Proposition I.2.15.

Indeed we can substantially weaken (III.2.1) and have that the following is a one-to-one correspondence, for any $p \in [1, \infty)$.

$$f \in L^p(\Omega) \quad \leftrightarrow \quad \{(\varphi, f[\varphi]) \quad \varphi \in C_c^\infty(\Omega)\} \tag{III.2.2}$$

One-to-one means in particular that $f[\varphi] = g[\varphi]$ implies $f = g$ in L^1, which is the fundamental theorem of calculus of variations, Lemma I.2.24.

Clearly we have that the map

$$\varphi \mapsto f[\varphi]$$

is linear in φ. Moreover, by Hölder's inequality, we have boundedness of $f[\varphi]$ as a linear operator acting on $L^{p'}(\Omega)$.

$$|f[\varphi]| \leq \|f\|_{L^p(\Omega)} \|\varphi\|_{L^{p'}(\Omega)} \quad \forall \varphi \in C_c^\infty(\Omega).$$

In view of Riesz Representation Theorem, Corollary III.1.6 linearity and boundedness of $f[\varphi]$ are enough to *find* f as a map in $L^p(\Omega)$.

Let us now talk about derivatives. For smooth functions $f \in C^1(\Omega)$ we have, by integration by parts,

$$(\partial_i f)[\varphi] = \int_\Omega \partial_i f\, \varphi = -\int_\Omega f\, \partial_i \varphi = f[-\partial_i \varphi] \quad \forall \varphi \in C_c^\infty(\Omega).$$

Why not define this for everything. If $T : C_c^\infty(\Omega) \to \mathbb{R}$ is a *distribution* (i.e. linear in φ) then we can just *define* the *distributional derivative* $\partial_i T$ as

$$(\partial_i T)[\varphi] := T[-\partial_i \varphi].$$

In this sense every distribution has infinitely many derivatives (in distributional sense). But not every *function* has a derivative.

Example III.2.1. Take the Heaviside function:

$$f(x) := \begin{cases} 1 & x > 0 \\ 0 & x < 0. \end{cases}$$

[1]Observe that the Fourier Transform is a somewhat similar idea: For $f \in L^2(\mathbb{R}^n)$ it is equivalent to consider $(x, f(x))$ for almost all $x \in \mathbb{R}^n$, or $(\xi, \hat{f}(\xi))$

This function is measurable, locally integrable, but certainly not continuous in $x = 0$.

The pointwise derivative of f is almost everywhere zero (since f is almost everywhere constant). But this is not true for the distributional derivative: For any $\varphi \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned} f'[\varphi] &= -\int_{\mathbb{R}} f\,\varphi' = -\int_{\mathbb{R}} f\,\varphi' \\ &= -\int_{-\infty}^{0} f\,\varphi' - \int_{0}^{\infty} f\,\varphi' \\ &= -\int_{-\infty}^{0} 0\,\varphi' - \int_{0}^{\infty} 1\,\varphi' \\ &= -\int_{0}^{\infty} \varphi' \\ &= \varphi(0). \end{aligned}$$

So, in the distributional sense,

$$f' = \delta_0$$

where δ_0 is the *dirac measure*.

$$\int_{\mathbb{R}} \varphi d\delta_0 = \varphi(0).$$

This is an important lesson: pointwise derivative do not necessarily predict distributional derivatives.

Also we see, the distributional derivative of a function may not be again a function (there is no function representing the dirac-measure).

But sometimes distributional derivatives of a function is again a function.

For example for any C^1-function its distributional derivative $\partial_i f[]$ coincides with its usual derivative $\partial_i f$.

Sobolev spaces $W^{1,p}$ are the function spaces of L^p-functions whose distributional derivative is also an L^p-function. That means, $f \in W^{1,p}(\Omega)$ if $f \in L^p(\Omega)$ and if there exist $g_i \in L^p(\Omega)$, $1 \le i \le n$, such that $\partial_i f = g_i$ in the sense of distributions, i.e.

$$\int_\Omega f\,\partial_i \varphi = -\int_\Omega g\,\varphi \quad \forall \varphi \in C_c^\infty(\Omega).$$

When we have this condition satisfied we will not bother with saying g is $\partial_i f$ in the sense of distribution, but simply denote g by $\partial_i f$ (it is unique by the fundamental theorem of calculus of variations, Lemma I.2.24).

Again observe: *pointwise derivative are not the same as the distributional derivative*, e.g. the Heaviside-function above does not belong to the Sobolev space $W^{1,p}$, even though its pointwise derivative is almost everywhere zero.

III.3. Sobolev Spaces

A remark on literature: A standard reference for Sobolev spaces is [**Adams and Fournier, 2003**]. Very readable is also [**Evans and Gariepy, 2015**]. The introduction here takes a lot from the introduction to Sobolev spaces in [**Evans, 2010**]. A classical reference Sobolev spaces in PDEs is [**Gilbarg and Trudinger, 2001**]. Also [**Ziemer, 1989**]. For very delicate problems one might also consult [**Maz'ya, 2011**].

Definition III.3.1. (1) Let $1 \leq p \leq \infty$, $k \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ open, nonempty. The *Sobolev space* $W^{k,p}(\Omega)$ is the set of functions

$$u \in L^p(\Omega)$$

such that for any multiinidex γ, $|\gamma| \leq k$ we find a function (the *distributional γ-derivative* or *weak γ-derivative*) *"$\partial^\gamma u$"*$\in L^p(\Omega)$ such that

$$\int_\Omega u\, \partial^\gamma \varphi = (-1)^{|\gamma|} \int_\Omega \text{“}\partial^\gamma u\text{”}\, \varphi \quad \forall \varphi \in C_c^\infty(\Omega).$$

Such u are also sometimes called Sobolev-functions.

(2) For simplicity we write $W^{0,p} = L^p$.

(3) The norm of the Sobolev space $W^{k,p}(\Omega)$ is given as

$$\|u\|_{W^{k,p}(\Omega)} = \sum_{|\gamma| \leq k} \|\partial^\gamma u\|_{L^p(\mathbb{R}^n)}$$

or equivalently (exercise!)

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\gamma| \leq k} \|\partial^\gamma u\|^p_{L^p(\mathbb{R}^n)} \right)^{\frac{1}{p}}.$$

(4) We define another Sobolev space $H^{k,p}(\Omega)$ as follows

$$H^{k,p}(\Omega) = \overline{C^\infty(\overline{\Omega})}^{\|\cdot\|_{W^{k,p}(\Omega)}}.$$

that is the (metric) closure or completion of the space $(C^\infty(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$. In yet other words, $H^{k,p}(\Omega)$ consists of such functions $u \in L^p(\Omega)$ such that there exist approximations $u_k \in C^\infty(\overline{\Omega})$ with

$$\|u_k - u\|_{W^{k,p}(\Omega)} \xrightarrow{k \to \infty} 0.$$

We will later see that $H^{k,p}$ is the same as $W^{k,p}$ locally, or for nice enough domains; and use the notation H or W interchangeably. For $k = 0$ this fact follows from Theorem III.1.5 for any open set Ω.

(5) Now we introduce the Sobolev space $H_0^{k,p}(\Omega)$

$$H_0^{k,p}(\Omega) = \overline{C_c^\infty(\overline{\Omega})}^{\|\cdot\|_{W^{k,p}(\Omega)}}.$$

We will later see that this space consists of all maps $u \in H^{k,p}(\Omega)$ that satisfy $u, \nabla u, \ldots \nabla^{k-1} u \equiv 0$ on $\partial\Omega$ in a suitable sense (the *trace sense*, for a precise

formulation see Theorem III.3.21). – Again, later we see that $H = W$ and thus, $W_0^{k,p}(\Omega) = H_0^{k,p}(\Omega)$ for nice sets Ω.

Observe that in view of Theorem III.1.5, $L^p(\Omega) = W^{0,p}(\Omega) = W_0^{0,p}(\Omega)$.

(6) The local space $W_{loc}^{k,p}(\Omega)$ is similarly defined as $L_{loc}^p(\Omega)$. A map belongs to $u \in W_{loc}^{k,p}(\Omega)$ if for any $\Omega' \subset\subset \Omega$ we have $u \in W^{k,p}(\Omega')$.

Remark III.3.2. Some people write $H^{k,p}(\Omega)$ instead of $W^{k,p}(\Omega)$. Other people use $H^k(\Omega)$ for $H^{k,2}$ – notation is inconsistent...

Some people claim that W stand for Weyl, and H for Hardy or Hilbert.

Example III.3.3. For $s > 0$ let

$$f(x) := |x|^{-s}.$$

Observe that f is only defined for $x \neq 0$, but since measurable functions need only be defined outside of a null-set this is still a reasonable function.

We have already seen, when working with fundamental solutions, that $f \in L_{loc}^p(\mathbb{R}^n)$ for any $1 \leq p < \frac{n}{s}$.

We can compute for $x \neq 0$ that

$$\partial_i f(x) = -s\,|x|^{-s-2} x^i \tag{III.3.1}$$

and by the same argument as above we could conjecture that $\partial_i f \in L_{loc}^q(\mathbb{R}^n)$ for any $1 \leq q < \frac{n}{s+1}$.

It is an exercise to show that

(1) (III.3.1) holds in the distributional sense, i.e. that if $n \geq 2$ and $0 < s < n-1$ then for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f(x)\,\partial_i\varphi(x)\,dx = \int_{\mathbb{R}^n} s\,|x|^{-s-2} x^i \varphi(x)\,dx.$$

(2) to conclude that $f \in W_{loc}^{1,q}(\mathbb{R}^n)$ for any $1 \leq q < \frac{n}{s+1}$.

Example III.3.4. Let

$$f(x) := \log|x|.$$

One can show that $f \in L_{loc}^p(\mathbb{R}^n)$ for any $1 \leq p < \infty$, and $f \in W_{loc}^{1,p}(\mathbb{R}^n)$ for all $p \in [1, n)$, if $n \geq 2$.

Example III.3.5. Let

$$f(x) := \log\log\frac{2}{|x|} \quad \text{in } B(0,1)$$

One can show that for $n \geq 2$, $f \in W^{1,n}(B(0,1))$.

Moreover, for $n = 2$, in distributional sense

$$\Delta f = |Df|^2$$

Observe that this serves as an example for solutions to nice differential equations that are not continuous!

Proposition III.3.6 (Basic properties of weak derivatives)**.** *Let $u, v \in W^{k,p}(\Omega)$ and $|\gamma| \leq k$. Then*

(1) $\partial^\gamma u \in W^{k-|\gamma|,p}(\Omega)$.
(2) Moreover $\partial^\alpha \partial^\beta u = \partial^\beta \partial^\alpha u = \partial^{\alpha+\beta} u$ if $|\alpha| + |\beta| \leq k$.
(3) For each $\lambda, \mu \in \mathbb{R}$ we have $\lambda u + \mu v \in W^{k,p}(\Omega)$ and

$$\partial^\alpha(\lambda u + \mu v) = \lambda \partial^\alpha u + \mu \partial^\alpha v$$

(4) If $\Omega' \subset \Omega$ is open then $u \in W^{k,p}(\Omega')$
(5) For any $\eta \in C_c^\infty(\Omega)$, $\eta u \in W^{k,p}$ and (if $k \geq 1$), and we have the Leibniz formula (aka product rule)

$$\partial_i(\eta u) = \partial_i \eta \; u + \eta \partial_i u.$$

Proof. (1) We show that $\partial_i u \in W^{k-1,p}(\Omega)$, only. The general statement then follows accordingly. By definition of the distributional derivative we have that $\partial_i u \in L^p(\Omega)$. For any $|\beta| \leq k-1$ and $\varphi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega \partial_i u \, \partial^\beta \varphi = -\int_\Omega u \, \partial_i \partial^\beta \varphi = -(-1)^{|\beta|+1} \int_\Omega \partial_i \partial^\beta u \, \varphi = (-1)^{|\beta|} \int_\Omega \partial_i \partial^\beta u \, \varphi.$$

The first inequality comes from the fact that $\partial^\beta \varphi \in C_c^\infty(\Omega)$ and from the definition of the weak derivative ∂_i. The second equation comes from the definition of the weak derivative of $\partial_i \partial^\beta$ for $W^{k,p}$-functions.

(2) We show $\partial_i \partial_j u = \partial_j \partial_i u$, again the general case follows. And as above this is proven by deducing respective properties from the properties in the space of test-funtions: For $\varphi \in C_c^\infty(\Omega)$ we have $\partial_i \partial_j \varphi = \partial_j \partial_i \varphi$, and thus

$$\int_\Omega \partial_i \partial_j u \, \varphi = \int_\Omega u \, \partial_i \partial_j \varphi = \int_\Omega u \, \partial_j \partial_i \varphi = \int_\Omega \partial_j \partial_i u \, \varphi.$$

(3) Follows from the linearity of the definition of weak derivative and the equivalent statements for smooth functions $\varphi \in C_c^\infty(\Omega)$

(4) If $\Omega' \subset \Omega$ then any $\varphi \in C_c^\infty(\Omega')$ belongs also to $C_c^\infty(\Omega)$. That is any property true for test functions $\varphi \in C_c^\infty(\Omega)$ holds also for testfunctions in $\varphi \in C_c^\infty(\Omega')$.

(5) For $\varphi \in C_c^\infty(\Omega)$ we have by the usual Leibniz rule

$$\begin{aligned}
\int_\Omega \eta u \partial_i \varphi &= \int_\Omega u \, \partial_i(\eta \varphi) - \int_\Omega u \, \partial_i \eta \; \varphi \\
&= -\int_\Omega \partial_i u \, \eta \varphi - \int_\Omega u \, \partial_i \eta \; \varphi \\
&= -\int_\Omega (\partial_i u \, \eta + u \, \partial_i \eta) \; \varphi
\end{aligned}$$

The second equation is the definition of weak derivative $\partial_i u$ (since $\eta \varphi \in C_c^\infty(\Omega)$ is a permissible testfunction).

That is we have shown for all $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \eta u \partial_i \varphi = \int_\Omega u\, \partial_i(\eta\varphi) - \int_\Omega u\, \partial_i \eta\ \varphi.$$

This means that in distributional sense $\partial_i(\eta u) = \partial_i \eta\ u + \eta \partial_i u$. Now observe that $\eta u \in L^p(\Omega)$ and $\partial_i \eta\ u + \eta \partial_i u \in L^p(\Omega)$, so $\eta u \in W^{1,p}(\Omega)$.

□

Proposition III.3.7. $(W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$, $(H^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$, $(H_0^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$ *are all Banach spaces.*

For $p = 2$ they are Hilbert spaces, with inner product

$$\langle u, v\rangle = \sum_{|\gamma|\le k} \int \partial^\gamma u\, \partial^\gamma v.$$

Proof. $\|\cdot\|_{W^{k,p}(\Omega)}$ is a norm. By definition $(H^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$, $(H_0^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$ are complete and thus Banach spaces.

As for the completeness of $W^{k,p}(\Omega)$, it essentially follows from the completeness of $L^p(\Omega)$.

Let $(u_i)_{i\in\mathbb{N}} \subset W^{k,p}(\Omega)$ be a Cauchy sequence of $W^{k,p}$-functions, i.e.

$$\forall \varepsilon > 0 \exists N = N(\varepsilon) \in \mathbb{N} \quad \text{s.t. } \forall i, j \ge N: \quad \|u_i - u_j\|_{W^{k,p}(\Omega)} < \varepsilon.$$

We have to show that u_i converges to some $u \in W^{k,p}(\Omega)$ in the $W^{k,p}(\Omega)$-norm.

Observe that by the definition of the $W^{k,p}$-norm, if u_i is a Cauchy sequence for $W^{k,p}$, then for any $|\gamma| \le k$, $(\partial^\gamma u_i)_{i\in\mathbb{N}}$ are Cauchy sequences of $L^p(\Omega)$.

Since $L^p(\Omega)$ is a Banach space, i.e. complete, each $\partial^\gamma u_i$ converges in $L^p(\Omega)$ to some object which we call $\partial^\gamma u$,

$$\|\partial^\gamma u_i - \partial^\gamma u\|_{L^p(\Omega)} \xrightarrow{i\to\infty} 0 \quad \forall |\gamma| \le k.$$

Observe that as of now we do not know that $\partial^\gamma u$ is actually the weak derivative of u! But we can check this is the case.

Since $\partial^\gamma u_i$ is the weak derivative of u_i, we have

$$\int_\Omega \partial^\gamma u_i\, \varphi = (-1)^{|\gamma|} \int_\Omega u_i\, \partial^\gamma \varphi \quad \forall \varphi \in C_c^\infty(\Omega).$$

But on both sides we have strong convergence in $L^p(\Omega)$. For any (fixed) $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \partial^\gamma u\, \varphi \xleftarrow{i\to\infty} \int_\Omega \partial^\gamma u_i\, \varphi = (-1)^{|\gamma|} \int_\Omega u_i\, \partial^\gamma \varphi \xrightarrow{i\to\infty} (-1)^{|\gamma|} \int_\Omega u\, \partial^\gamma \varphi$$

and thus for any $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \partial^\gamma u\, \varphi = (-1)^{|\gamma|} \int_\Omega u\, \partial^\gamma \varphi.$$

That is, $\partial^\gamma u$ is indeed the weak derivative of u, thus $u \in W^{k,p}(\Omega)$ and by the definition of the $W^{k,p}$-norm

$$\|u_i - u\|_{W^{k,p}(\Omega)} \xrightarrow{i\to\infty} 0.$$

□

III.3.1. Approximation by smooth functions. We mentioned above the $H = W$ problem, i.e. we would like to approximate Sobolev functions by smooth functions. Why? Because then we don't have to deal that many times with the weak definition of derivatives, but show desired results for smooth functions, then pass to the limit and hopefully obtain the result for Sobolev maps. Observe that since $W^{k,p}(\Omega)$ is a Banach space, and $C^\infty(\overline{\Omega}) \subset W^{k,p}(\Omega)$ (exercise!) we clearly have $H^{k,p}(\Omega) \subset W^{k,p}(\Omega)$. for the other direction we now obtain the first result:

Proposition III.3.8 (Local approximation by smooth functions)**.** *Let $u \in W^{k,p}(\Omega)$, $1 \le p < \infty$. Set*

$$u_\varepsilon(x) := \eta_\varepsilon * u(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(y-x)\, u(y)\, dy.$$

Here $\eta_\varepsilon(z) = \varepsilon^{-n}\eta(z/\varepsilon)$ for the usual bump function $\eta \in C_c^\infty(B(0,1),[0,1])$, $\int_{B(0,1)} \eta = 1$. Then

(1) $u_\varepsilon \in C^\infty(\Omega_{-\varepsilon})$, where as before

$$\Omega_{-\varepsilon} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$$

for each $\varepsilon > 0$ such that $\Omega_{-\varepsilon} \neq \emptyset$.

(2) Moreoever for any $\Omega' \subset\subset \Omega$,

$$\|u_\varepsilon - u\|_{W^{k,p}(\Omega')} \xrightarrow{\varepsilon\to 0} 0.$$

Proof. (1) As in the proof of Theorem I.2.16 we have $u_\varepsilon \in C^\infty(\Omega_{-\varepsilon})$ – we do not need that u is a Sobolev function, but merely that $u \in L^p(\Omega)$.

(2) Next we claim that $\partial^\gamma u_\varepsilon(x) = (\partial^\gamma u)_\varepsilon(x)$ for $x \in \Omega_-$. Indeed, for $x \in \Omega_{-\varepsilon}$,

$$\partial^\gamma u_\varepsilon(x) = \int_\Omega \partial_x^\gamma \left(\eta_\varepsilon(x-z)\right) u(z)\, dz = (-1)^{|\gamma|} \int_\Omega \partial_z^\gamma \left(\eta_\varepsilon(x-z)\right) u(z)\, dz.$$

Now we observe that $\eta_\varepsilon(x-\cdot) \in C_c^\infty(\Omega)$ if $x \in \Omega_{-\varepsilon}$: observing size of the support of η_ε, $\operatorname{supp}\eta_\varepsilon \subset B(0,\varepsilon)$.

Thus by the definition of weak derivative,

$$(-1)^{|\gamma|} \int_\Omega \partial_z^\gamma \left(\eta_\varepsilon(x-z)\right) u(z)\, dz = \int_\Omega \eta_\varepsilon(x-z)\partial^\gamma u(z)\, dz = (\partial^\gamma u)_\varepsilon(x).$$

Now similar to the argument in the proof of Theorem I.2.16, for any $\Omega' \subset\subset \Omega$ and $\varepsilon < \operatorname{dist}(\Omega', \partial\Omega)$, for any $1 \le p < \infty$[2]

$$\|(\partial^\gamma u)_\varepsilon - \partial^\gamma u\|_{L^p(\Omega')} \xrightarrow{\varepsilon\to 0} 0.$$

[2]but not for $p = \infty$!

This holds for any γ such that $\partial^\gamma u \in L^p$, i.e. for all $|\gamma| \le k$. We conclude that

$$\|u_\varepsilon - u\|_{W^{k,p}(\Omega')} \xrightarrow{\varepsilon\to 0} 0.$$

□

Even though Proposition III.3.8 is only about local approximation, it is very useful to prove properties of Sobolev function.

Lemma III.3.9. *For $1 \le p < \infty$[3], if $v \in W^{1,p}(\Omega)$ and $f \in C^1(\mathbb{R},\mathbb{R})$ with $[f]_{\mathrm{Lip}\,(\mathbb{R})} \equiv \|f'\|_{L^\infty(\mathbb{R}^n)} < \infty$ then $f(v) \in W^{1,p}(\Omega)$, and we have in distributional sense*

$$\partial_\alpha(f(v)) = f'(v)\,\partial_\alpha v. \tag{III.3.2}$$

Proof. Let v_ε be the (local) approximation of v in $W^{1,p}_{loc}(\Omega)$ from Proposition III.3.8. First we observe that (III.3.2) is true if v was a differentiable function, in particular,

$$\partial_\alpha(f(v_\varepsilon)) = f'(v_\varepsilon)\,\partial_\alpha v_\varepsilon \quad \text{in } \Omega_{-\varepsilon}.$$

Now let $\varphi \in C^\infty_c(\Omega)$, and take ε_0 so small such that $\Omega' := \operatorname{supp}\varphi \subset \Omega_{-\varepsilon}$ for all $\varepsilon \in (0,\varepsilon_0)$. Then we have for all $\varepsilon < \varepsilon_0$,

$$\int_\Omega f(v_\varepsilon)\,\partial_\alpha\varphi = -\int_\Omega f'(v_\varepsilon)\,\partial_\alpha v_\varepsilon\,\varphi \tag{III.3.3}$$

Now we observe that $f(v_\varepsilon) \xrightarrow{\varepsilon\to 0} f(v)$ with respect to the $L^p(\Omega')$-norm. Indeed, observe that $\Omega' \subset\subset \Omega$, so by Proposition III.3.8,

$$\|f(v_\varepsilon) - f(v)\|_{L^p(\Omega')} \le \|f'\|_{L^\infty(\Omega')}\,\|v_\varepsilon - v\|_{L^p(\Omega')} \xrightarrow{\varepsilon\to 0} 0.$$

That is, the left-hand side of (III.3.3) converges (recall $\operatorname{supp}\partial_\alpha\varphi \subset \operatorname{supp}\varphi = \Omega'$)

$$\int_\Omega f(v)\,\partial_\alpha\varphi \equiv \int_{\Omega'} f(v)\,\partial_\alpha\varphi = \lim_{\varepsilon\to 0}\int_{\Omega'} f(v_\varepsilon)\,\partial_\alpha\varphi \equiv \lim_{\varepsilon\to 0}\int_\Omega f(v_\varepsilon)\,\partial_\alpha\varphi.$$

As for the right-hand side of (III.3.3) we have that $\partial_\alpha v_\varepsilon \xrightarrow{\varepsilon\to 0} v$ in $L^p(\Omega')$, and $f'(v_\varepsilon) \xrightarrow{\varepsilon\to 0} f'(v)$ almost everywhere in Ω (up to taking a subsequence $\varepsilon \to 0$)[4]. By dominated convergence this implies

$$\int_\Omega f'(v_\varepsilon)\,\partial_\alpha v_\varepsilon\,\varphi \xrightarrow{\varepsilon\to 0} \int_\Omega f'(v)\,\partial_\alpha v\,\varphi$$

Then from (III.3.3) we get the claim, observing that $f'(v)\partial_\alpha v \in L^p(\Omega)$, since $f' \in L^\infty$. □

[3]we can later conclude, using Theorem III.3.17, that this also holds for $p = \infty$, since then Sobolev maps are simply Lipschitz maps

[4]these are results from measure theory:, since f' is continuous, and since L^1-convergence implies almost everywhere convergence, see [**Evans and Gariepy, 2015**, Theorem 1.21]

Actually, a stronger statement is true: if $u \in W^{1,p}(\Omega)$ and $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, $f \in C^{0,1}$ then $f \circ u \in W^{1,p}(\Omega)$. Again, formally this is no problem since $\nabla(f \circ u) = Df(u)\,\nabla u$– since f is almost everywhere differentiable. But the precise proof is tedious. We just sketch the proof of a special case:

Lemma III.3.10. *Let $u \in W^{1,1}(\Omega)$, then $|u| \in W^{1,1}(\Omega)$.*

Moreover we have $Du = 0$ almost everywhere in $\{u(x) = 0\}$[5]*.*

Also we have

$$D|u| = \frac{u}{|u|} Du.$$

Proof. We only sketch the proof.

The difficulty lies in the fact that $|\cdot|$ is merely Lipschitz continuous, so we mollify it:

$$f_{\varepsilon,\theta}(t) := \sqrt{(t+\theta\varepsilon)^2 + \varepsilon^2} - \sqrt{(\theta\varepsilon)^2 + \varepsilon^2}.$$

$f_{\varepsilon,\theta}$ is a smooth function.

One approximates $|u|$ by $u_\varepsilon := f_{\varepsilon,\theta}(u)$ for some $\theta \in \mathbb{R}$

Since $f_{\varepsilon,\theta}$ is smooth we have in distributional sense, by Lemma III.3.9,

$$Du_\varepsilon = \frac{u + \varepsilon\theta}{\sqrt{(u+\varepsilon\theta)^2 + \varepsilon^2}}\, Du \xrightarrow{\varepsilon \to 0} Du \cdot \begin{cases} 1 & \text{in } \{u > 1\} \\ \frac{\theta}{\sqrt{\theta^2+1}} & \text{in } \{u = 0\} \\ -1 & \text{in } \{u < 1\} \end{cases}$$

Now $u_\varepsilon \to |u|$ in $L^1(\Omega)$, and Du_ε converges also in $L^1(\Omega)$. Using test functions and the convergence as $\varepsilon \to 0$ we get that

$$L^1(\Omega) \ni D|u| = Du \cdot \begin{cases} 1 & \text{in } \{u > 1\} \\ \frac{\theta}{\sqrt{\theta^2+1}} & \text{in } \{u = 0\} \\ -1 & \text{in } \{u < 1\} \end{cases}$$

But weak derivatives are unique as L^1-functions. The nonunique looks independent in θ. This means either $Du = 0$ almost everywhere in $\{u = 0\}$ or $\{u = 0\}$ is a zeroset (which still means that $Du = 0$ almost everywhere in $\{u = 0\}$). □

In general, crazy sets, it might be difficult to extend Proposition III.3.8 to the boundary (think of an open set whose boundary is the Koch-curve). To rule this out we make the following definition of C^k-boundary data

[5]Check this for smooth functions: Either $\{u(x) = 0\}$ is a zeroset. On the other hand, on the "substantial" parts of $\{u(x) = 0\}$ we should think of u as constant

Definition III.3.11 (Regularity of boundary of sets)**.** Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that $\partial\Omega \in C^k$ (more generally in $C^{k,\alpha}$) if $\partial\Omega \subset \mathbb{R}^n$ is a C^k (or $C^{k,\alpha}$, respectively) manifold, that is if

for any $x \in \partial\Omega$ there exists a radius $r > 0$ and a C^k-diffeomorphism $\Phi : B(x,r) \to B(0,r)$ (i.e. the map Φ is a bijection between $B(x,r)$ and $B(0,r)$ and Φ and Φ^{-1} are both of class C^k) such that

- $\Phi(x) = 0$
- $\Phi(\Omega \cap B(x,r)) = B(0,r) \cap \mathbb{R}^n_+$
- $\Phi(B(x,r)\backslash\Omega) = B(0,r) \cap \mathbb{R}^n_-$.

Theorem III.3.12 (Smooth approximation for Sobolev functions). *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and $\partial\Omega \in C^1$. For any $u \in W^{k,p}(\Omega)$ there exist a smooth approximating sequence $u_i \in C^\infty(\overline{\Omega})$ such that*

$$\|u_i - u\|_{W^{k,p}(\Omega)} \xrightarrow{i\to\infty} 0.$$

Proof. First we consider the situation close to the boundary.

Let $x_0 \in \partial\Omega$.

Observe first the following: If $x \in B(0,r)^+$ and $|z - x| < \varepsilon$ (for $\varepsilon \ll r$) then $x + \varepsilon e_n \subset B(x, 2r)^+$. Since $\partial\Omega$ belongs to C^1 one can show that the same holds (on sufficiently small balls $B(x_0, r)$) as well: For some $\lambda = \lambda(x_0)$, a unit vector $\nu = \nu(x_0)$, if $z \in B(x,\varepsilon)$ and $x \in \Omega \cap B(x_0, r)$ then

$$z + \lambda\varepsilon\nu \in \Omega.$$

One should think of ν the inwards facing unit normal at x_0 (which can be computed from the derivatives of Φ and is continuous around x_0).

That is for $x \in \Omega' := B(x_0, r/2) \cap \Omega$ we may set

$$u_\varepsilon(x) := \int_{\mathbb{R}^n} \eta_\varepsilon(z - x)\, u(z + \lambda\nu\varepsilon)dz = \int_{\mathbb{R}^n} \eta_\varepsilon(z - \lambda\nu\varepsilon - x)\, u(z)dz.$$

Clearly, u_ε is still smooth, but now in all of of $\overline{\Omega'}$. Moreover observe that if we set

$$v_\varepsilon(x) := u(z + \lambda\nu\varepsilon).$$

we have

$$\|v_\varepsilon - u\|_{W^{k,p}(\Omega')} \xrightarrow{\varepsilon\to 0} 0$$

since v_ε is merely a translation. Moreover, $u_\varepsilon = \eta_\varepsilon * v_\varepsilon$, and thus as before

$$\|u_\varepsilon - v_\varepsilon\|_{W^{k,p}(\Omega')} = \xrightarrow{\varepsilon\to 0} 0.$$

We conclude that $u_\varepsilon \to u$ in $W^{k,p}(\Omega')$.

Now we cover all of $\partial\Omega$ by (finitely many, by compactness) balls $B(x_i, r_i)$ and choose the approximation $u_{\varepsilon,i}$ on $\Omega_i := B(x_i, r_i) \cap \Omega$ as above. In $\Omega_0 := \Omega\backslash \bigcup B(x_i, r_i) \subset\subset \Omega$ we can find another approximation $u_{\varepsilon,0}$.

Now we pick a smooth decomposition of unity η_i with support in $\Omega_i \cap \partial\Omega$ such that

$$\sum_{i\in\mathbb{N}} \eta_i \equiv 1 \quad \text{in } \Omega.$$

Setting

$$u_\varepsilon := \sum_i \eta_i u_{\varepsilon,i} \in C^\infty(\overline{\Omega}).$$

We then use the Leibniz rule to conclude that

$$\|u_\varepsilon - u\|_{W^{k,p}(\Omega)} \xrightarrow{\varepsilon\to 0} 0.$$

□

On $\mathbb{R}^n$ approximation is much easier, indeed we can approximate with respect to the $W^{k,p}$-norm any $u \in W^{k,p}(\mathbb{R}^n)$ by functions $u_k \in C_c^\infty(\mathbb{R}^n)$. That is, $W^{k,p}(\mathbb{R}^n) = W_0^{k,p}(\mathbb{R}^n)$. We could describe this as "$u \in W^{k,p}(R^n)$ implies that u and $k-1$-derivatives of u all vanish at infinity".

Proposition III.3.13. *(1) Let $u \in W^{k,p}(\Omega)$, $p \in [1,\infty)$. If $\operatorname{supp} u \subset\subset \mathbb{R}^n$ then there exists $u_k \in C_c^\infty(\Omega)$ such that*

$$\|u - u_k\|_{W^{k,p}(\Omega)} \xrightarrow{k\to\infty} 0.$$

(2) Let $u \in W^{k,p}(\mathbb{R}^n)$, $p \in [1,\infty)$. Then there exists $u_k \in C_c^\infty(\mathbb{R}^n)$ such that

$$\|u - u_k\|_{W^{k,p}(\mathbb{R}^n)} \xrightarrow{k\to\infty} 0.$$

(3) Let $u \in W^{k,p}(\mathbb{R}^n_+) = \mathbb{R}^{n-1} \times (0,\infty))$. Then there exists $u \in C_c^\infty(\mathbb{R}^{n-1} \times [0,\infty)$ (i.e.e u may not be zero on $(x',0)$ for small x') such that

$$\|u - u_k\|_{W^{k,p}(\mathbb{R}^n_+)} \xrightarrow{k\to\infty} 0.$$

Proof. (1) follows from the proof of Proposition III.3.8: Observe that $\operatorname{supp} u \subset\subset \Omega$ implies that $\eta_\varepsilon * u \in C_c^\infty(\Omega)$ if ε is only small enough.

(3) is an exercise, a combination of the proof of (2) and Theorem III.3.12.

So let us discuss (2). Let $\eta \in C_c^\infty(B(0,1))$ again be the typical mollifier bump function, $\eta_\varepsilon(x) = \varepsilon^{-n}\eta(x/\varepsilon)$. We have already seen that

$$\eta_\varepsilon * u \in C^\infty(\mathbb{R}^n).$$

But there is no reason that $\eta_\varepsilon * u \in C_c^\infty(\mathbb{R}^n)$. Set (without rescaling by R^n!)

$$\varphi_R(x) := \eta(x/R) \in C_c^\infty(B(0,R)).$$

Then we set

$$u_{\varepsilon,R} := \eta_\varepsilon * (\varphi_R\, u)$$

Now before $u_{\varepsilon,R} \in C_c^\infty(B(0,R+\varepsilon)) \subset C_c^\infty(\mathbb{R}^n)$.

Moreover we have for any $\ell = 0, \dots, k$

$$\begin{aligned}\|\nabla^\ell(u - u\varphi_R)\|_{L^p(\mathbb{R}^n)} =& \|\nabla^\ell(1-\varphi_R)u\|_{L^p(\mathbb{R}^n)}\\ \le& C(\ell)\sum_{i=0}^{\ell}\|\nabla^i(1-\varphi_R)\nabla^{\ell-i}u\|_{L^p(\mathbb{R}^n)}\\ \le& C(\ell)\,\|(1-\varphi_R)\nabla^\ell u\|_{L^p(\mathbb{R}^n)} + C(\ell)\sum_{i=0}^{\ell}\|\nabla^i(1-\varphi_R)\|_\infty\|\nabla^{\ell-i}u\|_{L^p(\mathbb{R}^n)}\\ \le& C(\ell,\eta)\,\|(1-\varphi_R)\nabla^\ell u\|_{L^p(\mathbb{R}^n\setminus B(0,R))} + C(\ell,\eta)\sum_{i=0}^{\ell}R^{-i}\|u\|_{W^{k,p}(\mathbb{R}^n)}\\ \xrightarrow{R\to\infty}& 0\end{aligned}$$

by Lebesgue dominated convergence theorem.

On the other hand, as already seen for R fixed,

$$\|\eta_\varepsilon * (\varphi_R\, u) - \varphi_R u\|_{W^{k,p}(R^n)} \xrightarrow{\varepsilon\to 0} 0.$$

Now we show that for any $\ell > 0$ there exists ε_ℓ, R_ℓ such that for $u_\ell := u_{\varepsilon_\ell, R_\ell}$ we have

$$\|u_\ell - u\|_{W^{k,p}(\mathbb{R}^n)} < \frac{1}{\ell} \xrightarrow{\ell\to\infty} 0, \tag{III.3.4}$$

that is $C_c^\infty(\mathbb{R}^n) \ni u_\ell \to u$ in $W^{k,p}(\mathbb{R}^n)$.

First, by the arguments above we can choose R_ℓ large enough such that

$$\|u - u\varphi_{R_\ell}\|_{W^{k,p}(\mathbb{R}^n)} \le \frac{1}{2\ell}.$$

Next we can choose ε_ℓ small enough such that

$$\|u_\ell - u\varphi_{R_\ell}\|_{W^{k,p}(\mathbb{R}^n)}\,\|\eta_{\varepsilon_\ell} * (u\varphi_{R_\ell}) - u\varphi_{R_\ell}\|_{W^{k,p}(\mathbb{R}^n)} \le \frac{1}{2\ell}.$$

Thus, by triangular inequality,

$$\|u_\ell - u\|_{W^{k,p}(\mathbb{R}^n)} \le \|u_\ell - u\varphi_{R_\ell}\|_{W^{k,p}(\mathbb{R}^n)} + \|u\varphi_{R_\ell} - u\|_{W^{k,p}(\mathbb{R}^n)} \le 2\frac{1}{2\ell} = \frac{1}{\ell}.$$

This proves (III.3.4), and thus (2) is established. □

III.3.2. Difference Quotients. We used above, e.g. for the Cauchy estimates, Proof of Lemma I.2.17 the method of differentiating the equation (e.g. that if $\Delta u = 0$ then also for $v := \partial_i u$ we have $\Delta v = 0$ – so we can easier estimates for $\partial_i u$). In the Sobolev space category this is also a useful technique. Sometimes, the "first assume that everything is smooth, then use mollification"-type argument as for Lemma I.2.17 is difficult to put into practice. In this case, a technique developed by Nirenberg, is discretely differentiating the equation (which does not require the function to be a priori differentiable):

$$\Delta u = 0 \Rightarrow v(x) := (\Delta_h^{e_i} u)(x) := \frac{u(x + he_i) - u(x)}{h} : \quad \Delta v = 0$$

For this to work, we need some good estimates. Recall that (by the fundamental theorem of calculus), for C^1-functions u,

$$\|\Delta_h^{e_\ell} u\|_{L^\infty} \leq \|\partial_\ell u\|_{L^\infty}.$$

This also holds in L^p for $W^{1,p}$-functions u, which is a result attributed to Nirenberg, see Proposition III.3.15.

One important ingredient is that the fundamental theorem of calculus holds for Sobolev functions:

Lemma III.3.14. *Let $u \in W^{1,1}_{loc}(\Omega)$. Fix $v \in \mathbb{R}^n$. Then for almost every $x \in \Omega$ such that the path $[x, x+v] \subset \Omega$ we have*

$$u(x+v) - u(x) = \int_0^1 \partial_\alpha u(x+tv) v^\alpha \, dt.$$

Proof. Let $\Omega' \subset \Omega$. In view of Proposition III.3.8 we can approximate u by $u_k \in C^\infty(\overline{\Omega'})$ such that

$$\|u_k - u\|_{W^{1,1}(\Omega')} \xrightarrow{k\to\infty} 0.$$

The claim holds for the smooth functions u_k, namely we have that whenever $[x, x+v] \subset \Omega'$,

$$u_k(x+v) - u_k(x) = \int_0^1 \partial_\alpha u_k(x+tv) v^\alpha \, dt. \tag{III.3.5}$$

Now we have

$$\|u_k(\cdot + v) - u_k(\cdot) - (u(\cdot + v) - u(\cdot))\|_{L^p(\Omega')} \xrightarrow{k\to\infty} 0,$$

in particular (up to taking a subsequence),

$$u_k(x+v) - u_k(x) \xrightarrow{k\to\infty} u(x+v) - u(x) \quad \text{for almost every } x \in \Omega' \text{ such that } [x, x+v] \subset \Omega'.$$

Also the right-hand side converges. Observing that[6]

$$\left(\int_\Omega \left(\int_0^1 |f(x,t)| \, dt \right)^p \right)^{\frac{1}{p}} \leq \left(\int_0^1 \int_\Omega |f(x,t)|^p \, dt \right)^{\frac{1}{p}}$$

we have

$$\begin{aligned} &\| \int_0^1 \partial_\alpha u_k(\cdot + tv) v^\alpha \, dt - \int_0^1 \partial_\alpha u(\cdot + tv) v^\alpha \, dt \|_{L^p(\Omega')} \\ &\leq \left(\int_0^1 |v| \|(Du_k - Du\|^p_{L^p(\Omega')} \, dt \right)^{\frac{1}{p}} = \|Du_k - Du\|_{L^p(\Omega')} \xrightarrow{k\to\infty} 0. \end{aligned}$$

So again, up to possibly a subsubsequence,

$$\int_0^1 \partial_\alpha u_k(x+tv) v^\alpha \, dt \xrightarrow{k\to\infty} \int_0^1 \partial_\alpha u(x+tv) v^\alpha \, dt$$

[6]This can be seen by Jensens inequalty: For any $p \in [1, \infty)$,

$$(\fint_A |f|)^p \leq \fint_A |f|^p,$$

this can also be shown by Hölder's inequality. Then Fubini gets to the claim.

for all x such that $[x, x+v] \subset \Omega$.

Taking the limit in (III.3.5) we conclude. $\square$

Proposition III.3.15. *(1) Let $k \in \mathbb{N}$, (i.e. $k \neq 0$), and $1 \leq p < \infty$. Assume that $\Omega' \subset\subset \Omega$ are two open (nonempty) sets, and let $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$. For $u \in W^{k,p}(\Omega)$ we have*

$$\|\Delta_h^{e_\ell} u\|_{W^{k-1,p}(\Omega')} \leq \|\partial_\ell u\|_{W^{k-1,p}(\Omega)}.$$

Moreover we have

$$\|\Delta_h^{e_\ell} u - \partial_\ell u\|_{W^{k-1,p}(\Omega')} \xrightarrow{h\to 0} 0.$$

(2) Let $u \in W^{k-1,p}(\Omega)$, $1 < p \leq \infty$. Assume that for any $\Omega' \subset\subset \Omega$ and any $\ell = 1, \dots, n$ there exists a constant $C(\Omega')$ such that

$$\sup_{|h|<\operatorname{dist}(\Omega',\partial\Omega)} \|\Delta_h^{e_\ell} u\|_{W^{k-1,p}(\Omega')} \leq C(\Omega', \ell)$$

Then we $u \in W^{k,p}_{loc}(\Omega)$, and for any $\Omega' \subset \Omega$ we have

$$\|\partial_\ell u\|_{W^{k-1,p}(\Omega')} \leq \sup_{|h|<\operatorname{dist}(\Omega',\partial\Omega)} \|\Delta_h^{e_\ell} u\|_{W^{k-1,p}(\Omega')}. \tag{III.3.6}$$

If $p = \infty$ we even have $u \in W^{k,\infty}(\Omega)$ with the estimate

$$\|\partial_\ell u\|_{W^{k-1,\infty}(\Omega)} \leq \sup_{\Omega'\subset\subset\Omega} \sup_{|h|<\operatorname{dist}(\Omega',\partial\Omega)} \|\Delta_h^{e_\ell} u\|_{W^{k-1,\infty}(\Omega')}. \tag{III.3.7}$$

Proof of Proposition III.3.15(1). The proof of (1) is essentially the same as for differentiable function, we use the fundamental theorem of calculus.

By the fundamental theorem of calculus, Lemma III.3.14,

$$\Delta_h^{e_\ell} u(x) = \frac{1}{h}\int_0^1 \frac{d}{dt}\left(u(x+the_\ell)\right)\,dt = \frac{1}{h} h \int_0^1 \partial_\ell u(x+the_\ell)\,dt = \int_0^1 \partial_\ell u(x+the_\ell)\,dt,$$

Similarly, for any $|\gamma| \leq k-1$, $\ell = 1, \dots, n$

$$|\Delta_h^{e_\ell} \partial^\gamma u(x)| \leq \int_0^1 |\partial_\ell \partial^\gamma u(x+the_\ell)|.$$

Taking the L^p-norm, observing that[7]

$$\left(\int_\Omega \left(\int_0^1 |f(x,t)|\,dt\right)^p\right)^{\frac{1}{p}} \leq \left(\int_0^1 \int_\Omega |f(x,t)|^p\,dt\right)^{\frac{1}{p}}$$

[7]This can be seen by Jensens inequalty: For any $p \in [1, \infty)$,

$$(\fint_A |f|)^p \leq \fint_A |f|^p,$$

this can also be shown by Hölder's inequality. Then Fubini gets to the claim.

we have

$$\|\Delta_h^{e_\ell}\partial^\gamma u\|_{L^p(\Omega')} \leq \left(\int_0^1 \|\partial_\ell\partial^\gamma u(\cdot + the_\ell)\|^p_{L^p(\Omega')}\,dt\right)^{\frac{1}{p}}.$$

Now observe that by substitution and $|h| < \operatorname{dist}(\Omega', \partial\Omega)$,

$$\|\partial_\ell\partial^\gamma u(\cdot + the_\ell)\|_{L^p(\Omega')} = \|\partial_\ell\partial^\gamma u(\cdot)\|_{L^p(\Omega'+he_\ell)} \leq \|\partial_\ell\partial^\gamma u(\cdot)\|_{L^p(\Omega)}$$

Consequently, for any $|h| < \operatorname{dist}(\Omega', \partial\Omega)$.

$$\|\Delta_h^{e_\ell}\partial^\gamma u\|_{L^p(\Omega')} \leq \left(\int_0^1 \|u\|^p_{W^{k,p}(\Omega)}\,dt\right)^{\frac{1}{p}} = \|u\|_{W^{k,p}(\Omega)}.$$

This shows (1), under the assumption that the fundamental theorem of calculus works. □

Before we proof Proposition III.3.15(2), we need results from Functional Analysis:

Let X be a Banach space, then $X^* = L(X)$ denotes the dual space (namely the space of all linear, continuous functionals $X \to \mathbb{R}$. The bi-dual space $X^{**} = L(X^*)$ has X as a canonical subset, meaning there is a canonical isometric embedding $J_X : X \to X^{**}$ given by (for $x \in X$ and $x^* \in X^*$,

$$J_X(x)[x^*] := x^*[x].$$

If J_X is surjective, then we say that X is a *reflexive* space. The important fact for us is that L^p-spaces are reflexive if $p \in (1, \infty)$ (L^1 and L^∞ are not reflexive).

Then we have the important theorem (essentialy a consequence of Banach-Alaoglu theorem) that if X is a reflexive Banach space then every bounded set es *weakly* (sequentially) pre-compact, which means that whenever $(x_k) \subset X$ is a bounded sequence, i.e. $\sup_k \|x_k\|_X < \infty$ then there exists a subsequence x_{k_i} such that x_{k_i} is *weakly* convergent to some $x \in X$. Weak convergence means that for any dual element $y^* \in X^*$ we have

$$y^*(x_{k_i}) \xrightarrow{i\to\infty} y^*(x) \quad \text{as numbers in } \mathbb{R}$$

Lucky for us, the dual space of L^p-spaces, $1 < p < \infty$, is characterized by the Riesz representation theorem, Theorem III.1.4. This leads to the following theorem

THEOREM III.3.16 (Weak compactness). *Let $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ open, and assume that $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in $L^p(\Omega)$, that is*

$$\sup_{k\in\mathbb{N}} \|f_k\|_{L^p(\Omega)} < \infty.$$

Then there exists a function $f \in L^p(\Omega)$ and a subsequence f_{k_i} such that f_{k_i} weakly L^p-converges to f, that is for any $g \in L^{p'}(\Omega)$, where $p' = \frac{p}{p-1}$ is the Hölder dual of p, we have

$$\int_\Omega f_{k_i}\, g \xrightarrow{i\to\infty} \int_\Omega f\, g.$$

In particular we have

$$\|f\|_{L^p(\Omega)} \leq \sup_k \|f_k\|_{L^p(\Omega)}.$$

Proof of Proposition III.3.15(2). First let us assume that $p < \infty$.

Assume that for all $\ell \in \{1, \dots, n\}$ we have

$$\sup_{|h|<\operatorname{dist}(\Omega',\partial\Omega)} \|\Delta_h^{e_\ell} u\|_{W^{k-1,p}(\Omega')} < \infty.$$

In view of Theorem III.3.16 we can choose a sequence $h_i \xrightarrow{i\to\infty} 0$ such that for any $|\gamma| \leq k-1$,

$$\Delta_{h_i}^{e_\ell} \partial^\gamma u \xrightarrow{i\to\infty} f_{\ell,\gamma} \quad \text{weakly in } L^p(\Omega').$$

Since we are optimists, we call $\partial_\ell \partial^\gamma u := f_{\ell,\gamma} \in L^p(\Omega')$. We still need to show that $\partial_\ell \partial^\gamma u$ is actually the distributional derivative of u! Also, for simplicity of notation we drop the i in h_i and write $h \to 0$ (meaning always this subsequence). Weak convergence means in particular, that for any $\varphi \in C_c^\infty(\Omega') \subset L^p(\Omega')$,

$$\int_{\Omega'} \Delta_h^{e_\ell} \partial^\gamma u \ \varphi \xrightarrow{h\to 0} \int_{\Omega'} \partial_\ell \partial^\gamma u \ \varphi \tag{III.3.8}$$

Since $\operatorname{supp}\varphi \subset\subset \Omega'$ for $|h|$ small enough we have that $\Delta_{-h}^{e_\ell}\varphi \in C_c^\infty(\Omega')$. Now we perform a discrete integration by parts, namely by substitution,

$$\int_{\Omega'} \Delta_h^{e_\ell} \partial^\gamma u \ \varphi = -\int_{\Omega'} \partial^\gamma u \ \Delta_{-h}^{e_\ell}\varphi$$

Now since $\Delta_{-h}^{e_\ell}\varphi \in C_c^\infty(\Omega')$ is a testfunction and $u \in W^{k-1,p}$,

$$\int_{\Omega'} \Delta_h^{e_\ell} \partial^\gamma u \ \varphi = -\int_{\Omega'} \partial^\gamma u \ \Delta_{-h}^{e_\ell}\varphi = (-1)^{|\gamma|+1} - \int_{\Omega'} u \ \Delta_{-h}^{e_\ell} \partial^\gamma \varphi \xrightarrow{h\to 0} (-1)^{|\gamma|+1} \int_{\Omega'} u \ \partial_\ell \partial^\gamma \varphi,$$

in the last step we used dominated convergence and the smoothness of φ.

But then in (III.3.8) we obtain

$$(-1)^{|\gamma|+1} \int_{\Omega'} u \ \partial_\ell \partial^\gamma \varphi = \int_{\Omega'} \partial_\ell \partial^\gamma u \ \varphi$$

This holds for any $\ell \in \{1, \dots, n\}$ and so we have shown that $\partial_\ell \partial^\gamma u$ is indeed the weak derivative of u which belongs to L^p, and thus $u \in W^{k,p}(\Omega')$. Since this holds for any $\Omega' \subset \Omega$ we conclude that $u \in W^{k,p}_{loc}(\Omega)$. The estimate (III.3.6) follows from the estimate of Theorem III.3.16.

As for the case $p = \infty$, we observe first that for $\Omega' \subset\subset \Omega$ the estimate

$$\sup_{|h|<\operatorname{dist}(\Omega',\partial\Omega)} \|\Delta_h^{e_\ell} u\|_{W^{k-1,\infty}(\Omega')} \leq C(\Omega', \ell) \tag{III.3.9}$$

implies (by Hölders inequality) also

$$\sup_{|h|<\operatorname{dist}(\Omega',\partial\Omega)} \|\Delta_h^{e_\ell} u\|_{W^{k-1,2}(\Omega')} \leq C(\Omega', \ell)$$

Thus (III.3.9) implies $u \in W^{k,2}_{loc}(\Omega)$ and in view of Proposition III.3.15(1) we have that $\Delta_h^{e_\ell} u \to \partial_\ell u$ in $W^{k-1,2}_{loc}(\Omega)$.

In particular, we already have the existence of the distributional derivative $\partial_\ell u \in W^{k-1,2}_{\text{loc}}(\Omega)$.

Set

$$\Lambda := \sup_{\Omega' \subset\subset \Omega} \sup_{|h| < \mathrm{dist}\,(\Omega', \partial\Omega)} \|\Delta_h^{e_\ell} u\|_{W^{k-1,\infty}(\Omega')}.$$

For simplicity of notation in the following we shall assume $k = 1$.

We now claim that the above observations, together with (III.3.7), for any $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \partial_\ell u\, \varphi \leq \Lambda\, \|\varphi\|_{L^1(\Omega)}. \tag{III.3.10}$$

Indeed, since for $\varphi \in C_c^\infty(\Omega)$ let $\operatorname{supp} \varphi \subset \Omega' \subset\subset \Omega$, then we have

$$\int_\Omega \partial_\ell u\ \varphi = \lim_{|h| \to 0} \int_\Omega \underbrace{\Delta_h^{e_\ell} u}_{\leq \Lambda}\ \varphi \leq \Lambda \|\varphi\|_{L^1(\Omega)}.$$

which is exactly (III.3.10).

Let $x \in \Omega$ be a Lebesgue point of $\partial_\ell u$ in Ω, i.e.

$$\partial_\ell u(x) = \lim_{r \to 0} \fint_{B(x,r)} \partial_\ell u.$$

Observe that almost all points in Ω are Lebesgue points (since $\partial_\ell u \in L^2_{loc}(\Omega)$).

Set

$$\Omega' = \{z \in \Omega : \mathrm{dist}\,(x, \partial\Omega) < \frac{1}{2} \mathrm{dist}\,(x, \partial\Omega)\} \subset\subset \Omega.$$

Then for all $r < \frac{1}{4}\mathrm{dist}\,(x, \partial\Omega)$ we can set $\varphi := |B(x,r)|^{-1} \chi_{B(x,r)} \in L^2(\Omega)$ which can be approximated by smooth $C_c^\infty(\Omega')$ functions $\varphi_i \to \varphi$ in $L^2(\Omega)$. Since $\Omega' \subset\subset \Omega$ we also have $\varphi_i \to \varphi$ in $L^1(\Omega)$ (observe $\|\varphi\|_{L^1(\Omega)} = 1$ by construction of φ). Then

$$\fint_{B(x,r)} \partial_\ell u = \lim_{i \to \infty} \int_{B(x,r)} \partial_\ell u \varphi_i,$$

which leads to

$$|\fint_{B(x,r)} \partial_\ell u| \leq \Lambda \lim_{i \to \infty} \|\varphi_i\|_{L^1(\Omega')} \leq \Lambda \|\varphi\|_{L^1(\Omega')} = \Lambda.$$

Since x was chosen to be a Lebesgue point of $\partial_\ell u$, and since the last estimate holds for any $r > 0$, we find

$$|\partial_\ell u(x)| = \lim_{r \to 0} |\fint_{B(x,r)} \partial_\ell u| \leq \Lambda.$$

This again holds for any Lebesgue point $x \in \Omega$, and since almost all points in Ω are Lebesgue points,

$$|\partial_\ell u(x)| \leq \Lambda \quad \text{a.e. } x \in \Omega,$$

which implies

$$\|\partial_\ell u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_\Omega |\partial_\ell u| \leq \Lambda,$$

which was the claim. □

THEOREM III.3.17 ($C^{k-1,1} \approx W^{k,\infty}$). *(1) Let $\Omega \subset \mathbb{R}^n$ be open and nonempty, $k \in \mathbb{N}$ then*

$$C^{k-1,1}(\overline{\Omega}) \subset W^{k,\infty}(\Omega),$$

and

$$\|D^k u\|_{L^\infty(\Omega)} \leq C\,[D^{k-1}u]_{\mathrm{Lip}\,(\Omega)}$$

(2) Let $\Omega \subset\subset \mathbb{R}^n$ connected, $\partial\Omega \in C^{0,1}$. Then for $k \in \mathbb{N}$

$$W^{k,\infty}(\Omega) \subset C^{k-1,1}(\overline{\Omega}),$$

and

$$[D^{k-1}u]_{\mathrm{Lip}\,(\Omega)} \leq C(k,\Omega)\ \|D^k u\|_{L^\infty(\Omega)}.$$

The above holds in the following sense: recall that functions in L^p (and thus in particular in $W^{k,\infty}$ are classes (namely: two functions $f, g \in L^p(\Omega)$ are the same if they coincide almost everywhere). So what we mean above is: For every $f \in W^{k,\infty}(\Omega)$ there exists a representative $g \in C^{k-1,1}(\Omega)$ that coinides with f a.e.

PROOF. We restrict our attention to $k = 1$ and leave the other cases as exercise.

For (1): Let $u \in C^{0,1}(\overline{\Omega})$. Since u is Lipschitz,

$$\sup_{\Omega' \subset\subset \Omega}\ \sup_{|h|<\mathrm{dist}\,(\Omega',\partial\Omega)} \|\Delta_h^{e_\ell} u\|_{L^\infty(\Omega')} \leq [u]_{\mathrm{Lip}\,(\Omega)}.$$

From Proposition III.3.15(2) we then obtain $u \in W^{1,\infty}(\Omega)$ with the claimed estimate.

For (2):

First we assume that $\Omega = B(0,1)$ is a ball, $u \in W^{1,\infty}(B(0,1))$. We argue by mollification (what else can we do): Let u_ε be the usual mollification $u_\varepsilon = \eta_\varepsilon * u$ which, as we already know, converges in $W^{1,2}_{loc}(B(0,1))$ to u. Moreover (also as seen before), for any $\delta \in (0,1)$, $x \in B(0,\delta)$, if $\varepsilon < \delta$ then

$$\partial_\ell u_\varepsilon(x) = \int \partial_\ell u(y)\ \eta_\varepsilon(y-x)\,dy,$$

and thus whenever $x \in B(0,\delta)$, if $\varepsilon < \delta$

$$|\partial_\ell u_\varepsilon(x)| \leq \|\partial_\ell u(y)\|_{L^\infty(B(0,1))}\|\eta_\varepsilon\|_{L^1(B(0,1))} = \|\partial_\ell u(y)\|_{L^\infty(B(0,1))}.$$

In particular, by the fundamental theorem of calculus (recall: u_ε is differentiable), whenever $\varepsilon < \delta$

$$[u_\varepsilon]_{\mathrm{Lip}\,,B(0,1-\delta)} \leq \|Du\|_{L^\infty(B(0,1))}.$$

Observe there is no constant on the right-hand side. Since moreover $\|u_\varepsilon\|_{L^\infty(B(0,\delta 0)} \leq \|u\|_{L^\infty(B(0,1))}$ we have that u_ε is equicontinuous and bounded, and thus by Arzela-Ascoli (up to a subsequence $\varepsilon \to 0$) we have $u_\varepsilon \to u$ in $C^0(B(0,\delta))$ (Here is where we find the "continuous representative of u", the limit of u_ε coinicides a.e. with u). In particular, u is continuous in $B(0,1-\delta)$. Also observe that for any $x \neq y \in B(0,1-\delta)$, for any $\varepsilon < \delta$,

$$|u(x)-u(y)| \leq 2\|u-u_\varepsilon\|_{L^\infty(B(0,1-\delta))}+|u_\varepsilon(x)-u_\varepsilon(y)| \leq 2\|u-u_\varepsilon\|_{L^\infty(B(0,1-\delta))}+|x-y|\|Du\|_{L^\infty(B(0,1))}.$$

This holds for any $\varepsilon < \delta$, so letting $\varepsilon \to 0$ we obtain by the uniform convergence $u_\varepsilon \to u$ in $B(0, 1-\delta)$.

$$|u(x) - u(y)| \leq |x-y| \|Du\|_{L^\infty(B(0,1))} \quad \text{for all } x, y \in B(0, 1-\delta)$$

This again holds for any $\delta > 0$ so that

$$|u(x) - u(y)| \leq |x-y| \|Du\|_{L^\infty(B(0,1))} \quad \text{for all } x, y \in B(0, 1).$$

That is, u is Lipschitz continuous and we have

$$[u]_{\mathrm{Lip}\,(B(0,1))} \leq \|Du\|_{L^\infty(B(0,1))}.$$

So Theorem III.3.17(2) is established for $\Omega = B(0,1)$.

Next assume that $\underline{\Omega \subset\subset \mathbb{R}^n \text{ and } \partial\Omega \in C^{0,1}. \text{ Moreover we assume } \Omega \text{ is path-connected.}}$

The regularity of the boundary is used in the following way: For any two points $x, y \in \Omega$ there exists a continuous path γ connecting x and y inside Ω such that the lenght of γ, $\mathcal{L}(\gamma) \leq C(\Omega)|x-y|$ (essentially take the straight line connecting x and y, when it hits $\partial\Omega$ follow $\partial\Omega$, then regularize and shift it away from $\partial\Omega$).

Since Ω is open, and by the argument above in every open ball we can replace u by its continuous representative we may assume that u w.l.o.g. is continous, and we just want to show that u is Lipschitz continuous.

Let $x, y \in \Omega$ and let γ be such a path connecting x and y. Set $\delta := \frac{1}{2}\mathrm{dist}\,(\gamma, \partial\Omega) > 0$. Setting $L := \lceil \frac{\mathcal{L}(\gamma)}{\delta} \rceil + 2$ points $(x_i)_{i=1}^L$ in γ, such that

$$\bigcup_{i=1}^{L} B(x_i, \delta) \supset \gamma.$$

and such that $B(x_i, \delta) \cap B(x_{i+1}, \delta) \neq \emptyset$, $x \in B(x_1, \delta)$ and $y \in B(x_L, \delta)$. In every $B(x_i, \delta)$ we use the argument from above, and have

$$[u]_{\mathrm{Lip}\,(B(x_i,\delta))} \leq \|Du\|_{L^\infty(B(x_i,\delta))} \leq \|Du\|_{L^\infty(\Omega)}.$$

Now, by triangular inequality

$$\begin{aligned} |u(x) - u(y)| \leq & |u(x) - u(x_0)| + |u(y) - u(x_L)| + \sum_{i=1}^{L-1} |u(x_i) - u(x_{i+1})| \\ \leq & \|Du\|_{L^\infty(\Omega)} (L+1) 2\delta \leq C\mathcal{L}(\gamma) \leq C(\Omega) \|Du\|_{L^\infty(\Omega)} |x-y|. \end{aligned}$$

This implies that u is Lipschitz continuous with

$$[u]_{\mathrm{Lip}} \leq C(\Omega) \|Du\|_{L^\infty(\Omega)}.$$

which establishes the theorem. $\square$

III.3.3. Weak compactness in $W^{k,p}$. In the proof of of Proposition III.3.15(2) we derived and used the following consequence of Theorem III.3.16, which we want to record (so we don't have to argue always with Theorem III.3.16.

THEOREM III.3.18 (Weak compactness). *Let $1 < p < \infty$, $k \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open. Assume that $(f_i)_{i\in\mathbb{N}}$ is a bounded sequence in $W^{k,p}(\Omega)$, that is*

$$\sup_{i\in\mathbb{N}} \|f_i\|_{W^{k,p}(\Omega)} < \infty.$$

Then there exists a function $f \in W^{k,p}(\Omega)$ and a subsequence f_{i_j} such that f_{i_j} weakly $W^{k,p}$-converges to f, that is for any $|\gamma| \le k$ and any $g \in L^{p'}(\Omega)$, where $p' = \frac{p}{p-1}$ is the Hölder dual of p, we have

$$\int_\Omega \partial^\gamma f_{i_j}\, g \xrightarrow{i\to\infty} \int_\Omega \partial^\gamma f\, g.$$

In particular we have

$$\|f\|_{W^{k,p}(\Omega)} \le \sup_i \|f_i\|_{W^{k,p}(\Omega)}.$$

III.3.4. Extension Theorems. If f is a Lipschitz function on a set $\Omega \subset \mathbb{R}^n$, then f can be thought of as a restriction of a map $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$, $f = \tilde{f}\big|_\Omega$. This is (a special case of) the so-called Kirszbraun theorem. This is in general not true for Sobolev functions, even if Ω is open.

Definition III.3.19. Let $\Omega \subset \mathbb{R}^n$ be open. Ω is called a $W^{k,p}$-extension domain, if there exists a linear operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ such that

$$Eu(x) = u(x) \quad \text{for all } x \in \mathbb{R}^n,\ u \in W^{k,p}(\Omega)$$

and E is bounded, i.e.

$$\sup_{\|u\|_{W^{k,p}(\Omega)}\le 1} \|Eu\|_{W^{k,p}(\mathbb{R}^n)} < \infty.$$

THEOREM III.3.20. *Any open set $\Omega \subset\subset \mathbb{R}^n$ with boundary $\partial\Omega \in C^k$ is a $W^{k,p}(\Omega)$ extension domain for $k \in \mathbb{N}$, $1 \le p < \infty$.*

More precisely, for any $\tilde{\Omega} \supset\supset \Omega$ there exists an operator $E : W^{k,p}(\Omega) \to W_0^{k,p}(\tilde{\Omega})$ with $Eu = u$ in Ω and

$$\|Eu\|_{W^{k,p}(\tilde{\Omega})} \le C(\Omega, \tilde{\Omega}, n, k)\, \|u\|_{W^{k,p}(\Omega)}.$$

PROOF. We will first show how to extend $W^{k,p}$-functions from $\mathbb{R}^n_+$ to all of $\mathbb{R}^n$. Then by "flattening the boundary" (for this we need the regularity $\partial\Omega$) we extend this argument to general Ω as claimed.

<u>From $\mathbb{R}^n_+$ to $\mathbb{R}^n$:</u>

Denote the variables in $\mathbb{R}^n$ by (x', x_n) where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$.

We can explicitely define $E_0 : W^{k,p}(\mathbb{R}^n_+) \to W^{k,p}(\mathbb{R}^n)$ by a type of reflection.

The main point is that we know (from the heaviside function example) that $W^{1,k}$-functions cannot have a jump, so at least for smooth functions u, if we hope for $E_0 u \in W^{1,p}$ we need that

$$\lim_{y_n \to 0^-} E_0 u(y', y_n) \overset{!}{=} \lim_{y_n \to 0^+} E_0 u(y', y_n) = \lim_{y_n \to 0^+} u(y', y_n)$$

So, for $k = 1$ we could simply use the even reflection,

$$E_0 u(y', y_n) := u(y', |y_n|) = \begin{cases} u(y', y_n) & \text{if } y_n > 0 \\ u(y, -y_n) & \text{if } y_n < 0. \end{cases}$$

which indeed takes $C^\infty(\overline{\mathbb{R}^n_+})$-functions into Lipschitz-functions (i.e. $W^{1,\infty}_{loc}(\mathbb{R}^n)$-functions, hence $W^{1,p}_{loc}(\mathbb{R}^n)$).

More generally, for $W^{k,p}$-functions, $k \geq 1$ we then need that for any $\ell = 1, \dots, k$ the $(\ell - 1)$-th derivatives in y_n-direction coincide:

$$\lim_{y_n \to 0^-} (\partial_n)^{\ell-1} E_0 u(y', y_n) \overset{!}{=} \lim_{y_n \to 0^+} (\partial_n)^{\ell-1} E_0 u(y', y_n) = \lim_{y_n \to 0^+} (\partial_n)^{\ell-1} u(y', y_n). \tag{III.3.11}$$

So again, we use a reflection, but a more complicated one,

$$E_0 u(y', y_n) := \begin{cases} u(y', y_n) & \text{if } y_n > 0 \\ \sum_{i=1}^k \sigma_i \, u(y, -i y_n) & \text{if } y_n < 0. \end{cases}$$

Here, $(\sigma_i)_{i=1}^k$ are constants to be chosen, such that (III.3.11) is true for smooth functions: For all $\ell = 1, \dots, k$

$$\sum_{i=1}^k \sigma_i (-i)^{\ell-1} (\partial_n)^\ell u(x', 0) = (\partial_n)^\ell u(x', 0) \quad \Leftarrow \quad \sum_{i=1}^k \sigma_i (-i)^{\ell-1} = 1.$$

Such a σ exists by linear algebra: Defining a matrix A by $A_{i\ell} := (-i)^{\ell-1}$, and interpreting σ as a vector in $\mathbb{R}^k$ we want to solve

$$A\sigma = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

which is possible if A is invertible (to check this is the case is left as an exercise).

Now we argue as follows: Let $u \in W^{k,p}(\mathbb{R}^n_+)$. By Proposition III.3.13 there exists $u_j \in C^\infty_c(\mathbb{R}^{n-1} \times [0, \infty))$ that approximate u in $W^{k,p}(\mathbb{R}^n_+)$.

One now checks that $E_0 u_j \in C^{k-1,1}(\mathbb{R}^n)$, moreover we have for almost any $x \in \mathbb{R}^n$ (namely whenever $x = (x', x_n)$, with $x_n \neq 0$), for any $|\gamma| \leq k$,

$$|\partial^\gamma (E_0 u_j)(x', x_n)| \leq C(\sigma, k) \begin{cases} |\partial^\gamma u_j(x', x_n)| & x_n > 0 \\ \sum_{i=1}^k |\partial^\gamma u(y, -i y_n)| & x_n < 0. \end{cases}$$

Let us illustrate this fact for $k = 1$, for $k > 1$ it is an exercise.

$$\partial_{x_\alpha}(E_0 u_j)(x', x_n) = \partial_{x_\alpha} u_j(x', |x_n|) = (\partial_{x_\alpha} u_j)(x', |x_n|) \quad \alpha = 1, \dots, n-1.$$

$$\partial_{x_n}(E_0 u_j)(x', x_n) = \partial_{x_\alpha} u_j(x', |x_n|) = (\partial_{x_n} u_j)(x', |x_n|)\frac{x_n}{|x_n|}.$$

In particular we get that $u \in W^{k,p}(\mathbb{R}^n)$ and

$$\|E_0 u_j\|_{W^{k,p}(\mathbb{R}^n)} \leq C(k)\ \|u_j\|_{W^{k,p}(\mathbb{R}^n_+)}$$

In particular we get

$$\limsup_{j\to\infty} \|E_0 u_j\|_{W^{k,p}(\mathbb{R}^n)} \leq C(k)\ \limsup_{j\to\infty} \|u_j\|_{W^{k,p}(\mathbb{R}^n_+)} = C(k)\ \|u\|_{W^{k,p}(\mathbb{R}^n_+)}.$$

Thus, in view of Theorem III.3.18 we find $g \in W^{k,p}$ which is the weak $W^{k,p}$-limit of $E_0 u_j$. By strong L^p-convergence of u_j to u we see that indeed $E_0 g = u$, and thus we get

$$\|E_0 u\|_{W^{k,p}(\mathbb{R}^n)} \leq C(k)\ \|u\|_{W^{k,p}(\mathbb{R}^n_+)}.$$

as claimed.

<u>From Ω to $\mathbb{R}^n$</u> We only sketch the remaining arguments. If $\partial\Omega \in C^k$ then from small balls B centered at boundary points there exists C^k-charts $\phi : B \to \mathbb{R}^n$ such that $\phi(B \cap \Omega) \subset \mathbb{R}^n_+$ $\phi(B \cap \Omega^c) \subset \mathbb{R}^n_-$. By a decomposition of unity, we set $u = \sum_i \eta_i u$ such that η_i are supported only in one of these balls $B_i \cap \Omega$. Then $(\eta_i u) \circ \phi_i \in W^{k,p}(\mathbb{R}^n_+)$ (since it is locally in $W^{k,p}$ and then it is constantly zero). Here we use that (we haven't shown it) the Transformation rule still holds for Sobolev functions. Then we extend $(\eta_i \circ u) \circ \phi_i$ to all of $\mathbb{R}^n$, i.e. consider$E_0((\eta_i u) \circ \phi_i)$. Finally we set

$$E_1 u := \sum_i (E_0((\eta_i u) \circ \phi_i)) \circ \phi_i^{-1}.$$

The transformation rule shows that $Eu \in W^{k,p}(\mathbb{R}^n)$.

<u>From Ω to Ω'</u> To get $E_2 u \in W_0^{k,p}(\Omega')$ we simply take a cuttoff function $\eta \in C_c^\infty(\Omega')$, $\eta \equiv 1$ in Ω, and set

$$E_2 u := \eta E_1 u.$$

□

III.3.5. Traces. Let $\Omega \subset \mathbb{R}^n$ and $\partial\Omega \in C^\infty$.

If $u \in C^\alpha(\Omega)$, $\alpha \in (0, 1]$ with

$$\|u\|_{C^\alpha(\Omega)} < \infty$$

then we find a unique map $u\Big|_{\partial\Omega} \in C^\alpha(\partial\Omega)$. Indeed, for any $\overline{x} \in \partial\Omega$ there exists exactly one value $u(\overline{x})$ such that $\overline{u}(\overline{x}) = \lim_{\Omega \ni x \to \overline{x}} u(x)$, because $\|u\|_{L^\infty} < \infty$ and since we have uniform continuity

$$|u(x) - u(y)| \leq \|u\|_{C^\alpha(\Omega)} |x - y|^\alpha \xrightarrow{|x-y|\to 0} 0.$$

Moreover, for any $\overline{x}, \overline{y} \in \partial\Omega$ and $x, y \in \Omega$ we have

$$\begin{aligned}|\overline{u}(\overline{x}) - \overline{u}(\overline{y})| \leq & |\overline{u}(\overline{x}) - u(x)| + |u(x) - u(y)| + |u(x) - \overline{u}(\overline{y})| \\ \leq & |\overline{u}(\overline{x}) - u(x)| + \|u\|_{C^\alpha}|x-y|^\alpha + |u(x) - \overline{u}(\overline{y})|\end{aligned}$$

Taking $x \to \overline{x}$ and $y \to \overline{y}$ we thus find

$$|\overline{u}(\overline{x}) - \overline{u}(\overline{y})| \leq \|u\|_{C^\alpha}|\overline{x} - \overline{y}|^\alpha$$

that is

$$\|\overline{u}\|_{C^\alpha(\partial\Omega)} \leq \|u\|_{C^\alpha(\Omega)}.$$

The map that computes from u the trace map $\overline{u}$ we may call T, $\overline{u} = Tu$. Then we have a linear operator

$$T : C^{k,\alpha}(\overline{\Omega}) \to C^{k,\alpha}(\partial\Omega),$$

By the computations above, T is linear and bounded

$$\|Tu\|_{C^{k,\alpha}(\partial\Omega)} \leq \|u\|_{C^{k,\alpha}(\Omega)}.$$

On the other hand, when $u \in L^p(\Omega)$ there is absolutely no reasonable (unique) sense of a trace $u\Big|_{\partial\Omega}$.

One interesting and important fact of Sobolev spaces is that there is such a trace operator T if $k - \frac{1}{p} > 0$, that associates to a Sobolev function $u \in W^{k,p}(\Omega)$ a map $Tu \in W^{k-\frac{1}{p},p}(\partial\Omega)$. Observe that formally, if $p = \infty$ (i.e. in the Lipschitz case, the trace map is of the same class as the interior map, but for $p < \infty$ the trace map has less differentiability than the interior map. We do not want to deal with fractional Sobolev spaces here, so instead of proving the sharp trace estimate

$$T : W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\partial\Omega)$$

we will only show the following:

Theorem III.3.21. *Let $\Omega \subset\subset \mathbb{R}^n$, $\partial\Omega \in C^1$, $1 \leq p < \infty$. There exists a (unique) bounded and linear Trace operator T*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that

(1) $Tu = u\Big|_{\partial\Omega}$ whenever $u \in C^0(\overline{\Omega}) \cap W^{1,p}(\Omega)$.
(2) for each $u \in W^{1,p}(\Omega)$ we have

$$\|Tu\|_{L^p(\partial\Omega)} \leq C(\Omega, p)\ \|u\|_{W^{1,p}(\Omega)}.$$

Proof. For $u \in C^1(\overline{\Omega}) \cap W^{1,p}(\Omega)$ we define

$$Tu := u\Big|_{\partial\Omega}.$$

It now suffices to show that for all $u \in C^1(\overline{\Omega}) \cap W^{1,p}(\Omega)$ we have

$$\|Tu\|_{L^p(\partial\Omega)} \leq C(\Omega, p)\ \|u\|_{W^{1,p}(\Omega)}. \tag{III.3.12}$$

Then, by density of smooth functions $C^\infty(\overline{\Omega})$ in $W^{1,p}(\Omega)$, Theorem III.3.12, linearity and boundedness of the trace operator, there exists a (unique) extension of T to all of $W^{1,p}(\Omega)$.

To see (III.3.12) we argue again first on a flat boundary $\Omega = \mathbb{R}^n_+$. A flattening the boundary argument as above, then leads to the claim.

Observe the following, which holds by the integration-by-parts formula:

$$\|u\|^p_{L^p(\mathbb{R}^{n-1})} = \int_{\mathbb{R}^{n-1}\times\{0\}} |u(x')|^p d\mathcal{H}^{n-1}(x) = \int_{\mathbb{R}^n_+} \partial_n \left(|u(x)|^p\right) dx = \int_{\mathbb{R}^n_+} p|u(x)|^{p-2}u(x)\, \partial_n u(x)\, dx$$

Then by Young's inequality, $ab \leq C(a^p + b^{p'})$ (where $p' = \frac{p}{p-1}$ is the Hölder dual of p),

$$\int_{\mathbb{R}^n_+} p|u(x)|^{p-2}u(x)\, \partial_n u(x)\, dx \leq C \int_{\mathbb{R}^n_+} (|u|^p + |\partial_n u|^p) \leq C\|u\|^p_{W^{1,p}(\mathbb{R}^n_+)}.$$

This establishes (III.3.12) for $\Omega = \mathbb{R}^n_+$. For general Ω we use a decomposition of unity and flattening the boundary argument as in the theorems above. □

THEOREM III.3.22 (Zero-boundary data and traces). *Let $\Omega \subset\subset \mathbb{R}^n$ and $\partial\Omega \in C^1$. Let $u \in W^{1,p}(\Omega)$.*

Then $u \in H^{1,p}_0(\Omega)$ is equivalent to $u \in W^{1,p}_0(\Omega)$, where

$$W^{1,p}_0(\Omega) = \{u \in W^{1,p}(\Omega) : Tu = 0\}$$

for the trace operator T from Theorem III.3.21.

Remark III.3.23. By induction one obtains that if $\partial\Omega \in C^\infty$ then $H^{k,p}_0(\Omega)$ are exactly those functions where $T(\partial^\gamma u) = 0$ for any $|\gamma| \leq k-1$.

For time reasons we will not give the proof here. For a proof see [**Evans, 2010**, §5.5, Theorem 2].

III.3.6. Embedding theorems. Let X, Y be two Banach spaces. $T : X \to Y$ is a (we assume always: linear) embedding if T is injective. We say that the *embedding* $X \subset Y$ is continuous under the operator T, if T is a linear embedding and T is *continuous* (i.e. a bounded operator). If (as it often happens) T is (in a reasonable sense) the identity map, then we say that X embedds into Y continuously, and write $X \hookrightarrow Y$. E.g., clearly (by definition)

$$W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$$

since, by definition of the norm

$$\|u\|_{L^p(\Omega)} \leq \|u\|_{W^{1,p}(\Omega)}.$$

We say than an embedding $X \hookrightarrow Y$ is *compact* if the operator $T : X \to Y$ is compact, i.e. if for any bounded sequence $(x_n)_{n\in\mathbb{N}} \subset X$, $\sup_n \|x_n\|_X < \infty$, we have that $(T(x_n))_{n\in\mathbb{N}} \subset Y$ has a convergent subsequence in Y.

From functional analysis we also have: If $T : X \to Y$ is compact, then if $(x_n)_{n\in\mathbb{N}}$ is weakly convergent in X then Tx_n is strongly convergent in Y.

By Arzela-Ascoli, it is easy to check that $C^{k,\alpha}(\overline{\Omega})$ embedds compactly into $C^{\ell,\beta}(\overline{\Omega})$ if $k \geq \ell$ and $k + \alpha > \ell + \beta$.

The first implortant theorem is that for bounded sets Ω with smooth boundary we have $W^{1,p}(\Omega)$ embedds compactly into $L^p(\Omega)$. (By induction: $W^{k,p}(\Omega)$ embedds compactly into $W^{\ell,p}(\Omega)$ whenever $k \geq p$).

Observe that by Theorem III.3.18 we have weak compactness in $W^{1,p}(\omega)$ for bounded set, but strong convergence in $L^p(\Omega)$.

Theorem III.3.24 (Rellich-Kondrachov). *Let $\Omega \subset\subset \mathbb{R}^n$, $\partial\Omega \subset C^{0,1}$, $1 \leq p \leq \infty$. Assume that $(u_k)_{k\in\mathbb{N}} \in W^{1,p}(\Omega)$ is bounded, i.e.*

$$\sup_{k\in\mathbb{N}} \|u_k\|_{W^{1,p}(\Omega)} < \infty.$$

Then there exists a subsequence $k_i \to \infty$ and $u \in L^p(\Omega)$ such that u_{k_i} is (strongly) convergent in $L^p(\Omega)$, moreover the convergence is pointwise a.e..

Proof. If $p = \infty$, from Theorem III.3.17 we have $W^{1,\infty}(\Omega) = C^{0,1}(\overline{\Omega})$. By Arzela-Ascoli it is clear that $C^{0,1}$ is compactly embedded in $C^0(\overline{\Omega})$, so in particular in L^∞ (which has the same norm as $C^0(\overline{\Omega})$.

Now let $p \in [1, \infty)$. By the extension theorem, Theorem III.3.20 we may assume that $u_k \in W^{1,p}(\mathbb{R}^n)$ with $\operatorname{supp} u_k \subset B(0, R)$ for some (fixed) large $R > 0$.

The main idea is to use Arzela-Ascoli for mollified versions of u_k. Denote by $\eta \in C_c^\infty(B(0, 1))$ the usual bump function, $\int \eta = 1$, and $\eta_\varepsilon = \varepsilon^{-n}\eta(\cdot/\varepsilon)$. Set

$$u_{k,\varepsilon} := \eta_\varepsilon * u_k \in C_c^\infty(B(0, 2R)).$$

Observe that

$$|u_{k,\varepsilon}(x)| \leq C(R)\varepsilon^{-n}\|u_k\|_{L^p(B(0,R))}$$

$$|Du_{k,\varepsilon}(x)| \leq C(R)\varepsilon^{-n-1}\|u_k\|_{L^p(B(0,R))},$$

so since u_k is bounded (even L^p-boundedness is enough for now) we have

$$\sup_{k\in\mathbb{N}} \|u_{k,\varepsilon}\|_{\mathrm{Lip}\,(\mathbb{R}^n)} \leq C(\varepsilon).$$

That is, for any $\varepsilon_j := \frac{1}{j}$ there exists a subsequence $u_{k_{i(\varepsilon_j)},\varepsilon_j}$ that is convergent in $L^\infty(\mathbb{R}^n)$.

By a diagonalizing this subsequences we obtain only one subsequence u_{k_i,ε_j} so that for any fixed ε_j we have convergence in $L^\infty(\mathbb{R}^n)$, i.e. for any $j \in \mathbb{N}$ and any $\delta > 0$ there exists $N_{j,\delta} \in \mathbb{N}$ such that

$$\|u_{k_{i_1},\varepsilon_j} - u_{k_{i_2},\varepsilon_j}\|_{L^\infty(\mathbb{R}^n)} \le \delta \quad \forall i_1, i_2 > N_{j,\delta}.$$

Next we observe, by the fundamental theorem of Calculus,

$$\begin{aligned}\left|u_{k_{i_1}}(x) - u_{k_{i_1},\varepsilon_j}(x)\right| =& \int_{\mathbb{R}^n} |\eta_\varepsilon(z)| \left|u_{k_{i_1}}(x-z) - u_{k_{i_1}}(x)\right| \le \int_0^1 \int_{B(0,\varepsilon)} |\eta_\varepsilon(z)||Du_{k_{i_1}}(x-tz))|\,|z| dz\,dt \\ \le& \varepsilon^{1-n} \int_0^1 \left(\int_{B(0,\varepsilon)} |Du_{k_{i_1}}(x-tz))|^p\right)^{\frac{1}{p}} \varepsilon^{n-\frac{n}{p}} dz\,dt\end{aligned}$$

Thus, by Fubini

(III.3.13)

$$\begin{aligned}\|u_{k_{i_1}} - u_{k_{i_1},\varepsilon_j}\|_{L^p(\mathbb{R}^n)} =& \|u_{k_{i_1}} - u_{k_{i_1},\varepsilon_j}\|_{L^p(B(0,2R))} \le \varepsilon^{1-\frac{n}{p}} \left(\int_0^1 \int_{B(0,R)} \int_{B(0,\varepsilon)} |Du_{k_{i_1}}(x-tz) dx dz dt\right)^{\frac{1}{p}} \\ \le& \varepsilon \|Du_{k_{i_1}}\|_{L^p(B(0,2R))}.\end{aligned}$$

Now we claim that this leads to a Cauchy-sequence for the (non-mollified) u_{k_i}: Let $\delta > 0$.

$$\begin{aligned}\|u_{k_{i_1}} - u_{k_{i_2}}\|_{L^p(\mathbb{R}^n)} \le& \|u_{k_{i_1}} - u_{k_{i_1},\varepsilon_j}\|_{L^p(\mathbb{R}^n)} + \|u_{k_{i_1,\varepsilon_j}} - u_{k_{i_2,\varepsilon_j}}\|_{L^p(\mathbb{R}^n)} + \|u_{k_{i_2},\varepsilon_j} - u_{k_{i_2}}\|_{L^p(\mathbb{R}^n)} \\ \overset{(III.3.13)}{\le}& 2\,C\,\varepsilon_j \sup_k \|u_k\|_{W^{1,p}(\mathbb{R}^n)} + C(R)\|u_{k_{i_1,\varepsilon_j}} - u_{k_{i_2,\varepsilon_j}}\|_{L^\infty(B(2R))}\end{aligned}$$

Choosing now first ε_j small enough so that

$$2\,C\,\varepsilon_j \sup_k \|u_k\|_{W^{1,p}(\mathbb{R}^n)} < \frac{\delta}{2}$$

and then choosing for this ε_j the $N(\varepsilon_j, \delta)$ large enough so that for any $i_1, i_2 > N(\varepsilon_j, \delta)$

$$C(R)\|u_{k_{i_1,\varepsilon_j}} - u_{k_{i_2,\varepsilon_j}}\|_{L^\infty(B(2R))} < \frac{\delta}{2}$$

we see that

$$\|u_{k_{i_1}} - u_{k_{i_2}}\|_{L^p(\mathbb{R}^n)} \le \delta \quad \text{for any } i_1, i_2 > N(\varepsilon_j, \delta).$$

That is, u_{k_i} is a Cauchy sequence in $L^p(\mathbb{R}^n)$ and thus converges. □

One important consequence of Rellich's theorem, Theorem III.3.24 is Poincarè's inequality. In 1D it is called sometimes Wirtinger's inequality – and it is quite easy to prove. Let $I = (a, b) \subset \mathbb{R}$, then for any $u \in W^{1,p}(I)$,

$$\|u - (u)_I\|_{L^p(I)} \le C(I, p)\|u'\|_{L^p(I)}. \tag{III.3.14}$$

Here

$$(u)_I := \fint_I u$$

denotes the mean value of u on I.

The proof of (III.3.14) is done by the fundamental theorem of calculus, Lemma III.3.14. We have (using Hölder's inequality and Fubini many times)

$$\|u-(u)_I\|^p_{L^p(I)} \leq |I|^{-1}\int_I\int_I |u(x)-u(y)|^p \leq |I|^{-1}\int_0^1\int_I\int_I |u'(tx+(1-t)y)|^p\,|x-y|^p\,dx\,dydt$$

Now observe that by substituting $\tilde{y} := tx+(1-t)y$, we have

$$\begin{aligned}
&|I|^{-1}\int_0^{\frac{1}{2}}\int_I\int_I |u'(tx+(1-t)y)|^p\,|x-y|^p\,dx\,dydt\\
\leq& C(I,p)\int_0^{\frac{1}{2}}\int_I\int_I \frac{1}{1-t}|u'(\tilde{y})|^p\,d\tilde{y}\,dxdt\\
=& C(I,p)\int_0^{\frac{1}{2}}\frac{1}{1-t}\,dt\;|I|\;\|u'\|^p_{L^p(I)}\\
=& \tilde{C}(I,p)\|u'\|^p_{L^p(I)}.
\end{aligned}$$

In the same way, substituting $\tilde{x} := tx+(1-t)y$

$$\begin{aligned}
&|I|^{-1}\int_{\frac{1}{2}}^1\int_I\int_I |u'(tx+(1-t)y)|^p\,|x-y|^p\,dx\,dydt\\
\leq& \tilde{C}(I,p)\|u'\|^p_{L^p(I)}.
\end{aligned}$$

So (III.3.14) is established.

The Poincarè inequality says that (III.3.14) holds also in higher dimensions,

$$\|u-(u)_\Omega\|_{L^p(\Omega)} \leq C(\Omega,p)\|\nabla u\|_{L^p(\Omega)}. \tag{III.3.15}$$

If Ω is convex, the above proof works almost verbatim, in general open sets Ω this is more tricky.

Clearly, (III.3.15) does not hold if we remove $(u)_\Omega$ from the left-hand side. Indeed, just take $u \equiv const$ to find a counterexample. And indeed, a $W^{1,p}$-Poincaré-type inequality holds whenever *constants are excluded* in a reasonable sense.

THEOREM III.3.25 (Poincaré). *Let $\Omega \subset\subset \mathbb{R}^n$ be open and connected, $\partial\Omega \in C^{0,1}$, $1 \leq p \leq \infty$.*

Let $K \subset W^{1,p}(\Omega)$ be a closed (with respect to the $W^{1,p}$-norm) cone with only one constant function $u \equiv 0$. That is, let $K \subset W^{1,p}(\Omega)$ be a closed set such that

(1) $u \in K$ implies $\lambda u \in K$ for any $\lambda \geq 0$.
(2) if $u \in K$ and $u \equiv const$ then $u \equiv 0$.

Then there exists a constant $C = C(K,\Omega)$ such that

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)} \quad \forall u \in K. \tag{III.3.16}$$

The proof is a standard method from analysis, called a *blow-up proof*. One assumes that the claim is false, and then tries to compute/construct the "most extreme" counterexample – which one then hopes to see cannot exist. Before we begin, we need the following small Lemma.

Lemma III.3.26. *For $\Omega \subset \mathbb{R}^n$ open assume that $u \in W^{1,1}_{loc}(\Omega)$. If $\nabla u \equiv 0$ then u is constant in every connected component of Ω.*

Proof. This follows from (local) approximation by smooth functions. If $\nabla u \equiv 0$ then $\nabla u_\varepsilon \equiv 0$ in $\Omega_{-\varepsilon}$, where $u_\varepsilon = \eta_\varepsilon * u$. This implies that $u_\varepsilon \equiv const$ in every connected component of $\Omega_{-\varepsilon}$. Pointwise a.e. convergence of u_ε to u gives the claim. □

Proof of Theorem III.3.25. Assume the claim is false for a given K as above. That means however we choose the constant C there will be some countexample u that dails the claimed inequality (III.3.16).

That is, for any $m \in \mathbb{N}$ there exists $u_m \in K$ such that (III.3.16) is false for $C = m$, i.e.

$$\|u_m\|_{L^p(\Omega)} > m \, \|\nabla u_m\|_{L^p(\Omega)}. \tag{III.3.17}$$

Now we construct the "extreme/blown up" counterexample (that, as we shall see, does not exist – leading to a contradicition).

Firstly, we can assume w.l.o.g.

$$\|u_m\|_{L^p(\Omega)} = 1, \quad \|\nabla u_m\|_{L^p(\Omega)} \leq \frac{1}{m}. \tag{III.3.18}$$

Indeed otherwise we can just take $\tilde{u}_m := \frac{u_m}{\|u_m\|_{L^p(\Omega)}}$ which satisfies (III.3.18).

(III.3.18) implies in particular,

$$\sup_m \|u_m\|_{W^{1,p}(\Omega)} < \infty.$$

In view of Rellich's theorem, Theorem III.3.24, we can thus assume w.l.o.g. (otherwise taking a subsequence) that u_m is convergent in $L^p(\Omega)$. In particular u_m is a Cauchy sequence in $L^p(\Omega)$. Observe that also ∇u_m is a cauchy sequence in $L^p(\Omega)$, indeed by (III.3.18) $\nabla u_m \xrightarrow{m\to\infty} 0$ in $L^p(\Omega)$. In particular, u is a Cauchy sequence in $W^{1,p}(\Omega)$. Since $W^{1,p}(\Omega)$ is a Banach space we find a limit map $u \in W^{1,p}(\Omega)$ such that

$$\|u_m - u\|_{W^{1,p}(\Omega)} \xrightarrow{m\to\infty} 0. \tag{III.3.19}$$

In view of (III.3.18) this implies that $\nabla u \equiv 0$. From Lemma III.3.26 and since Ω is connected, u is a constant map. But since K is closed we have that $u \in K$, and since the only constant map in K is the constant zero map, we find $u \equiv 0$ in Ω. But then by (III.3.19)

$$\|u_m\|_{W^{1,p}(\Omega)} \xrightarrow{m\to\infty} 0.$$

which contradicts the conditions in (III.3.18), namely

$$\|u_m\|_{W^{1,p}(\Omega)} \geq \|u_m\|_{L^p(\Omega)} \overset{\text{(III.3.18)}}{=} 1.$$

We have found a contradiction, and thus the assumption above (that for any m there exists u_m that contradicts the claimed equation) is false. So there must be some number m such that for $C := m$ the equation (III.3.16) holds. □

Corollary III.3.27 (Poincaré type lemma)**.** *Let $\Omega \subset\subset \mathbb{R}^n$ be open, connected, and $\partial\Omega \in C^{0,1}$.*

(1) There exists $C = C(\Omega)$ such that for all $u \in W^{1,p}(\Omega)$ we have

$$\|u - (u)_\Omega\|_{L^p(\Omega)} \leq C(\Omega)\|\nabla u\|_{L^p(\Omega)}$$

(2) For any $\Omega' \subset\subset \Omega$ open and nonempty there exists $C = C(\Omega, \Omega')$ such that for all $u \in W^{1,p}(\Omega)$ we have

$$\|u - (u)_{\Omega'}\|_{L^p(\Omega)} \leq C(\Omega, \Omega')\|\nabla u\|_{L^p(\Omega)}$$

(3) There exists $C = C(\Omega)$ such that for all $u \in W^{1,p}_0(\Omega)$

$$\|u\|_{L^p(\Omega)} \leq C(\Omega)\|\nabla u\|_{L^p(\Omega)}$$

If $\Omega = B(x, r)$ (and in the second claim $\Omega' B(x, \lambda r)$) then $C(\Omega) = C(B(0, 1))\, r$ (and for the second claim: $C(\Omega, \Omega') = C(B(0, 1), B(0, \lambda))\, r$).

Proof. The last claim can be proven by a scaling argument, and it is given as an exercise.

Regarding the first claim, we simply let

$$K := \{u \in W^{1,p}(\Omega), (u)_\Omega = 0\}.$$

By Rellich's theorem, Theorem III.3.24 this is a closed cone in $W^{1,p}$. Observe that if $u \in K$ is constant, $u \equiv C$ then $(u)_\Omega = C = 0$ by assumption, so $C = 0$. That is, the only constant function in K is the zero-function. Clearly $u - (u)_\Omega$ belongs to K, so we get the claim.

Regarding the second claim, we argue similarly setting

$$K := \{u \in W^{1,p}(\Omega), (u)_{\Omega'} = 0\}.$$

Regarding the third claim, observe that $W^{1,p}_0(\Omega)$ is (by definition) a closed set, and since it is a linear space it is in particular a cone. Now if $u \in W^{1,p}_0(\Omega)$ is constant, $u \equiv c$ then u is in particular continuous, but then by the zero trace theorem, Theorem III.3.22 $c \equiv 0$. Again, the only constant function in K is the zero-function. □

We have seen in Theorem III.3.24 and used in the Poincaré inequality that $W^{1,p}_{loc}(\Omega)$ embedds *compactly* into $L^p_{loc}(\Omega)$. There is a meta-theorem/feeling that "above" any compact

embedding there is a merely continuous embedding, for more precise versions of this effect see [**Hajłasz and Liu, 2010**].

In our case it is that $W^{1,p}$ embedds into L^{p^*} where p^* follows the following rule

$$1 - \frac{n}{p} = 0 - \frac{n}{p^*} \tag{III.3.20}$$

(we will see this numerology appear later again for Morrey and Sobolev-Poincaré embedding, Corollary III.3.31 and Theorem III.3.35). Observe that $p^* = \frac{np}{n-p} \in (1, \infty)$ for $p < n$. We set $p^* := \infty$ for $p \geq n$. p^* is called the *Sobolev exponent*. What happens if $p^* > n$ (which should be interpreted from this numerological point of view as $p^* > \infty$? Theorem III.3.35 will tell us: u is Hölder continuous.

THEOREM III.3.28 (Sobolev inequality). *Let $p \in [1, \infty)$ such that $p^* := \frac{np}{n-p} \in (1, \infty)$ (equivalently: $p \in [1, n)$). Then $W^{1,p}(\mathbb{R}^n)$ embedds into $L^{p^*}(\mathbb{R}^n)$. That is, if $u \in W^{1,p}(\mathbb{R}^n)$ then $u \in L^{p^*}(\mathbb{R}^n)$ and we have*[8]

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C(p, n) \, \|Du\|_{L^p(\mathbb{R}^n)}.$$

PROOF OF THEOREM III.3.28. There are more than one way to prove Sobolev's inequality. One (the "Harmonic Analysis" one) is by convolution, using the Riesz potential representation and boundedness of Riesz transform on L^p-spaces. It is very strong and general but beyond the scope of these lectures.

The one we present here is an elegant trick due to Nirenberg (here we are again!). It is much less stable, relies heavily on the structure of $\mathbb{R}^n$, etc., but it obtains the case $p = 1$ (that in general is much more difficult to obtain), see e.g. [**Schikorra et al., 2017**].

By approximation it suffices to assume that $u \in C_c^\infty(\mathbb{R}^n)$.

Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Then we have by the fundamental theorem of calculus

$$u(x_1, x_2, \ldots, x_n) = u(y_1, x_2, \ldots, x_n) + \int_{x_1}^{y_1} \partial_1 u(z_1, x_2, \ldots, x_n) \, dz_1.$$

Taking y_1 large enough we have $u(y_1, x_2, \ldots, x_n) = 0$, since $\operatorname{supp} u \subset\subset \mathbb{R}^n$. Thus we obtain the estimate

$$|u(x_1, x_2, \ldots, x_n)| \leq \int_{\mathbb{R}} |Du(z_1, x_2, \ldots, x_n)| \, dz_1.$$

The same way we obtain, for any $\ell = 1, \ldots, n$,

$$|u(x_1, x_2, \ldots, x_n)| \leq \int_{\mathbb{R}} |Du(x_1, x_2, \ldots, z_\ell, \ldots x_n)| \, dz_\ell.$$

Multiplying these estimates for $\ell = 1, \ldots, n$ we obtain

$$|u(x_1, x_2, \ldots, x_n)|^n \leq \Pi_{\ell=1}^n \int_{\mathbb{R}} |Du(x_1, x_2, \ldots, z_\ell, \ldots x_n)| \, dz_\ell.$$

[8] The optimal constant $C(p, n)$ has actually a geometric meaning, and is related to the isoperimetric inequality, cf. [**Talenti, 1976**]

Now we prove the case $p = 1$, when $p^* = \frac{n}{n-1}$. We have

$$\int_{\mathbb{R}} |u(x_1, x_2, \dots, x_n)|^{\frac{n}{n-1}} dx_1 \le \int_{\mathbb{R}} \Pi_{\ell=1}^n \left(\int_{\mathbb{R}} |Du(x_1, x_2, \dots, z_\ell, \dots x_n)| \, dz_\ell \right)^{\frac{1}{n-1}} \, dx_1$$

$$\le \left(\int_{z_1 \in \mathbb{R}} |Du(z_1, x_2, \dots, x_n)| \, dz_1 \right)^{\frac{1}{n-1}} \int_{\mathbb{R}} \Pi_{\ell=2}^n \left(\int_{\mathbb{R}} |Du(x_1, x_2, \dots, z_\ell, \dots x_n)| \, dz_\ell \right)^{\frac{1}{n-1}} \, dx_1$$

Now by Hölder's inequality[9]

$$\int_{\mathbb{R}} \Pi_{\ell=2}^n \left(\int_{\mathbb{R}} |Du(x_1, x_2, \dots, z_\ell, \dots x_n)| \, dz_\ell \right)^{\frac{1}{n-1}} \, dx_1 \le \left(\Pi_{\ell=2}^n \int_{\mathbb{R}} \int_{\mathbb{R}} |Du(x_1, x_2, \dots, z_\ell, \dots x_n)| \, dz_\ell \, dx_1 \right)^{\frac{1}{n-1}}$$

and thus

$$\int_{\mathbb{R}} |u(x)|^{\frac{n}{n-1}} dx_1 \le \left(\int_{z_1 \in \mathbb{R}} |Du(z_1, x_2, \dots, x_n)| \, dz_1 \right)^{\frac{1}{n-1}} \left(\Pi_{\ell=2}^n \int_{\mathbb{R}} \int_{\mathbb{R}} |Du(x_1, x_2, \dots, z_\ell, \dots x_n)| \, dz_\ell \, dx_1 \right)^{\frac{1}{n-1}}$$

Now we integrate this with respect to x_2, and again by Hölder's inequality,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |u(x)|^{\frac{n}{n-1}} dx_1 dx_2$$

$$\le \int_{x_2 \in \mathbb{R}} \left(\int_{z_1 \in \mathbb{R}} |Du(z_1, x_2, \dots, x_n)| \, dz_1 \right)^{\frac{1}{n-1}} \left(\Pi_{\ell=2}^n \int_{\mathbb{R}} \int_{\mathbb{R}} |Du(x_1, x_2, \dots, z_\ell, \dots x_n)| \, dz_\ell \, dx_1 \right)^{\frac{1}{n-1}} \, dx_2$$

$$\le \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |Du(x_1, z_2, x_3, \dots x_n)| \, dz_2 \, dx_1 \right)^{\frac{1}{n-1}}$$

$$\cdot \int_{x_2 \in \mathbb{R}} \left(\int_{z_1 \in \mathbb{R}} |Du(z_1, x_2, \dots, x_n)| \, dz_1 \right)^{\frac{1}{n-1}} \left(\Pi_{\ell=3}^n \int_{\mathbb{R}} \int_{\mathbb{R}} |Du(x_1, x_2, \dots, z_\ell, \dots x_n)| \, dz_\ell \, dx_1 \right)^{\frac{1}{n-1}} \, dx_2$$

$$\le \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |Du(x_1, z_2, x_3, \dots x_n)| \, dz_2 \, dx_1 \right)^{\frac{1}{n-1}}$$

$$\cdot \left(\int_{x_2 \in \mathbb{R}} \int_{z_1 \in \mathbb{R}} |Du(z_1, x_2, \dots, x_n)| \, dz_1 \, dx_2 \right)^{\frac{1}{n-1}} \left(\Pi_{\ell=3}^n \int_{x_2 \in \mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} |Du(x_1, x_2, \dots, z_\ell, \dots x_n)| \, dz_\ell \, dx_1, dx_2 \right)$$

If $n = 2$ we are done (the $\Pi_{\ell=3}^n$-term is one). If $n \ge 3$ we see a pattern, continuing to integrate in $x_3, \dots, x_n$ we obtain

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |u(x)|^{\frac{n}{n-1}} dx_1 dx_2 \le \Pi_{\ell=1}^n \left(\int_{\mathbb{R}^n} |Du(x_1, \dots, x_{\ell-1}, z_\ell, x_{\ell+1} \dots x_n)| dx_1, \dots, x_{\ell-1}, z_\ell, x_{\ell+1} \dots x_n) \right)^{\frac{1}{n-1}}$$

$$= \left(\int_{\mathbb{R}^n} |Du| \right)^{\frac{n}{n-1}}.$$

Taking the exponent $\frac{n-1}{n}$ on both sides we obtain

$$\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le \|Du\|_{L^1(\mathbb{R}^n)} \tag{III.3.21}$$

[9] the generalized version for $k := n-1$ and all $p_i := n-1$: whenever $p_1, \dots, p_k \in [1, \infty]$ and $\sum_i \frac{1}{p_i} = 1$,

$$\int_{\mathbb{R}^d} \Pi_{i=1}^k |f_i| \le \Pi_{i=1}^k \left(\int_{\mathbb{R}^d} |f_i|^{p_i} \right)^{\frac{1}{p_i}}$$

This is the claim for $p = 1$ (i.e. $p^* = \frac{n}{n-1}$).

The general claim follows when we apply the $p = 1$ Sobolev inequality to $v := |u|^\gamma$ for some $\gamma > 1$ that we choose later. We have

$$|Dv| = |D|u|^\gamma| \leq \gamma |u|^{\gamma-1}|Du|,$$

thus (III.3.21) applied to v

$$\||u|^\gamma\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq C(\gamma)\||u|^{\gamma-1}\,|Du|\|_{L^1(\mathbb{R}^n)} \tag{III.3.22}$$

Now observe that

$$\||u|^\gamma\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} = \|u\|^\gamma_{L^{\gamma\frac{n}{n-1}}(\mathbb{R}^n)}$$

Moreover, by Hölder's inequality, $p' = \frac{p}{p-1}$,

$$\||u|^{\gamma-1}\,|Du|\|_{L^1(\mathbb{R}^n)} \leq \||u|^{\gamma-1}\|_{L^{p'}(\mathbb{R}^n)}\|Du\|_{L^p(\mathbb{R}^n)} = \|u\|^{\gamma-1}_{L^{p'(\gamma-1)}(\mathbb{R}^n)}\|Du\|_{L^p(\mathbb{R}^n)}$$

So (III.3.22) becomes

$$\|u\|^\gamma_{L^{\gamma\frac{n}{n-1}}(\mathbb{R}^n)}\|u\|^{1-\gamma}_{L^{p'(\gamma-1)}(\mathbb{R}^n)} \leq C(\gamma)\|Du\|_{L^p(\mathbb{R}^n)}$$

Choosing $\gamma := \frac{p(n-1)}{n-p} > 1$ we have $\gamma\frac{n}{n-1} = p'(\gamma-1) = p^*$, and then

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C(\gamma)\|Du\|_{L^p(\mathbb{R}^n)}.$$

□

The relation between p and p^* in Theorem III.3.28 is sharp in the following sense

Lemma III.3.29. *Assume that $p, q \in (1, \infty)$ are such that for all $u \in C_c^\infty(\mathbb{R}^n)$*

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C(p, n)\ \|Du\|_{L^p(\mathbb{R}^n)}. \tag{III.3.23}$$

Then $p = q^*$.

Proof. This is proven by a *scaling argument*. Assume (III.3.23) holds. Take an arbitrary $u \in C_c^\infty(\mathbb{R}^n)$ such that $\|Du\|_{L^p(\mathbb{R}^n)} \geq 1$, $\|u\|_{L^p(\mathbb{R}^n)} \geq 1$.

We rescale u and set for $\lambda > 0$,

$$u_\lambda(x) := u(\lambda x).$$

We apply (III.3.23) to u_λ. Observe that by substitution

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} = \lambda^{-\frac{n}{q}}\|u\|_{L^q(\mathbb{R}^n)},$$

and since $\nabla u_\lambda = \lambda(\nabla u)_\lambda$ we have

$$\|\nabla u_\lambda\|_{L^p(\mathbb{R}^n)} = \lambda^{1-\frac{n}{p}}\|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

From (III.3.23) applied to u_λ we then obtain for any $\lambda > 0$,

$$\lambda^{-\frac{n}{q}}\|u\|_{L^q(\mathbb{R}^n)} \leq \lambda^{1-\frac{n}{p}}\|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

Equivalently, setting $\Lambda := \|\nabla u\|_{L^q(\mathbb{R}^n)}/\|u\|_{L^q(\mathbb{R}^n)} > 0$ we obtain

$$\lambda^{0-\frac{n}{q}-(1-\frac{n}{p})} \leq \Lambda \quad \forall \lambda > 0$$

The exponent above the λ is exactly the numerology of (III.3.20)! In particular, if $q \neq p^*$ then $\sigma := 0 - \frac{n}{q} - (1 - \frac{n}{p}) \neq 0$, and we have

$$\lambda^\sigma \leq \Lambda \quad \forall \lambda > 0$$

If $\sigma > 0$ we let $\lambda \to \infty$, if $\sigma < 0$ we let $\lambda \to 0^+$ to get a contradiction. Thus, necessarily $\sigma = 0$, that is $q = p^*$. □

Corollary III.3.30 (Sobolev-Poincaré embedding)**.** *Let $u \in W^{1,p}(\mathbb{R}^n)$, $1 \leq p < n$. For any $q \in [p, p^*]$ we have $u \in L^q(\mathbb{R}^n)$ with the estimate*

$$\|f\|^q_{L^q(\mathbb{R}^n)} \leq C(q,n) \left(\|f\|^p_{L^p(\mathbb{R}^N)} + \|Df\|^{p^*}_{L^p(\mathbb{R}^n)} \right).$$

Proof. Clearly the claim holds for $q = p$ and, by Theorem III.3.28, for $q = p^*$.

Now observe that for $q \in [p, p^*]$ we can estimate the L^q-norm by the L^p-norm and the L^{p^*}-norm (this technique is called *interpolation*).

$$\int_{\mathbb{R}^n} |f|^q = \int_{\mathbb{R}^n} |f|^q \chi_{|f|>1} + \int_{\mathbb{R}^n} |f|^q \chi_{|f|\leq 1} \leq \int_{\mathbb{R}^n} |f|^{p^*} + \int_{\mathbb{R}^n} |f|^p.$$

That is,

$$\|f\|^q_{L^q(\mathbb{R}^n)} \leq \|f\|^p_{L^p(\mathbb{R}^n)} + \|f\|^{p^*}_{L^{p^*}(\mathbb{R}^n)} \lesssim \|f\|^p_{L^p(\mathbb{R}^n)} + \|Df\|^{p^*}_{L^p(\mathbb{R}^n)}.$$

□

Corollary III.3.31 (Sobolev-Poincaré embedding on domains)**.** *Let $\Omega \subset \mathbb{R}^n$ and $\partial\Omega$ be C^1. For $1 \leq p < n$ we have for any $u \in W^{1,p}(\Omega)$,*

$$\|u\|_{L^{p^*}(\Omega)} \leq C(\Omega) \left(\|u\|_{L^p(\Omega)} + \|Du\|_{L^p(\Omega)} \right)$$

Also, for any $q \in [p, p^*]$ [10]

$$\|u\|_{L^q(\Omega)} \leq C(\Omega, q, \|u\|_{W^{1,p}(\Omega)}).$$

If moreover $\Omega \subset\subset \mathbb{R}^n$ and $u \in W^{1,p}_0(\Omega)$ then

$$\|u\|_{L^{p^*}(\Omega)} \leq C(\Omega) \|Du\|_{L^p(\Omega)}.$$

Lastly, if $1 \leq p < \infty$ and $\Omega \subset\subset \mathbb{R}^n$, $u \in W^{1,p}(\Omega)$ then for any $q \in [1, p^*]$ (if $p < n$) or for any $q \in [1, \infty)$ (if $p \geq n$)

$$\|u\|_{L^q(\Omega)} \leq C(\Omega, q, p, n) \ \|u\|_{W^{1,p}(\Omega)}.$$

[10] This means the following: For any $\Lambda > 0$ there exists a constant $C(\Omega, q, \Lambda)$ such that

$$\|u\|_{L^q(\Omega)} \leq C(\Omega, q, \Lambda) \quad \forall u : \ \|u\|_{W^{1,p}(\Omega)} \leq \Lambda.$$

PROOF. By the extension theorem, Theorem III.3.20, we can extend u to $Eu \in W^{1,p}(\mathbb{R}^n)$. Then from Sobolev inequality, Theorem III.3.28, we get

$$\|u\|_{L^{p^*}(\Omega)} \leq \|Eu\|_{L^{p^*}(\mathbb{R}^n)} \lesssim \|DEu\|_{L^p(\mathbb{R}^n)} \lesssim C(\Omega)\|u\|_{W^{1,p}(\Omega)} \leq C(\Omega)\left(\|u\|_{L^p(\Omega)} + \|Du\|_{L^p(\Omega)}\right).$$

The second claim follows from the same argument using Corollary III.3.30. The third claim follows from Poincaré inequality, Corollary III.3.27, since for $u \in W_0^{1,p}(\Omega)$, $\Omega \subset\subset \mathbb{R}^n$ we have

$$\|u\|_{L^p(\Omega)} \leq C(\Omega)\|Du\|_{L^p(\Omega)}.$$

The last claim follows by additionall using Hölder's inequality: if $p < n$, $q \in [1, p^*]$,

$$\|u\|_{L^q(\Omega)} \leq C(|\Omega|, q, p)\|u\|_{L^{p^*}(\Omega)} \leq C(|\Omega|, q, p)\|u\|_{W^{1,p}(\Omega)}.$$

If $p \geq n$ and $q \in [p, \infty)$ we can find $r \leq p$ such that $r^* > q$. Thus, from first Hölder's inequality, then Sobolev inequality, and then again Hölder's inequality,

$$\|u\|_{L^q(\Omega)} \leq C(|\Omega|, q, p)\|u\|_{L^{r^*}(\Omega)} \leq C(|\Omega|, q, p)\|u\|_{W^{1,r}(\Omega)} \leq C(|\Omega|, q, p)\|u\|_{W^{1,p}(\Omega)}.$$

□

THEOREM III.3.32 (Sobolev Embedding). *Let $\Omega \subset\subset \mathbb{R}^n$ be open, $\partial\Omega \in C^{0,1}$, $k \geq \ell$ for $k, \ell \in \mathbb{N} \cup \{0\}$, and $1 \leq p, q < \infty$ such that (compare with (III.3.20))*

$$k - \frac{n}{p} \geq \ell - \frac{n}{q}. \tag{III.3.24}$$

Then the identity is a continuous embedding $W^{k,p}(\Omega) \hookrightarrow W^{\ell,q}(\Omega)$. That is,

$$\|u\|_{W^{\ell,q}(\Omega)} \leq C(\|u\|_{W^{k,p}(\Omega)}) \tag{III.3.25}$$

If $k > \ell$ and we have the strict inequality

$$k - \frac{n}{p} > \ell - \frac{n}{q}, \tag{III.3.26}$$

then the embedding above is compact. That is, whenever $(u_i)_{i\in\mathbb{N}} \subset W^{k,p}(\Omega)$ such that

$$\sup_i \|u_i\|_{W^{k,p}(\Omega)} < \infty$$

then there exists a subsequence $(u_{i_j})_{j\in\mathbb{N}}$ such that $(u_{i_j})_{j\in\mathbb{N}}$ is convergent in $W^{\ell,q}(\Omega)$.

PROOF. If $k = \ell$, then (III.3.24) implies $p \geq q$. Thus, in that case (III.3.25) follows from the Hölder's inequality:

$$\|u\|_{W^{\ell,q}(\Omega)} \leq C(|\Omega|, n)\ \|u\|_{W^{\ell,p}(\Omega)} = C(|\Omega|, n)\ \|u\|_{W^{k,p}(\Omega)}$$

Next we us assume $k = \ell + 1$. Then (III.3.24) implies that $q \leq p^*$ (if $p < n$) or $q < \infty$ (for $p > n$), wher we recall the Sobolev exponent $p^* := \frac{np}{n-p}$. Then by Sobolev inequality, Corollary III.3.31,

$$\|f\|_{L^q(\Omega)} \leq C(q, \Omega)\|f\|_{W^{1,p}(\Omega)}.$$

Applying this inequality to $f := \partial^\gamma u$ for $|\gamma| \leq \ell$ we obtain (III.3.25) for $k = \ell + 1$, namely for $q \leq p^*$,

$$\|u\|_{W^{\ell,q}(\Omega)} \leq C(p, q, \Omega)\|u\|_{W^{\ell+1,p}(\Omega)}$$

More generally If $k = \ell + N$ for some $N \in \mathbb{N}$, set $r_i := (r_{i-1})^*$ for $i = 1, \dots, N$ with $r_0 := p$. This works well if all of the $r_i^* \neq \infty$ (otherwise we choose $r_i \leq (r_{i_1})^*$ and $r_0 < p$, but large enough such that $r_N > q$). Then (III.3.24) implies that $q \leq r_N$, and we get first by Hölder's inequality then by the argument above iterated

$$\|u\|_{W^{\ell,q}(\Omega)} \lesssim \|u\|_{W^{\ell,r_N}(\Omega)} \lesssim \|u\|_{W^{\ell+1,r_{N-1}}(\Omega)} \lesssim \dots \lesssim \|u\|_{W^{k,r_0}(\Omega)} \lesssim \|u\|_{W^{k,p}(\Omega)}.$$

This proves the continuous embedding, (III.3.26) in full generality.

As for the compact embedding, it suffices to assume $k = \ell+1$. This is because combinations of continuous and compact embeddings are compact, so if we show the compactness of the embedding satisfying (III.3.26) for $k = \ell + 1$ then we can build a chain of embeddings as above to get a compact embedding for all $k > \ell$.

Moreover, we can assume w.l.o.g. $k = 1, \ell = 0$. The general case then follows by considering $\partial^\gamma u$ for $|\gamma| \leq \ell$.

So let $1 \leq q < p^*$ (i.e. (III.3.26) and assume that we have a sequence (u_i) such that

$$\sup_i \|u_i\|_{W^{1,p}(\Omega)} < \infty.$$

By Sobolev's inequality, Corollary III.3.31 this implies also for some $r \in (q, p^*)$,

$$\Lambda := \sup_i \|u_i\|_{L^r(\Omega)} < \infty. \tag{III.3.27}$$

By Rellichs theorem, Theorem III.3.24, we can find a subsequence u_{i_j} that is strongly convergent in $L^p(\Omega)$ and in particular we can choose the subsequence such that u_{i_j} converges pointwise a.e. to some $u \in L^q(\Omega)$ (that u belongs to L^r, and thus to L^q follows from the weak compactness, Theorem III.3.16).

Now we use Vitali's convergence theorem[11]. To show the uniform absolute continuity of the integral let $\varepsilon > 0$ and for some δ to be chosen (independent of j) let $E \subset \Omega$ be measurable with $|E| < \delta$. Then we have by Hölder's inequality (recall Λ from (III.3.27))

$$\sup_j \|u_{i_j}\|_{L^q(E)} \leq |E|^{\frac{1}{q}-\frac{1}{r}} \sup_j \|u_{i_j}\|_{L^r(E)} \leq \delta^{\frac{1}{q}-\frac{1}{r}}\Lambda.$$

So if we choose $\delta = \delta(\varepsilon, \Lambda) > 0$ small, so that

$$\delta^{\frac{1}{q}-\frac{1}{p}}\Lambda < \varepsilon,$$

then

$$\sup_j \|u_{i_j}\|_{L^q(E)} < \varepsilon \quad \text{whenever } E \subset \Omega \text{ measurable and } |E| < \delta.$$

This is uniform absolute continuity, and by Vitali's theorem u_{i_j} is convergent in $L^q(\Omega)$. This shows compactness, and Sobolev's embedding theorem is proven. □

[11]See wikipedia. It essentially shows that almost everywhere convergent sequences f_i of functions converge also in $L^q(\Omega)$ if (and only if) we have uniform absolute continuity of the integrals

Our next goal is Morrey's embedding theorem, Theorem III.3.35. For this we use a characterization of Hölder functions by so-called Campanato spaces.

THEOREM III.3.33 (Campanato's theorem). *Let $u \in L^1(\mathbb{R}^n)$ and assume that for some $\lambda > 0$*

$$\Lambda := \sup_{B(x,r)\subset\mathbb{R}^n} r^{-\lambda} \fint_{B(x,r)} |u - (u)_{B(x,r)}| < \infty, \tag{III.3.28}$$

where

$$(u)_{B(x,r)} = \fint_{B(x,r)} u.$$

Then $u \in C^\lambda_{loc}(\mathbb{R}^n)$ and we have for some uniform constant $C = C(n,\lambda) > 0$

$$\sup_{x\neq y} \frac{|u(x) - u(y)|}{|x-y|^\lambda} \leq C\ \Lambda.$$

Remark III.3.34. The converse also holds, if $u \in C^\alpha$ than $\Lambda < [u]_{C^\alpha}$, which is an easy exercise to check.

PROOF OF THEOREM III.3.33. First we claim that for any $R > 0$ and almost any $x \in \mathbb{R}^n$ we have for some uniform constant $C > 0$

$$|u(x) - (u)_{B(x,R)}| \leq C\ R^\lambda \Lambda. \tag{III.3.29}$$

To see this, observe that for almost every $x \in \mathbb{R}^n$, by Lebesgue's theorem, $\lim_{k\to\infty}(u)_{B(x,2^{-k}R)} = u(x)$. Thus, by a telescoping sum

$$|u(x) - (u)_{B(x,R)}| \leq \sum_{k=0}^\infty \left|(u)_{B(x,2^{-k}R)} - (u)_{B(x,2^{-(k+1)}R)}\right| \tag{III.3.30}$$

Now,

$$\begin{aligned}
&\left|(u)_{B(x,2^{-k}R)} - (u)_{B(x,2^{-(k+1)}R)}\right| \\
\leq &\fint_{B(x,2^{-(k+1)}R)} \left|u(x) - (u)_{B(x,2^{-k}R)}\right| \\
\leq &\underbrace{\frac{|B(x,2^{-k}R)|}{|B(x,2^{-k+1}R)|}}_{=C(n)} \fint_{B(x,2^{-k}R)} \left|u(x) - (u)_{B(x,2^{-k}R)}\right| \\
\overset{(III.3.28)}{\leq} &C(n)\,\Lambda\,(2^{-k}R)^\lambda.
\end{aligned}$$

Plugging this into (III.3.30) we get

$$|u(x) - (u)_{B(x,R)}| \leq C(n)\Lambda R^\lambda \sum_{k=0}^\infty 2^{-k\lambda} \overset{\lambda\geq 0}{=} C(\lambda,n)\Lambda\, R^\lambda,$$

i.e. (III.3.29) is established.

Now let $x, y \in \mathbb{R}^n$. Set $R := |x-y|$. Then

$$\begin{aligned} |u(x)-u(y)| \leq & |u(x)-(u)_{B(x,R)}| + |u(x)-(u)_{B(y,R)}| + |(u)_{B(y,R)}-(u)_{B(x,R)}| \\ \overset{\text{(III.3.29)}}{\leq} & C(n,\lambda)|x-y|^\lambda + |(u)_{B(y,R)}-(u)_{B(x,R)}|. \end{aligned} \tag{III.3.31}$$

We have to estimate the last term, which we do as above: Observe that $B(x,2R) \supset B(y,R) \cup B(x,R)$,

$$\begin{aligned} & |(u)_{B(y,R)}-(u)_{B(x,R)}| \\ \leq & \fint_{B(x,R)} \fint_{B(y,R)} |u(z_1)-u(z_2)| dz_1 dz_2 \\ \leq & \underbrace{\frac{|B(x,2R)|}{|B(y,R)|}\frac{|B(x,2R)|}{|B(x,R)|}}_{=C(n)} \fint_{B(x,2R)} \fint_{B(x,2R)} |u(z_1)-u(z_2)| dz_1 dz_2 \\ = & C(n) \fint_{B(x,2R)} \fint_{B(x,2R)} |u(z_1)-(u)_{B(x,2R)}| dz_1 dz_2 + C(n) \fint_{B(x,2R)} \fint_{B(x,2R)} |(u)_{B(x,2R)}-u(z_2)| dz_1 dz_2 \\ = & C(n) \fint_{B(x,2R)} \fint_{B(x,2R)} |u(z_1)-(u)_{B(x,2R)}| dz_1 dz_2 + C(n) \fint_{B(x,2R)} \fint_{B(x,2R)} |(u)_{B(x,2R)}-u(z_2)| dz_1 dz_2 \\ = & 2C(n) \fint_{B(x,2R)} |u(\tilde{z})-(u)_{B(x,2R)}| d\tilde{z} \\ \overset{\text{(III.3.28)}}{\leq} & 2C(n,\lambda)R^\lambda. \end{aligned}$$

Since $R = |x-y|$, together with (III.3.31) we have shown

$$|u(x)-u(y)| \leq C(n,\lambda)|x-y|^\lambda,$$

and can conclude. □

THEOREM III.3.35 (Morrey Embedding). *Let $\Omega \subset\subset \mathbb{R}^n$ with $\partial\Omega \in C^k$, $k \in \mathbb{N}$. Assume that for $p \in (1,\infty)$, $\alpha \in (0,1)$ and $\ell < k$ we have*

$$k - \frac{n}{p} \geq \ell + \alpha.$$

Then the embedding $W^{k,p}(\Omega) \hookrightarrow C^{\ell,\alpha}(\overline{\Omega})$ is continuous.

If $k - \frac{n}{p} > \ell + \alpha$ then the embedding is compact.

PROOF. Let $u \in W^{k,p}(\Omega)$. By Extension Theorem, Theorem III.3.20, we can assume $u \in W^{k,p}(\mathbb{R}^n)$ and $\operatorname{supp} u \subset\subset B(0,R)$ for some large $R > 0$.

As in the Sobolev theorem it suffices to assume $\ell = k-1$, and indeed we can reduce to the case $k = 1$ and $\ell = 0$.

We use Campanato's Theorem, Theorem III.3.33. For $B(x,r) \subset \mathbb{R}^n$, we have by Poincaré's inequality, Corollary III.3.27, and then Hölder's inequality,

$$\fint_{B(x,r)} |u-(u)_{B(x,r)}| \leq r^{1-\lambda} \fint_{B(x,r)} |Du| = Cr^{1-n} \int_{B(x,r)} |Du| \leq Cr^{1-n} r^{n-\frac{n}{p}} \left(\int_{B(x,r)} |Du|^p \right)^{\frac{1}{p}}$$

That is,

$$\sup_{B(x,r)\subset\mathbb{R}^n} r^{-\left(1-\frac{n}{p}\right)} \fint_{B(x,r)} |u-(u)_{B(x,r)}| \leq C\|Du\|_{L^p(\mathbb{R}^n)}$$

Thus, by Campanato's theorem, if $1-\frac{n}{p} = 0+\alpha \in (0,1)$, then (using also the extension theorem estimate),

$$[u]_{C^\alpha(\Omega)} \leq [u]_{C^\alpha(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega)},$$

which is the continuity of the embedding of $W^{1,p}(\Omega)$ in $C^{0,\alpha}$ if $1-\frac{n}{p} = 0+\alpha$.

If on the other hand $1-\frac{n}{p} > 0+\alpha$, then we use Arzela-Ascoli to show that the embedding $L^\infty \cap C^\beta(\mathbb{R}^n) \hookrightarrow L^\infty \cap C^\alpha(\mathbb{R}^n)$ is compact if $\beta > \alpha$, and from this we conclude the compactness of the embedding $W^{1,p}(\Omega) \hookrightarrow C^\alpha(\overline{\Omega})$ if $1-\frac{n}{p} > \alpha$. $\square$

CHAPTER 4

Existence and Regularity for linear elliptic PDE

Our main goal in this section is the following.

THEOREM IV.0.1. *Let $\Omega \subset\subset \mathbb{R}^n$ with $\partial\Omega \in C^\infty$. For any $f \in L^2(\Omega)$ there exists a unique solution $u \in W_0^{1,2}(\Omega)$*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

I.e.

$$\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f\varphi \quad \forall \varphi \in C_c^\infty(\Omega).$$

Moreover we have interior regularity: if for any $k \geq 0$ we have additionally $f \in W_{loc}^{k,2}(\Omega)$ then $u \in W_{loc}^{k+2,2}(\Omega)$ and the equation above holds almost everywhere in Ω.

In particular if $f \in C^\infty(\Omega)$ then $u \in C^\infty(\Omega)$ and the equation holds pointwise[1]

We split the proof of Theorem IV.0.1 into two parts. First we prove existence and uniqueness for f in dual space, $f \in (W^{1,2}(\Omega))^*$. That is assume that f is a linear, bounded functional on $W^{1,2}(\Omega)$ with

$$\|f\|_{(W^{1,2}(\Omega))^*} := \sup_{\|\varphi\|_{W^{1,2}(\Omega)} \leq 1} f[\varphi] < \infty.$$

One should think of two standard objects: For some $g \in L^2(\Omega)$

$$f[\varphi] := \int_\Omega g\varphi$$

and

$$f[\varphi] := \int_\Omega g\partial_i\varphi$$

Oberserve that in both cases (by Poincaré inequality),

$$\|f\|_{(W^{1,2}(\Omega))^*} \leq \|f\|_{L^2(\Omega)}.$$

THEOREM IV.0.2. *Let $\Omega \subset\subset \mathbb{R}^n$ with $\partial\Omega \in C^\infty$.*

[1] this last part follows from the Morrey embedding, Theorem III.3.35

Assume that $f \in (W^{1,2}(\Omega))^*$, i.e. assume that f is a linear, bounded functional on $W^{1,2}(\Omega)$ with

$$\|f\|_{(W^{1,2}(\Omega))^*} := \sup_{\|\varphi\|_{W^{1,2}(\Omega)\leq 1}} f[\varphi] < \infty.$$

Then there exists a unique (weak) solution to

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{IV.0.1}$$

and it satisfies

$$\|u\|_{W^{1,2}(\Omega)} \leq C \ \|f\|_{(W^{1,2}(\Omega))^*}.$$

Here the equation

Proof. We use what is called the direct method of Calculus of Variations[2]: Set

$$\mathcal{E}(u) := \frac{1}{2}\int_\Omega |Du|^2 + f[u].$$

As in Section I.2.7 one can check that there is at most one minimizer in $W^{1,2}_0(\Omega)$ of this functional, that any minimizer is a solution to (IV.0.1) and that any solution is a minimizer.

So all that is needed to show is the existence of a minimizer with the claimed estimate.

Let $u_k \in W^{1,2}_0(\Omega)$ be a sequence that approximates $\inf \mathcal{E}$ (this exists by the very definition of inf),

$$\lim_{k\to\infty} \mathcal{E}(u_k) = \inf_{W^{1,2}_0(\Omega)} \mathcal{E}.$$

In particular, we can assume that $\mathcal{E}(u_k) \leq \mathcal{E}(0) = 0$ for all $k \in \mathbb{N}$. Now observe that

$$\frac{1}{2}\|Du_k\|^2_{L^2(\Omega)} = \mathcal{E}(u_k) + f[u_k] \leq 0 + \|f\|_{(W^{1,2}(\Omega))^*} \|u_k\|_{W^{1,2}(\Omega)}.$$

That is, by Poincaré inequality, Corollary III.3.27,

$$\|u_k\|^2_{W^{1,2}(\Omega)} \leq C\|f\|_{(W^{1,2}(\Omega))^*} \|u_k\|_{W^{1,2}(\Omega)}.$$

Dividing both sides by $\|u_k\|_{W^{1,2}(\Omega)}$ we get

$$\sup_k \|u_k\|_{W^{1,2}(\Omega)} \leq C\|f\|_{W^{1,2}(\Omega)}.$$

That is u_k is uniformly bounded in $W^{1,2}(\Omega)$. By the weak compactness theorem, Theorem III.3.18, we can thus (up to taking a subsequence) assume u_k weakly converging to $u \in W^{1,2}_0(\Omega)$, which in particular implies

$$f[u_k] \xrightarrow{k\to\infty} f[u].$$

Moroever,

$$\|Du\|_{L^2(\Omega)} \leq \liminf_{k\to\infty} \|Du_k\|_{L^2(\Omega)} \leq \|f\|_{(W^{1,2}(\Omega))^*}.$$

[2]we did something very similar in the variational methods section, Section I.2.7, but we did not have the tools to show existence of a minimizer

Thus, we conclude

$$E(u) \leq \liminf_{k\to\infty} E(u_k) = \inf_{W_0^{1,2}(\Omega)} E.$$

But since $u \in W_0^{1,2}(\Omega)$ we also have

$$E(u) \geq \inf_{W_0^{1,2}(\Omega)} E,$$

and thus

$$E(u) = \inf_{W_0^{1,2}(\Omega)} E.$$

That is we have found a minimizer of E. □

THEOREM IV.0.3. *Let $u \in W_0^{1,2}(\Omega)$ solve*

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for some $f \in W_{loc}^{k,2}(\Omega)$, $k \geq 0$. Then $u \in W_{loc}^{k+2,0}(\Omega)$.

IDEA OF THE PROOF. Let $k = 0$.

Essentially we differentiate the equation,

$$\Delta \partial_\alpha u = \partial_\alpha f.$$

Observe that if $f \in L_{loc}^2$ then $\partial_\alpha f \in W_{loc}^{-1,2}$, where $W^{-1,2}$ denotes the dual $(W^{1,2})^*$.

By Theorem IV.0.2 we could hope that since there is only one solution to the above equation, we get

$$\|\partial_\alpha u\|_{W^{1,2}} \leq C \, \|\partial_\alpha f\|_{(W^{1,2})^*} \leq \|f\|_{L^2}.$$

For $k \geq 1$ we would then argue by an induction.

The above is a great idea, but there are different problems: $\partial_\alpha u$ does not a priori belong to $W^{1,2}$, but even if we knew that: $\partial_\alpha u$ has no reason to have the zero-boundary data $W_0^{1,2}(\Omega)$.

To overcome the first obstacle we use difference quotients, Proposition III.3.15, for the second one we cut off. □

PROOF. We show the claim by induction, for any $k = 0, 1 \ldots$, and any $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$

$$\|u\|_{W^{k+2,2}(\Omega_1)} \leq \|f\|_{W^{k,2}(\Omega_2)} + \|u\|_{W^{k+1,2}(\Omega_2)}.$$

For $k = -1$ this already holds by Theorem IV.0.2. Let $\eta \in C_c^\infty(\Omega_2)$ with $\eta \equiv 1$ in Ω_1: observe that

$$\Delta(\eta u) = \eta \Delta u + 2\nabla\eta \cdot \nabla u + u\Delta\eta = \eta f + 2\nabla\eta \cdot \nabla u + u\Delta\eta.$$

Set $g := \Delta(\eta u)$. If $f \in W^{k,2}$ and $u \in W^{k+1,2}$ then $g \in W^{k,2}(\Omega)$. So we can work with $v := \eta u$ and g instead of f and u to obtain our theorem (away from the boundary $\partial\Omega$).

The advantage of $\operatorname{supp} v \subset\subset \Omega_2$ is that now, for $|h| \ll 1$, that if $u \in W^{k+1,2}_{loc}(\Omega)$ then

$$\Delta^h_{e_{\ell_1}} \dots \Delta^h_{e_{\ell_k}} v \in W^{1,2}_0(\Omega).$$

Moreover,

$$\Delta(\Delta^h_{e_{\ell_1}} \dots \Delta^h_{e_{\ell_k}} v) = \Delta^h_{e_{\ell_1}} \dots \Delta^h_{e_{\ell_k}} \Delta v = \Delta^h_{e_{\ell_1}} \dots \Delta^h_{e_{\ell_k}} g.$$

In order to apply Theorem IV.0.2, we interpret $\Delta^h_{e_{\ell_1}} \dots \Delta^h_{e_{\ell_k}} g \in (W^{1,2}(\Omega))^*$, and

$$\left| \int_\Omega \Delta^h_{e_{\ell_1}} \dots \Delta^h_{e_{\ell_k}} g \ \varphi \right| = \left| \int_\Omega \Delta^h_{e_{\ell_2}} \dots \Delta^h_{e_{\ell_k}} g \ \Delta^h_{e_{\ell_1}} \varphi \right| \lesssim \|g\|_{W^{k-1,2}(\Omega)} \|\varphi\|_{W^{1,2}(\Omega)}.$$

Then Theorem IV.0.2 implies for any $|h| \ll 1$

$$\|\Delta^h_{e_{\ell_1}} \dots \Delta^h_{e_{\ell_k}} v\|_{W^{1,2}(\Omega_1)} \leq C \|g\|_{W^{k-1,2}(\Omega_2)}$$

By Proposition III.3.15 we get $v \in W^{k+1,2}(\Omega_1)$

$$\|v\|_{W^{k+1,2}(\Omega_1)} \lesssim C \|g\|_{W^{k-1,2}(\Omega_2)},$$

since $v \equiv u$ in Ω_1 we get for any $k \geq 1$,

$$\|u\|_{W^{k+1,2}(\Omega_1)} \lesssim C \|f\|_{W^{k-1,2}(\Omega_2)} + \|u\|_{W^{k,2}(\Omega_1)}.$$

□

CHAPTER 5

Fractional Sobolev spaces as trace spaces

V.1. The Fractional Sobolev space $W^{s,p}$

There is no fractional Sobolev space, there are many. Some of them are called Besov spaces. We restrict our attention on the so-called *Gagliardo-Slobodeckij-Sobolev* space that appears as trace operator. We call it $W^{s,p}$ but be aware that some authors denote by $W^{s,p}$ another space (and technically: $W^{s,p}$ for $s = 1$ is a different space that $W^{1,p}$ – it is very messy; The most general version are so-called Triebel-Lizorkin spaces $F^s_{p,q}$ and Besov-spaces $B^s_{p,q}$ – each of them is one fractional Sobolev space. Our choice is $F^s_{p,p} = B^s_{p,p}$).

As for references, many people like the Hitchhiker's guide [**Di Nezza et al., 2012**], see also [**Bucur and Valdinoci, 2016**]. More related to Interpolation theory is [**Tartar, 2007**]. Overviews on Besov- and Triebel spaces can be found beginner-friendly in Grafakos' Harmonic Analysis books [**Grafakos, 2014a, Grafakos, 2014b**]. The book [**Runst and Sickel, 1996**] gives in particular a good recollection with references on Triebels' books. A book very focussed on these Sobolev spaces is also [**Samko, 2002**].

Let Γ be a (smooth) n-dimensional set (since we will interpret $W^{s,p}$ as *trace-spaces* later, one can think of Γ as the boundary of a nice set $\partial\Omega$, but it is acceptable to think of Γ as a subset of $\mathbb{R}^n$.

For $s \in (0,1)$ and $p \in (1,\infty)$ we set

$$[f]_{W^{s,p}(\Gamma)} := \left(\int_\Gamma \int_\Gamma \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}} .$$

This object (depending usually on your geographic location) is called the *Gagliardo-norm* or *Slobodeckij-norm*.

It is only a seminorm, for $f = const$ we have $[f]_{W^{s,p}} = 0$.

It is a fun exercise to check that $[f]_{W^{s,p}(\Gamma)} < \infty$ for $s \geq 1$ means that f is constant, hence the restriction to $s \in (0,1)$.

In some sense one can interpret (and should do so) as $[f]_{W^{s,p}(\Gamma)}$ measuring some sort of L^p-norm of some sort of an s-derivative. Namely

$$\frac{|f(x) - f(y)|}{|x-y|^s}$$

measures some sort of the s-Hölder constant (at a point x and y), indeed if we were to take $\sup_{x,y}$ then we would get the Hölder norm of order s.

So

$$\frac{|f(x)-f(y)|^p}{|x-y|^{sp}} = \left(\frac{|f(x)-f(y)|}{|x-y|^s}\right)^p,$$

is some sort of s-Derivative to the power p.

Now

$$\frac{dx\,dy}{|x-y|^n}$$

is a slightly singular measure (it is singular on the axis $x=y$), but not very. Almost like a mean value or (for the harmonic analysis enthusiast: maximal function).

So

$$[f]_{W^{s,p}(\Gamma)} = \left(\int_\Gamma \int_\Gamma \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\right)^{\frac{1}{p}}.$$

is somehow the s-derivative of f in L^p (ish).

It is easy (and fun) to check for any $\varepsilon > 0$ then if Γ is compact (or f has compact support)

$$[f]_{W^{s,p}(\Gamma)} \leq \|f\|_{L^\infty(\Gamma)} + [f]_{C^{s+\varepsilon}(\Gamma)}. \tag{V.1.1}$$

The Sobolev space $W^{s,p}(\Gamma)$ is then defined as all $f \in L^p(\Gamma)$ such that $[f]_{W^{s,p}(\Gamma)} < \infty$, with norm

$$\|f\|_{W^{s,p}(\Gamma)} = \|f\|_{L^p(\Omega)} + [f]_{W^{s,p}(\Gamma)}$$

This is very similar to the $W^{1,p}$-norm which is L^p plus Gradient in L^p.

V.2. Its a trace space!

Why do we care about $W^{s,p}$? First and foremost they are *trace spaces*.

In Theorem III.3.21 we learned that if $u \in W^{1,p}(\Omega)$ then $u\Big|_{\partial\Omega} \in L^p(\partial\Omega)$. Indeed the precise rule is $u \in W^{1,p}(\Omega)$ then $u\Big|_{\partial\Omega} \in W^{1-\frac{1}{p},p}(\partial\Omega)$. And this is a sharp embedding, in the sense that for any $u \in W^{1-\frac{1}{p},p}(\partial\Omega)$ there exists an extension $u \in W^{1,p}(\Omega)$ with u as its trace.

Let us investigate this statement for $p=2$ and $\Omega = \mathbb{R}^{n+1}_+$ the upper halfplane. For simplicity we will now start to call functions on the boundary $\partial\Omega = \mathbb{R}^n \times \{0\}$ with small letters u, v, w etc. and functions in the set U, V, W.

Theorem V.2.1. *For any $u \in C^\infty_c(\mathbb{R}^n)$, $p \in (1,\infty)$, we have*

$$[u]_{W^{\frac{1}{2},2}(\mathbb{R}^n)} \approx \inf_{U|_{\mathbb{R}^n}=u} \|DU\|_{L^2(\mathbb{R}^{n+1}_+)}.$$

where $A \approx B$ means that there exist a constant $C > 0$ such that $C^{-1}A \leq B \leq CA$.

In particular,

- *If $U \in C_c^\infty(\mathbb{R}^n \times [0,\infty))$ then its trace belongs to $W^{\frac{1}{2},2}(\mathbb{R}^n)$ and we have the estimate*
$$[u]_{W^{\frac{1}{2},2}(\mathbb{R}^n)} \leq C \|DU\|_{L^2(\mathbb{R}^{n+1}_+)}.$$
- *For any $u \in C_c^\infty(\mathbb{R}^n)$ there exists $U \in \dot{W}^{1,2}(\mathbb{R}^n)$ (The dot in $\dot{W}^{1,2}$ means that U does not need to belong to L^2 but only $DU \in L^2$), such that*
$$\|DU\|_{L^2(\mathbb{R}^{n+1}_+)} \leq C[u]_{W^{\frac{1}{2},2}(\mathbb{R}^n)}$$
Indeed, the proof will show that such a U can be chosen as the variation minimizer of $\|DU\|_{L^2(\mathbb{R}^{n+1}_+)}$, i.e. the Harmonic extension of u
$$\begin{cases} \Delta_{\mathbb{R}^{n+1}} U = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ \\ U = u & \text{on } \mathbb{R}^n \times \{0\} \end{cases}$$
Here we assume (often implicitely) that $\lim_{|x|\to\infty} U(x) = 0$.

Proof. It is helpful to consider the variables in $\mathbb{R}^{n+1}$ by (x,t), $x \in \mathbb{R}^n$, $t \in \mathbb{R}$.

We need to show that a solution U to
$$\begin{cases} \Delta_{\mathbb{R}^{n+1}} U = \Delta_x U + \partial_{tt} U = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ \\ U = u & \text{on } \mathbb{R}^n \times \{0\} \end{cases}$$
with $\lim_{|(x,t)|\to\infty} U = 0$ (which thus minimizes the inf in the statement) satisfies
$$[u]_{W^{\frac{1}{2},2}(\mathbb{R}^n)} \approx \|DU\|_{L^2(\mathbb{R}^{n+1}_+)}.$$
Computing by an integration by parts, (observe that $\partial_t = -\partial_\nu$ on $\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \{0\}$),
$$\|DU\|^2_{L^2(\mathbb{R}^{n+1}_+)} = \int_{\mathbb{R}^{n+1}_+} DU \cdot DU = -\int_{\mathbb{R}^n \times \{0\}} \partial_t U\, U - \int_{\mathbb{R}^{n+1}_+} \underbrace{\Delta_{x,t} U}_{=0}\, U$$
So,
$$\|DU\|^2_{L^2(\mathbb{R}^{n+1}_+)} = -\int_{\mathbb{R}^n \times \{0\}} \partial_t U\, U. \tag{V.2.1}$$
So we really would like to know the Neumann-derivative $\partial_t U$ of a solution to the Dirichlet problem $\Delta U = 0$ and $U = u$ on $\mathbb{R}^n$ – this is why we talk about a *Dirichlet-to-Neumann property*.

Lucky for us, we have the Fourier transform.

Taking the Fouriertransform in x-direction, $\Delta_{x,t}U = 0$, recalling that Δ_x becomes $-c|\xi|^2$ after Fourier transform (Lets for simplicity pretend that $c = 1$),

$$\partial_{tt}\hat{U}(\xi,t) - |\xi|^2\hat{U}(\xi,t) = 0.$$

The only solution to the ordinary differential equations with $\lim_{t\to\infty} U = 0$, is

$$\hat{U}(\xi,t) = e^{-t|\xi|}\hat{U}(\xi,0) = e^{-t|\xi|}\hat{u}(\xi)$$

Observe that $|\xi|^2$ is the *Fourier symbol* of the Laplacian $(-\Delta)$, we could think of $|\xi|$ as the Fourier symbol of an operatoron we shall call $(-\Delta)^{\frac{1}{2}}$, the *half-laplace* or *fractional laplace*.

Thus, reverting the Fourier transform, we could write $\Delta U = 0$ from our assumption as

$$U(x,t) = e^{-t(-\Delta)^{\frac{1}{2}}}u(x).$$

However we want to interpret this, we have

$$\partial_t U(x,0) = -(-\Delta)^{\frac{1}{2}}u(x).$$

From (V.2.1) we get

$$\|DU\|^2_{L^2(\mathbb{R}^{n+1}_+)} = \int_{\mathbb{R}^n}(-\Delta)^{\frac{1}{2}}u\,u.$$

That does not look much better, but we can do an integration by parts,

$$\int_{\mathbb{R}^n}(-\Delta)^{\frac{1}{2}}u\,u = \int_{\mathbb{R}^n}(-\Delta)^{\frac{1}{4}}u\,(-\Delta)^{\frac{1}{4}}u = \|(-\Delta)^{\frac{1}{4}}u\|^2_{L^2(\mathbb{R}^n)}.$$

Indeed, using twice Plancherel theorem (Fouriertransform: yay! – a nice exercise: try to show integration by parts formula via Fourier transform for the classical derivative ∂_i)

$$\int_{\mathbb{R}^n}(-\Delta)^{\frac{1}{2}}u\,u = \int_{\mathbb{R}^n}|\xi|\hat{u}\,\overline{\hat{u}} = \int_{\mathbb{R}^n}|\xi|^{\frac{1}{2}}\hat{u}\,\overline{|\xi|^{\frac{1}{2}}\hat{u}} = \int_{\mathbb{R}^n}|(-\Delta)^{\frac{1}{4}}u|^2$$

So we have shown that for our U-u combination

$$\|DU\|^2_{L^2(\mathbb{R}^{n+1}_+)} = C\|(-\Delta)^{\frac{1}{4}}u\|^2_{L^2(\mathbb{R}^n)}.$$

Now counting derivatives: The Laplacian has two derivatives, so $(-\Delta)^{\frac{1}{4}}$ has "1/2"derivatives. So one would hope that

$$\|(-\Delta)^{\frac{1}{4}}u\|_{L^2(\mathbb{R}^n)} \approx [u]_{W^{\frac{1}{2},2}(\mathbb{R}^n)}$$

This is indeed true, but this is because we have L^2 (and thus: Plancherell). In general

(WARNING) $$\|(-\Delta)^{\frac{s}{2}}u\|_{L^p(\mathbb{R}^n)} \not\approx [u]_{W^{s,p}(\mathbb{R}^n)}.$$

unless $p = 2$. I.e. for $s \in (0,1)$,

$$\|(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^n)} \approx [u]_{W^{s,2}(\mathbb{R}^n)}.$$

For this let us compute a new formula for $(-\Delta)^{\frac{s}{2}}u$. Recall that after Fourier transform

$$((-\Delta)^{\frac{s}{2}}u)^\wedge(\xi) = |\xi|^s u(\xi)$$

. We computed in the lines after (I.2.3) how products and homogeneous functions behave under fourier transform, so we could (see the Newton potential which was derived from this for $s < 0$)

$$(-\Delta)^{\frac{s}{2}} u(x) = |\cdot|^{-n-s} * u(x).$$

This is almost true, (but in a much more precise sense it is false, for any function $u \not\equiv 0$.), one needs to take into account the (now hypersingular singularity of $|z|^{-n-s}$ for $z = 0$). Namely

$$(-\Delta)^{\frac{s}{2}} u(x) = c|\cdot|^{-n-s} * (u - u(x))(x) = c \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|x-y|^{n+s}}\, dy.$$

This is good for $s \in (0,1)$, for $s \in [1,2)$ one should to write a *principal value symbol*. With this formula one checks that

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} u(x)\, v(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|x-y|^{n+s}}\, v(x)\, dy\, dx = -\frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(y) - u(x))\,(v(y) - v(x))}{|x-y|^{n+s}}\, dy\, dx$$

where the last step follows from symmetry: using fubini with interchanging x and y we see that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(y) - u(x))\,(v(y) + v(x))}{|x-y|^{n+s}}\, dy\, dx = 0$$

and noting that $v(x) = \frac{1}{2}(v(x) - v(y)) + \frac{1}{2}(v(x) + v(y))$.

In particular,

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{1}{2}} u(x)\; u(x) = -[u]^2_{W^{\frac{1}{2},2}(\mathbb{R}^n)},$$

and the claim is shown. □

An adaptation (using some delicate harmonic analysis, like square function estimates) leads to the general statement of Theorem V.2.1:

THEOREM V.2.2. *For any $u \in C_c^\infty(\Omega)$, $s \in (0,1)$, $p \in (1,\infty)$, for some set Ω with nice boundary. Then*

$$[u]_{W^{s,p}(\partial\Omega)} \approx \inf_{U|_{\partial\Omega} = u} \|t^{1-\frac{1}{p}-s} DU\|_{L^p(\Omega)}.$$

Where U is a solution of

$$\begin{cases} \Delta U = 0 & \text{in } \Omega \\ U = u & \text{on } \partial\Omega \end{cases}$$

This is somewhat generally known fact in the Harmonic Analysis community and intrinsically contained in Harmonic Analysi classics like [**Stein, 1993**]. See also [**Bui and Candy, 2015**] and [**Lenzmann and Schikorra, 2019**].

V.3. Cool things one can do with this: Integration by parts revisited

Let us do an example (which surprisingly is not known to a wider community, but known for a long time to some experts):

Let $f, g \in C_c^\infty(\mathbb{R}, \mathbb{R})$, and consider

$$A := \int_{\mathbb{R}} f(x)\, g'(x)\, dx = -\int_{\mathbb{R}} f'(x)\, g(x)\, dx$$

By Hölder's inequality, we easily get the estimates

$$|A| \leq \|f\|_{L^2(\mathbb{R})}\, \|g'\|_{L^2(\mathbb{R})}$$

or

$$|B| \leq \|f'\|_{L^2(\mathbb{R})}\, \|g\|_{L^2(\mathbb{R})}.$$

But we can get an "intermediate" estimate where for f and g only half a derivative counts:

Proposition V.3.1. *Let $f, g \in C_c^\infty(\mathbb{R}, \mathbb{R})$, then*

$$\left| \int_{\mathbb{R}} f(x)\, g'(x)\, dx \right| \leq [f]_{W^{\frac{1}{2},2}(\mathbb{R})}\, [g]_{W^{\frac{1}{2},2}(\mathbb{R})}$$

One application of this estimate (which in its statement does not use the notion of fractional Sobolev spaces) is (cf. (V.1.1)):

Corollary V.3.2. *Let $f, g \in C_c^\infty(\mathbb{R}, \mathbb{R})$. For any $s \in (1/2, 1]$*

$$\left| \int_{\mathbb{R}} f(x)\, g'(x)\, dx \right| \leq C(s, \operatorname{supp} f, \operatorname{supp} g)\, \|f\|_{C^s} \|g\|_{C^s}$$

Also,

$$\left| \int_0^1 f(x)\, g'(x)\, dx \right| \leq C(s)\ \left(|f(0)| + [f]_{C^s}\right)\ \left(|g(0)| + [g]_{C^s}\right) \tag{V.3.1}$$

Proof of (V.3.1), proving also the Proposition and the Corollary. First we assume that $u(0) = v(0) = u(1) = v(1) = 0$. Then we can extend u, v by zero to a Lipschitz function on all of $\mathbb{R}$ (obviously with compact support). Denote by $U, V : \mathbb{R}^2_+ \to \mathbb{R}$ extensions of $u, v : \mathbb{R} \to \mathbb{R}$ with decay at infinity of U, V, and $\partial_t U$, $\partial_t V$, i.e. $\lim_{|(s,t)|\to\infty} U(s,t) = 0$, $\lim_{|(s,t)|\to\infty} |DU(s,t)| = 0$, and likewise for V.

Then, following the ideas in [**Lenzmann and Schikorra, 2019**], see also [**Brezis and Nguyen, 2011**],

$$\begin{aligned}
&\int_0^1 u(t)\, v'(t) dt \\
=&\int_{\mathbb{R}} u(t)\, v'(t) dt \\
=&\int_{\mathbb{R}} \int_0^\infty \partial_s \left(U(t,s) \partial_t V(t,s)\right)\, ds\, dt \\
=&\int_{\mathbb{R}} \int_0^\infty \left(\partial_s U(t,s) \partial_t V(t,s)\right)\, ds\, dt + \int_{\mathbb{R}} \int_0^\infty U(t,s) \partial_{s,t} V(t,s)\, ds\, dt
\end{aligned}$$

For the second term observe

$$\int_{\mathbb{R}} \int_0^\infty U(t,s)\partial_{s,t}V(t,s)\,ds\,dt = -\int_{\mathbb{R}} \int_0^\infty \partial_t U(t,s)\partial_s V(t,s)\,ds\,dt$$

That is, we obtain

$$\begin{aligned}&\left|\int_0^1 u(t)\,v'(t)dt\right|\\ \leq& 2\int_{\mathbb{R}^2_+} |DU|\,|DV|\\ \leq& 2\|DU\|_{L^2(\mathbb{R}^2_+)}\,\|DV\|_{L^2(\mathbb{R}^2_+)}\end{aligned}$$

Choosing U and V the harmonic extension we obtain

$$\left|\int_0^1 u(t)\,v'(t)dt\right| \leq C\,\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy\right)^{\frac{1}{2}}\,\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|v(x)-v(y)|^2}{|x-y|^2}\,dx\,dy\right)^{\frac{1}{2}}$$

From this one easily obtains for any $s > \frac{1}{2}$,

$$\left|\int_0^1 u(t)\,v'(t)dt\right| \leq C\,\left(\|u\|_{L^\infty} + [u]_{C^s}\right)\,\left(\|v\|_{L^\infty} + [v]_{C^s}\right).$$

Recall that u and v have support in $[0,1]$. Thus we arrive at

$$\left|\int_0^1 u(t)\,v'(t)dt\right| \leq C\,\left([u]_{C^s((0,1))}\right)\,\left([v]_{C^s((0,1))}\right).$$

This shows the claim of the proposition under the assumption that $u(0) = v(0) = u(1) = v(1) = 0$.

If this assumption is not satisfied, we set $\tilde{u}(t) := (1-t)u(0) + tu(1)$ and likewise $\tilde{v}(t) := (1-t)v(0) + tv(1)$. Then

$$\left|\int_0^1 u(t)v'(t)\right| \leq \left|\int_0^1 (u-\tilde{u})\,(t)\,(v-\tilde{v})'\,(t)\right| + \left|\int_0^1 u(t)\tilde{v}'(t)\,dt\right| + \left|\int_0^1 \tilde{u}(t)\tilde{v}'(t)\right| + \left|\int_0^1 \tilde{u}(t)v'(t)\right|$$

The first term we can estimate by the first part of the proof. Clearly

$$[\tilde{u}]_{C^s((0,1))} \leq |u(0)| + |u(1)| \leq |u(0)| + [u]_{C^s((0,1))},$$

and likewise for $\tilde{v}$. For the second term, observe

$$\left|\int_0^1 u(t)\tilde{v}'(t)\right| \leq |v(1)-v(0)|\|u\|_{L^\infty(0,1)} \leq [v]_{C^s((0,1))}\,\left(|u(0)| + [u]_{C^s((0,1))}\right)$$

We argue similarly for the third term. For the fourth term, we do an integration by parts, before arguing as above.

$$\left|\int_0^1 \tilde{u}(t)v'(t)\right| \leq |v(1)||\tilde{u}(1)| + |v(0)||\tilde{u}(0)| + \left|\int_0^1 \tilde{u}'(t)v(t)\right| \leq \left(|u(0)| + [u]_{C^s((0,1))}\right)\,\left(|v(0)| + [v]_{C^s((0,1))}\right).$$

□

These kind of estimates can be generalized to several situations that involve "commutator"-type structures (it falls under the realm of "compensated compactness"). See, e.g., [**Brezis and Nguyen, 2** for Jacobians, [**Lenzmann and Schikorra, 2019**] for general commutator estimates.

V.4. $W^{s,p}$ is not a gradient to the power p

Here we have a word of warning. We explained $W^{s,p}$ as "essentially" the s-derivative $\frac{f(x)-f(y)}{|x-y|^s}$ to the power p.

This leads to a good intuition for many things, but also one important false analogy:

Recall: If $f \in W^{1,p}(\Omega)$ then for any $q < p$ we have $W^{1,q}_{loc}(\Omega)$. This is simply because the same holds for $L^p(\Omega) \subset L^q_{loc}(\Omega)$ if $q < p$.

But this is false for $W^{s,p}$. This can be easily be seen by a characterization of $W^{s,p}$ as so-called Triebel-spaces. But we do not want to go there, but refer to [**Mironescu and Sickel, 2015**] which used this relation to construct a counterexample for $W^{s,p}(\Omega) \not\subset W^{s,q}_{loc}(\Omega)$ even thought $q < p$.

That is one odd result, and it should serve as a warning that $W^{s,p}$ behave sometimes a little bit different from $W^{1,p}$.

V.5. $W^{s,p}$ becomes $W^{1,p}$ as s goes to one

The notion of

$$[u]_{W^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}}$$

does not make sense as $s \to 1$. Indeed,

$$\left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+p}} \, dx \, dy \right)^{\frac{1}{p}} < \infty \Leftrightarrow u \text{ is constant.}$$

However, one can show, see [**Bourgain et al., 2001**], that

$$(1-s)^{\frac{1}{p}} [u]_{W^{s,p}} \xrightarrow{s \to 1} \|\nabla u\|_{L^p}.$$

V.6. "The" other fractional Sobolev space $H^{s,p}$

We found above the operator $(-\Delta)^{\frac{s}{2}}$. Another fractional Sobolev space, often denoted by $H^{s,p}(\mathbb{R}^n)$ is given as all functions such that

$$[u]_{H^{s,p}(\mathbb{R}^n)} := \|(-\Delta)^{\frac{s}{2}} u\|_{L^p(\mathbb{R}^n)} < \infty.$$

For $p = 2$ there is nothing new: $H^{s,2}$ is $W^{s,2}$, for $p \neq 2$ these two spaces are different.

The space $H^{s,p}$ is a more natural fractional Sobolev space in some features (for example $u \in H^{1,p}_{loc}$ implies $u \in H^{1,q}_{loc}$ for $q < p$) but it also has some disadvantages: it is a bit more complicated to define on subsets $\Omega \subset \mathbb{R}^n$ (since $(-\Delta)^{\frac{s}{2}}$ is a global operator), it is not a trace space (but there are some trace-space-type characterizations, see [**Lenzmann and Schikorra, 2019**]).

But for example it satisfies that $H^{s,p} \xrightarrow{s\to 1} W^{1,p}$ (and this is almost trivial to show).

Warning: The notation $H^{s,p}$ for one type, $W^{s,p}$ for the other type of Sobolev spaces is not uniform in the literature. Check carefully what space is meant if you find it in a research paper!

V.7. Embedding theorems, Trace-theorems etc. hold true

The good news is: All the embedding theorems we learned before have an analogue. Moreover, up to the Sobolev-inequality (where we used the very elegant proof by Nirenberg in Theorem III.3.28) even the proofs stay the same (up to minor adjustments).

Recall the numerology for Sobolev spaces: $W^{1,p}$ was locally embedded into L^q if

$$1 - \frac{n}{p} \geq 0 - \frac{n}{q}$$

The 1 is the derivative order of $W^{1,p}$ and 0 is the derivative order of L^q. Here are the main results:

- $W^{s,p}(\Omega)$ embedds (locally) continuously into $W^{t,q}(\Omega)$ if $s \geq t$ and
$$s - \frac{n}{p} \geq t - \frac{n}{q}.$$
- If the inequality above is strict, the embedding is compact.
- In particular we have Rellich: $W^{s,p}$ compactly embedds into L^p if $s > 0$ (locally)
- In particular we have Poincaré inequalty
$$\|u - (u)\|_{L^p(\Omega)} \leq C(\Omega)\, [u]_{W^{s,p}(\Omega)}$$
whenever $s > 0$ (and Ω is sufficiently nice).
- If $s - \frac{n}{p} \geq k + \alpha$ and $s > k$ then $W^{s,p}$ embedds into $C^{k,\alpha}$ (compactness if there is a strict inequality)

We mentioned before that $H^{s,p}$ and $W^{s,p}$ are different spaces. However they are in some sense "very close by". This can, e.g., be seen with Sobolev embedding: If if $s \geq t$ and

$$s - \frac{n}{p} \geq t - \frac{n}{q}.$$

then $W^{s,p}(\Omega)$ embedds into $H^{t,q}(\Omega)$ (locally) and $H^{s,p}(\Omega)$ embedds into $W^{t,q}(\Omega)$ (locally).

As for traces the numerology before was $W^{1,p}(\Omega)$ embedds into $W^{1-\frac{1}{p},p}(\partial\Omega)$. This stays the same, as long as $s-\frac{1}{p}>0$ then $W^{s,p}(\Omega)$ embedds into $W^{s-\frac{1}{p},p}(\partial\Omega)$.

When $s-\frac{1}{p}=0$ things get difficult. For an illustration: In the class $W^{\frac{1}{2},2}$ there are "two spaces" with 0 on the boundary. One, $W_0^{\frac{1}{2},2}(\Omega)$ where the trace is zero, and the *Lions-Magenes*-space $W_{00}^{\frac{1}{2},2}(\Omega)$ which are maps which can be extended by zero to a $W^{\frac{1}{2},2}(\mathbb{R}^n)$-map. See [**Tartar, 2007**].

CHAPTER 6

Parabolic PDEs

VI.1. The heat equation: Fundamental solution and Representation

We consider

$$\begin{aligned} \partial_t u - \Delta u &= f \quad \text{in} \quad \mathbb{R}^{n+1}_+ \\ u(0,\cdot) &= g \quad \text{on} \quad \mathbb{R}^n. \end{aligned} \tag{VI.1.1}$$

First assume $f = 0$. Then (VI.1.1) is called *homogeneous heat equation*. For $f \neq 0$ it is called *inhomogeneous*.

Trivial solutions of the homogeneous equation constant maps $u(x,t) \equiv c$, or (not completely trivial) time-independent harmonic functions $u(x,t) := v(x)$ with $\Delta v = 0$.

For elliptic equations we had the notion of a fundamental solution, Section I.2.1; There exists a similar concept for the heat equation, the *heat kernel*, which we will (formally) derive now.

If we fix $x \in \mathbb{R}^n$ and look at (VI.1.1) as an equation in time t then it looks like an ODE, and naïvely the solution should be

$$u(x,t) = e^{t\Delta} u(x,0).$$

Of course, $e^{t\Delta}$ does not make any sense for now, but we will define this later via *semi-group theory*, Chapter 8.

To make (still formally, but more precise sense) of the "ODE argument", we use the Fourier-transformation (with respect to the variables $x \in \mathbb{R}^n$):

Let u be a solution of $\partial_t u = \Delta u$. Taking the Fourier transform (in x) on both sides we find

$$\begin{aligned} \frac{d}{dt}\hat{u}(\xi,t) = \widehat{\partial_t u}(\xi,t) &= \widehat{\Delta u}(\xi,t) \\ &= -|\xi|^2 \hat{u}(\xi,t). \end{aligned}$$

Let ξ be fixed and let

$$v(t) = \hat{u}(\xi,t).$$

Then the above reads as

$$\frac{d}{dt} v(t) = -|\xi|^2 \hat{v}(t).$$

There is one solution to this ODE (starting from a given value $v(0)$):

$$v(t) = e^{-t|\xi|^2} v(0).$$

Observe that in particular $v(\infty) = 0$ (i.e. "decay at infinity").

Ansatz: $v(0) = 1$, resp. $u(0) = \delta_0$. This means

$$\hat{u}(\xi, t) = e^{-t|\xi|^2}.$$

In this case we have

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}},$$

which seems to be a special solution.

Definition VI.1.1.

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, & t > 0, x \in \mathbb{R}^n \\ 0, & t < 0, x \in \mathbb{R}^n. \end{cases}$$

is called *fundamental solution* or *heat kernel*.

One has

$$\partial_t \Phi - \Delta \Phi = 0, \quad \text{for} \quad t > 0$$

and

$$\lim_{t \to 0} \Phi(x_0, t) = \begin{cases} 0, & x_0 \neq 0 \\ \infty, & x_0 = 0. \end{cases}$$

Lemma VI.1.2.

$$\forall t > 0 : \int_{\mathbb{R}^n} \Phi(x,t)\ dx = 1.$$

Proof.

$$\int_{\mathbb{R}^n} \Phi(x,t)\ dx = \hat{\Phi}(0,t) = 1.$$

□

Analogously to the fundamental solution for the Laplace equation, the heat kernel Φ generates solutions to the heat equation. Indeed, if we set

$$\begin{aligned} u(x,t) &:= \Phi(\cdot, t) * g(x) \\ &= \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\ dy \end{aligned}$$

Then

$$\hat{u}(\xi, t) = (\widehat{\Phi(\cdot, t) * g})(\xi) = \hat{\Phi}(\xi, t)\hat{g}(\xi).$$

That is,

$$\hat{u}(\xi, 0) = \hat{g}(\xi), \quad (\frac{d}{dt} + |\xi|^2)\hat{u}(\xi, t) = 0.$$

Revert the Fourier-transformation to obtain

$$(\frac{d}{dt} - \Delta)u = 0 \quad \text{in} \quad \mathbb{R}^{n+1}_+, \quad u(x,0) = g(x), x \in \mathbb{R}^n.$$

Motivated by this calculation we set

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)\ dy.$$

THEOREM VI.1.3 (Potential representation). *Let $g \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Let u as in (VI.1). Then u is defined in $\mathbb{R}^n$ and there holds:*

(i) $u \in C^\infty(\mathbb{R}^{n+1}_+)$,

(ii) $\partial_t u - \Delta u = 0$ *in* $\mathbb{R}^{n+1}_+$ *und*

(iii)

$$\forall x_0 \in \mathbb{R}^n \colon \lim_{(x,t)\to(x_0,0)} u(x,t) = g(x_0).$$

Next we search a potential representation for

$$\begin{aligned} (\frac{d}{dt} - \Delta)u &= f \quad \text{in } \mathbb{R}^{n+1}_+ \\ u(\cdot,0) &= 0 \quad \text{on } \mathbb{R}^n. \end{aligned} \tag{VI.1.2}$$

Again we use the Fourier transformation for intuition. Set $v(t) = \hat{u}(\xi,t)$. Then

$$\partial_t v(t) + |\xi|^2 v(t) = \hat{f}(\xi,t) =: h(t).$$

We use what is called *Duhamel's formula*.

For $s > 0$ solve

$$\begin{aligned} \partial_t w_s(t) + |\xi|^2 w_s(t) &= 0, \quad t > s \\ w_s(s) &= h(s). \end{aligned}$$

Again, from ODE theory we know what w_s is,

$$w_s(t) = e^{-(t-s)|\xi|^2} h(s). \tag{VI.1.3}$$

Now we try to solve (VI.1.2) by the Ansatz (which we may recognize from ODE calculus)

$$v(t) = \int_0^t w_s(t)\ ds.$$

Then $v(0) = 0$, and moreover

$$\partial_t v(t) = w_t(t) + \int_0^t \partial_t w_s(t)\ ds = h(t) - |\xi|^2 \int_0^t w_s(t)\ ds = h(t) - |\xi|^2 v(t).$$

Thus, using (VI.1.3) we know that

$$v(t) = \int_0^t e^{-(t-s)|\xi|^2} h(s)\ ds$$

solves (VI.1.2).

Using the inverse Fourier transform,

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\, dyds.$$

THEOREM VI.1.4. *Let $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$ with compact support and let u as in (VI.1). Then*

(i) $u \in C_1^2(\mathbb{R}^n \times (0,\infty))$,

(ii) $(\frac{d}{dt} - \Delta)u = f$ *in* $\mathbb{R}^n \times (0,\infty)$

(iii) $\forall x_0 \in \mathbb{R}^n\colon \lim_{(x,t)\to(x_0,0)} u(x,t) = 0$.

VI.2. Mean-value formula

(cf. [**Evans, 2010**, Chapter 2.3])

Use the fundamental solution to construct a parabolic ball, or *heat ball*

$$E(x,t;r) \subset \mathbb{R}^{n+1}. \tag{VI.2.1}$$

Definition VI.2.1 (Heat ball)**.** Let $(x,t) \in \mathbb{R}^{n+1}$. Set

$$E(x,t;r) = \left\{(y,s) \in \mathbb{R}^{n+1}\colon s \le t, \Phi(x-y,t-s) \ge \frac{1}{r^n}\right\}. \tag{VI.2.2}$$

THEOREM VI.2.2 (mean value). *Let $X \subset \mathbb{R}^{n+1}$ be open and $u \in C_1^2(X)$ solve $(\partial_t - \Delta)u = 0$ in X. Then there holds*

$$u(x,t) = \frac{1}{4r^n} \int_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2}\, dyds \tag{VI.2.3}$$

for all $E(x,t;r) \subset X$.

VI.3. Maximum principle and Uniqueness

Definition VI.3.1. Let $\Omega \subset \mathbb{R}^n$ be an open set and denote with $\Omega_T := \Omega \times (0,T]$ for some time $T > 0$. It is important to note that the top $\Omega \times \{T\}$ belongs to Ω_T. The parabolic boundary Γ_T of Ω_T is the boundary of Ω_T without the top,

$$\Gamma_T = \overline{\Omega_T} \backslash \Omega_T = \partial\Omega \times [0,T) \cup \Omega \times \{0\}.$$

THEOREM VI.3.2. *Let U be bounded and $u \in C_1^2(U_T) \cap C^0(\bar{U}_T)$ be a solution of $u_t = \Delta u$ in U_T. Then there holds the weak maximum principle*

(i)

$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u \tag{VI.3.1}$$

and the strong maximum principle:

(ii) If U is connected and if there is $(x_0, t_0) \in U_T$ with

$$u(x_0, t_0) = \max_{\bar{U}_T} u, \tag{VI.3.2}$$

then

$$u(x, t) = u(x_0, t_0) \quad \forall (x, t) \in U_{t_0}. \tag{VI.3.3}$$

Proof. (ii)⇒(i), since if

$$\max_{\bar{U}_T} u > \max_{\Gamma_T} u \tag{VI.3.4}$$

then by (ii) u is constant at all prior times, which contradicts (VI.3.4).

Now we prove (ii). Suppose there is $(x_0, t_0) \in U_T$ with

$$u(x_0, t_0) = M = \max_{\bar{U}_T} u. \tag{VI.3.5}$$

Since $t_0 > 0$, there exists a small heat ball $E(x_0, t_0, r_0) \subset U_T$ and we have by Theorem VI.2.2

$$M = u(x_0, t_0) = \frac{1}{4{r_0}^n} \int_{E(x_0,t_0,r_0)} u(y, s) \frac{|y - x|^2}{(t - s)^2} \, dsdy \leq M. \tag{VI.3.6}$$

Hence $u \equiv M$ in $E(x_0, t_0; r_0)$.

Now we need to show $u = M$ in all of U_{t_0}. It suffices to show $u \equiv M$ in any U_{t_1} for any $t_1 < t_0$, by continuity $u \equiv M$ in all of U_{t_0}. So let $(x_1, t_1) \in U_{t_0}$., $t_1 < t_0$. Then there exists a continuous path $\gamma \colon [0, 1] \to U$ connecting x_0 and x_1. In the spacetime set

$$\Gamma(r) = (\gamma(r), rt_1 + (1 - r)t_0). \tag{VI.3.7}$$

Let

$$\rho = \max\{r \in [0, 1] \colon u(\Gamma(r)) = M\}. \tag{VI.3.8}$$

Show that $\rho = 1$. Suppose $\rho < 1$. Then we use the proof above to find a heat ball

$$E = E(\Gamma(\rho), r'), \tag{VI.3.9}$$

where $u = M$. Since Γ crosses E (time parameter is decreasing along Γ), we obtain a contradiction to the maximality of ρ. □

Remark VI.3.3. The same holds for $-u$ and hence we have a minimum principle. Hence, if in particular

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } U_T \\ u &= 0 \quad \text{on } \partial U \times [0, T] \\ u &= g \quad \text{in } U \times \{0\} \end{aligned} \tag{VI.3.10}$$

with $g(x) > 0$ for some $x \in U$ then $u > 0$ in U_T (infinite speed of propagation, non-relativistic).

Remark VI.3.4. For general $X \subset \mathbb{R}^{n+1}$ open we have a similar result, see exercises.

THEOREM VI.3.5 (Uniqueness on bounded domains). *Let $U \Subset \mathbb{R}^n$ bounded and $g \in C^0(\Gamma_T)$, $f \in C^0(U_T)$. Then there is at most one solution $C^2_1(U_T) \cap C^0(\bar{U}_T)$ to*

$$\begin{aligned} u_t - \Delta u &= f \quad \text{in } U_T \\ u &= g \quad \text{on } \Gamma_T. \end{aligned} \tag{VI.3.11}$$

PROOF. Apply the maximum (and minimum) principle to show that the difference of two solutions is zero. □

THEOREM VI.3.6. *Let $u \in C^2_1(\mathbb{R}^n \times (0,T]) \cap C^0(\mathbb{R}^n \times [0,T])$ be a solution of*

$$\begin{aligned} (\partial_t - \Delta)u &= 0 \quad \text{in } \mathbb{R}^n \times (0,T) \\ u &= g \quad \text{on } \mathbb{R}^n \times \{t = 0\} \end{aligned} \tag{VI.3.12}$$

with the growth condition

$$u(x,t) \le Ae^{a|x|^2} \tag{VI.3.13}$$

for some $a, A > 0$. Then there holds

$$\sup_{\mathbb{R}^n \times [0,T]} u \le \sup_{\mathbb{R}^n} g. \tag{VI.3.14}$$

PROOF. It suffices to show this estimate for small times, by splitting up the time interval into many small time steps. For this reason we assume first:

$$4aT < 1. \tag{VI.3.15}$$

For $\varepsilon > 0$ and μ chosen below, let

$$v(x,t) = u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{\frac{n}{2}}} e^{\frac{|x|^2}{4(T+\varepsilon-t)}} \tag{VI.3.16}$$

for some $\mu > 0$. Then $v_t - \Delta v = 0$. (observe that t appears in the negative above). Theorem VI.3.2 implies

$$\forall U \Subset \mathbb{R}^n\colon \max_{\bar{U}_T} v \le \max_{\Gamma_T} v \le \max(\max v(\cdot,0), \max_{\partial U \times [0,T]} v(x,t)). \tag{VI.3.17}$$

We have

$$v(x,0) = g(x) - \frac{\mu}{(T+\varepsilon)^{\frac{n}{2}}} e^{\frac{|x|^2}{4(T+\varepsilon)}} \le \sup_{\mathbb{R}^n} g. \tag{VI.3.18}$$

Let $U = B_R(0)$, then

$$\max_{\bar{B}_R(0) \times [0,T]} v \le \max\left(\sup_{\mathbb{R}^n} g, \max_{|x|=R, t \in [0,T]} v(x,t)\right). \tag{VI.3.19}$$

For $|x| = R$ and $t \in (0, T)$

$$\begin{aligned} v(x,t) &= u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{\frac{n}{2}}} e^{\frac{R^2}{4(T+\varepsilon-t)}} \\ &\leq Ae^{a|x|^2} - \frac{\mu}{(T+\varepsilon-t)^{\frac{n}{2}}} e^{\frac{R^2}{4(T+\varepsilon-t)}} \\ &\leq Ae^{a|x|^2} - \frac{\mu}{(T+\varepsilon)^{\frac{n}{2}}} e^{\frac{R^2}{4(T+\varepsilon)}} \end{aligned}$$

Since $4aT < 1$, there exist $\varepsilon > 0, \gamma > 0$, such that

$$a + \gamma = \frac{1}{4(T+\varepsilon)} \tag{VI.3.20}$$

and hence

$$v(x,t) \leq Ae^{aR^2} - \frac{\mu}{(T+\varepsilon)^{\frac{n}{2}}} e^{aR^2+\gamma R^2}. \tag{VI.3.21}$$

In particular, the right term dominates for $R >> 0$: in particular for all large $R > 0$ we have $v(x,t) \leq g(0)$. So for large R and $|x| = R$ we have for all $t \in (0, T]$,

$$v(x,t) \leq g(0) \leq \sup_{\mathbb{R}^n} g \tag{VI.3.22}$$

and so

$$\max_{(x,t)\in \overline{B_R(0)\times(0,T]}} v(x,t) \leq \sup_{\mathbb{R}^n} g \quad \forall R >> 1. \tag{VI.3.23}$$

Letting $R \to \infty$ we find that

$$\sup_{\mathbb{R}^n\times[0,T]} v(x,t) \leq \sup_{\mathbb{R}^n} g, \tag{VI.3.24}$$

i.e.

$$\sup_{\mathbb{R}^n\times[0,T]} \left(u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{\frac{n}{2}}} e^{\frac{|x|^2}{4(T+\varepsilon-t)}} \right) \leq \sup_{\mathbb{R}^n} g \tag{VI.3.25}$$

This holds for any any $\mu > 0$.

Letting $\mu \to 0$ for fixed x gives the claim under the assumption that $4aT < 1$.

If $4aT \geq 1$, we can slice the time interval $(0, T]$ into parts $(0, T_1] \cup (T_1, T_2] \cup \ldots \cup (T_K, T]$ with $4a(T_{i+1} - T_i) < 1$ for all i. Using the estimate in each of these time intervals we conclude. □

THEOREM VI.3.7. *Let $g \in C^0(\mathbb{R}^n)$, $f \in C^0(\mathbb{R}^n \times [0, T])$. Then there is at most one solution $u \in C^2_1(\mathbb{R}^n \times (0, T]) \cap C^0(\mathbb{R}^n \times [0, T])$ of*

$$\begin{aligned} (\partial_t - \Delta)u &= f \quad \text{in } \mathbb{R}^n \times (0,T) \\ u &= g \quad \text{on } \mathbb{R}^n \times \{0\} \end{aligned} \tag{VI.3.26}$$

with

$$|u(x,t)| \leq Ae^{a|x|^2} \quad \forall (x,t) \in \mathbb{R}^n \times (0,T). \tag{VI.3.27}$$

PROOF. Exercise VI.3.9 □

Exercise VI.3.8. *In Theorem VI.3.7 we learned of the strong maximum principle in parabolic Cylinders. Use this to obtain the strong maximum principle in general open sets X:*

let $X \subset \mathbb{R}^{n+1}$ be a bounded, open set. Assume that $u \in C^\infty(\overline{X})$ and

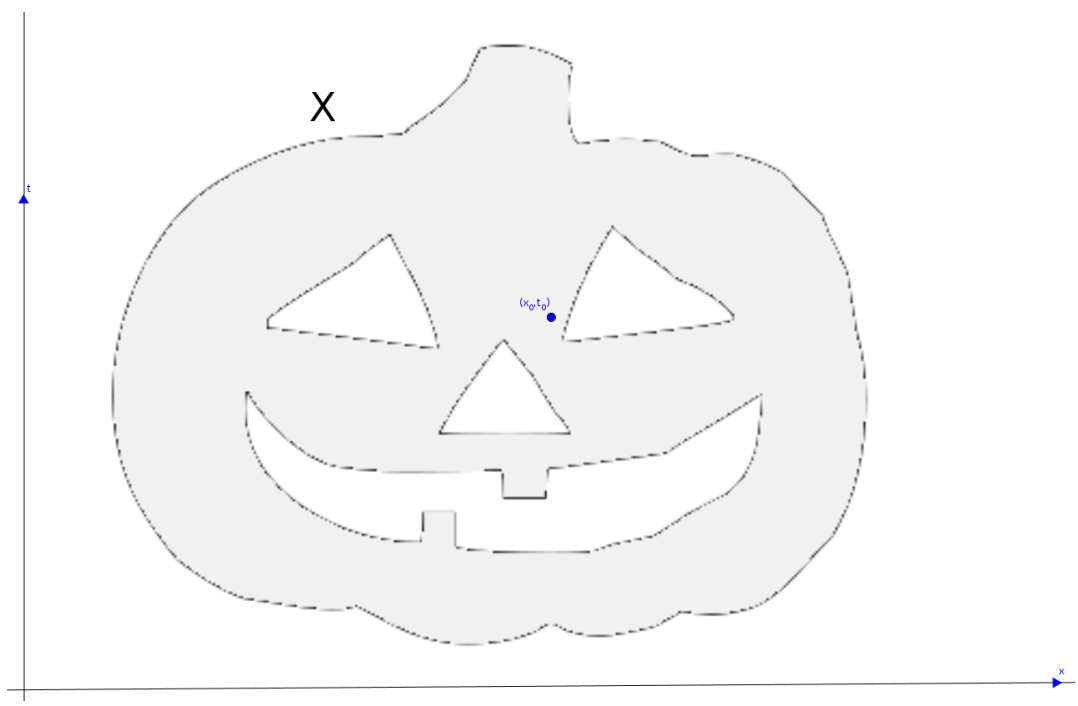
$$\partial_t u - \Delta u \quad \text{in } X.$$

Assume moreover that for some $(x_0, t_0) \in X$ we have

$$M := u(x_0, t_0) = \sup_{(x,t)\in X} u(x,t).$$

(1) Describe (in words) in which set C the function is necessarily constant

$$C := \{(x,t) \in X : \quad u(x,t) = M\}.$$



(2) Assume the set X (grey) and the point (x_0, t_0) are given in the picture. Draw (in orange) the set C from the question above.

Exercise VI.3.9. *Show Theorem VI.3.7: let $g \in C^0(\mathbb{R}^n)$, $f \in C^0(\mathbb{R}^n \times [0,T])$ for some $T > 0$.*

Assume there are two solutions u^1 *and* $u^2 \in C^2_1(\mathbb{R}^n \times (0,T)) \cap C^0(\mathbb{R}^n \times [0,T])$ *of the problem*

$$\begin{cases} (\partial_t - \Delta)u = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = g(x) & \text{für } x \in \mathbb{R}^n. \end{cases}$$

If moreover we know that there are a_1, a_2 *and* $A_1, A_2 > 0$ *with*

$$|u^1(x,t)| \le A_1\, e^{a_1 |x|^2}, \quad |u^2(x,t)| \le A_2\, e^{a_2 |x|^2} \quad \forall (x,t) \in \mathbb{R}^n \times [0,T],$$

show that then

$$u^1 \equiv u^2 \quad \text{auf } \mathbb{R}^n \times [0,T].$$

Hint: Use Theorem VI.3.6.

Exercise VI.3.10. *(cf.* [**John, 1991**]*) Define the following Tychonoff-function,*

$$u(x,t) := \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!}\, x^{2k}.$$

Here $g^{(k)}$ *denotes the* k*-th derivative of* g*, given as*

$$g(t) := \begin{cases} e^{(-t^{-\alpha})} & t > 0 \\ 0 & t \le 0. \end{cases}$$

(1) Show that $u \in C^2_1(\mathbb{R}^2_+) \cap C^0(\mathbb{R} \times [0,\infty))$.
(2) Show moreover that

$$\begin{cases} (\partial_t - \Delta)u = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = 0 & \text{für } x \in \mathbb{R}^n. \end{cases} \tag{VI.3.28}$$

(3) Find a different solution $v \not\equiv u$ *of* (VI.3.28).
(4) Why (without proof) does this not contradict Question VI.3.9?

VI.4. Harnack's Principle

In the parabolic setting an "immediate" Harnack principle is not true in general, to compare sup and inf of a function one needs to wait for an (arbitrary short) amount of time.

THEOREM VI.4.1 (Parabolic Harnack inequality). *Assume* $u \in C^2_1(\mathbb{R}^n \times (0,T]) \cap L^\infty(\mathbb{R}^n \times [0,T])$ *and solves*

$$\partial_t u - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0,T)$$

and

$$u \ge 0 \quad \text{in } \mathbb{R}^n \times (0,T)$$

Then for any compactum $K \subset \mathbb{R}^n$ *and any* $0 < t_1 < t_2 < T$ *there exists a constant* C*, so that*

$$\sup_{x \in K} u(x,t_1) \le C \inf_{y \in K} u(y,t_2)$$

Proof. By the representation formula, Section VI.1, and uniqueness of the Cauchy problem

$$u(x_2,t_2) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t_2)^{\frac{n}{2}}} e^{-\frac{|x_2-y|^2}{4t_2}}\ u_0(y)\ dy.$$

Now, for $t_1 < t_2$ whenever $|x_1|$, $|x_2| \le \Lambda < \infty$, there exists a constant $C = C(|t_1 - t_2|, \Lambda)$ so that

$$-\frac{|x_2-y|^2}{4t_2} \ge -\frac{|x_1-y|^2}{4t_1} - C. \quad \forall y \in \mathbb{R}^n$$

See Exercise VI.4.2.

Consequently,

$$u(x_2,t_2) \ge \left(\frac{t_1}{t_2}\right)^{\frac{n}{2}} e^{-C} \int_{\mathbb{R}^n} \frac{1}{(t_1)^{\frac{n}{2}}} e^{-\frac{|x_1-y|^2}{4t_1}}\ u_0(y)\ dy = \left(\frac{t_1}{t_2}\right)^{\frac{n}{2}} e^{-C} u(x_1,t_1).$$

□

Exercise VI.4.2. *Show the following estimate, which we used for Harnack-principle, Theorem VI.4.1:*

If $K \subset \mathbb{R}^n$ is compact and $0 < t_1 < t_2 < \infty$, then there exists a constant $C > 0$ depending on K and $(t_2 - t_1)$, such that

$$\frac{|x_1-y|^2}{t_2} \le \frac{|x_2-y|^2}{t_1} + C \quad \forall x_1, x_2 \in K,\ y \in \mathbb{R}^n.$$

Exercise VI.4.3 (Counterexample Harnack). *(1) Sei $u_0 : \mathbb{R}^n \to [0,\infty)$ eine glatte Funktion mit kompaktem support mit $u_0(0) = 1$. Setze*

$$u(x,t) := \int_{\mathbb{R}^n} \Phi(x-y,t)\ u_0(y) \quad t > 0$$

Zeigen Sie,

$$\inf_{x\in\mathbb{R}^n} u(x,t) = 0 \quad \text{für alle } t > 0.$$

Aber

$$\sup_{x\in\mathbb{R}^n} u(x,t) > 0 \quad \text{für alle } t > 0.$$

Warum ist dies kein Widerspruch zum Harnack-Prinzip, Theorem VI.4.1?

(2) Zeigen Sie, dass das folgende Sei $\xi \in \mathbb{R}^n$ gegeben, und u definiert als

$$u_\xi(x,t) := (t+1)^{-\frac{1}{2}} e^{-\frac{|x+\xi|^2}{4(t+1)}}.$$

Zeigen Sie dass u eine Lösung von $(\partial_t - \Delta)u = 0$ auf $\mathbb{R}^n \times (0,\infty)$ ist. Zeigen Sie aber auch, dass es jedes feste $t > 0$ keine Konstante $C = C(t) > 0$ gibt für die gilt

$$\sup_{x\in[-1,1]} u_\xi(x,t) \le C \inf_{y\in[-1,1]} u_\xi(y,t) \quad \forall \xi \in \mathbb{R}^n.$$

Warum ist dies kein Widerspruch zum Harnack-Prinzip, Theorem VI.4.1?

Hinweis: Wählen Sie $x = -\frac{\xi}{|\xi|}$ und $y = 0$. Was passiert, wenn $|\xi| \to \infty$?

VI.5. Regularity and Cauchy-estimates

THEOREM VI.5.1 (Smoothness). *Let $u \in C_1^2(U_T)$ satisfy*

$$u_t = \Delta u \quad \text{in } U_T. \tag{VI.5.1}$$

Then $u \in C^\infty(\mathrm{int}(U_T))$.

PROOF. This is a standard technique to transfer local questions to global situations, using a cut-off function. Let

$$C(x,t;r) = \{(y,s)\colon |x-y| \le r, t-r^2 \le s \le t\} \tag{VI.5.2}$$

and

$$C_1 = C(x_0,t_0;r), \quad C_2 = C\left(x_0,t_0;\frac{3}{4}r\right), \quad C_3 = C\left(x_0,t_0;\frac{r}{2}\right) \tag{VI.5.3}$$

for some r such that $C_1 \subset U_T$. Choose a cut-off function

$$\eta \in C^\infty(\mathbb{R}^n \times [0,t_0]) \tag{VI.5.4}$$

with $0 \le \eta \le 1$, $\eta_{|C_2} \equiv 1$, $\eta \equiv 0$ around $\mathbb{R}^n \times [0,t_0] \backslash C_1$. Suppose first that u is smooth. Set

$$v(x,t) = \eta(x,t)u(x,t) \quad \forall (x,t) \in \mathbb{R}^n \times (0,t_0], \tag{VI.5.5}$$

extended by 0. Then

$$\begin{aligned} \partial_t v - \Delta v &= u_t\eta + \eta_t u - \eta\Delta u - u\Delta\eta - 2\langle \nabla u, \nabla\eta\rangle \\ &= \eta_t u - u\Delta\eta - 2\langle \nabla u, \nabla\eta\rangle \\ &=: f(x,t) \end{aligned} \tag{VI.5.6}$$

with bounded v and $f \in C_1^2$ by smoothness of u. Let $(x,t) \in C_3$. Then by Theorem VI.1.4

$$\begin{aligned} v(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\, dyds \\ &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\Big(u(y,s)\eta_t(y,s) - u(y,s)\Delta\eta(y,s) \\ &\quad - 2\langle \nabla u(y,s), \nabla\eta(y,s)\rangle\Big)\, dyds \end{aligned} \tag{VI.5.7}$$

We note: The singularity $y = x$ and $s = t$ is cut off due to $(x,t) \in C_3$. Hence ($\eta \equiv 1$ around C_1)

$$\begin{aligned} v(x,t) &= \int_{C_1^c} \Phi(x-y,t-s)\Big((\partial_t - \Delta)\eta(y,s)u(y,s)\Big)\, dyds \\ &\quad + \int_{C_1^c} 2D\Phi(x-y,t-s)D\eta(y,s)u(y,s). \end{aligned} \tag{VI.5.8}$$

By convolution: If $u \in C_1^2(U_T)$, we have a representation

$$v(x,t) = \int_C K(x,y,s,t)u(y,s)\, dyds \tag{VI.5.9}$$

with no singularities in the kernel. Thus v is smooth and so is u around (x_0, t_0). $\square$

THEOREM VI.5.2 (Cauchy estimates). *For all $k, l \in \mathbb{N}$ there exists $C > 0$ such that for all $u \in C^{2,1}(U_T)$ ($u \in L^1_{\text{loc}}$ will be sufficient), solving*

$$(\partial_t - \Delta)\, u = 0, \tag{VI.5.10}$$

there holds

$$\max_{C(x_0,t_0;\frac{r}{2})} |D_x^k \partial_t^l u| \leq \frac{C}{r^{k+2l+n+2}} \|u\|_{L^1(C(x_0,t_0;r))} \tag{VI.5.11}$$

for all $C(x_0, t_0; r) \subset U_T$.

PROOF. Suppose first $(x_0, t_0) = (0, 0)$ and $r = 1$. Set

$$C(1) = C(0, 0; 1). \tag{VI.5.12}$$

Then as in the proof of Theorem VI.5.1 we have

$$u(x,t) = \int_{C(1)} K(x,t,y,s) u(y,s)\ dyds \quad \forall (x,t) \in C\left(\frac{1}{2}\right). \tag{VI.5.13}$$

Then

$$D_x^k \partial_t^l u(x,t) = \int_{C(1)} \left(D_x^k \partial_t^l K(x,t,y,s)\right) u(y,s)\ dyds \tag{VI.5.14}$$

and hence

$$|D_x^k \partial_t^l u(x,t)| \leq C_{k,l} \|u\|_{L^1(C(1))} \quad \forall (x,t) \in C\left(\frac{1}{2}\right). \tag{VI.5.15}$$

Thus the claim is proven for $r = 1$. For $r > 0$ and $(x_0, t_0) \in \mathbb{R}^{n+1}$ set

$$v(x,t) = u(x_0 + rx, t_0 + r^2 t). \tag{VI.5.16}$$

Then

$$\max_{C\left(\frac{1}{2}\right)} |D_x^k \partial_t^l v| \leq C_{k,l} \|v\|_{L^1(C(1))}. \tag{VI.5.17}$$

Hence

$$\max_{C(x_0,r_0;\frac{r}{2})} |D_x^k \partial_t^l u| r^{k+2l} \leq C_{k,l} r^{-(n+2)} \|u\|_{L^1(C(1))}. \tag{VI.5.18}$$

$\square$

CHAPTER 7

linear parabolic equations

VII.1. Definitions

The heat equation is the simplest or most pure *parabolic* equation. In general we want to study equations of the form

$$\partial_t u - Lu,$$

where L is a uniformly elliptic differential operator (for each time t). More precisely, we study L which for given coefficient functions $a_{ij}(x,t)$, $b_i(x,t)$ and $c(x,t)$ has the form

$$Lu(x,t) = a_{ij}(x,t)\,\partial_{ij}u(x,t) + b_i(x,t)\,\partial_i u(x,t) + c(x,t)\,u(x,t).$$

Recall that we use Einstein's summation convention,

$$= \sum_{i,j=1}^{n} a_{ij}(x,t)\,\partial_{ij}u(x,t) + \sum_{i=1}^{n} b_i(x,t)\,\partial_i u(x,t) + c(x,t)\,u(x,t).$$

We want L to be elliptic (and equivalently $\partial_t - L$ to be parabolic), which simply means that the leading order coefficients form a non-degenerate, positive matrix.

Definition VII.1.1 (Parabolic)**.** We say that an operator $\partial_t - L$ is uniformly parabolic, if there exists a constant $\lambda > 0$ so that

$$a_{ij}(x,t)\,\xi_i\,\xi_j \geq \lambda|\xi|^2 \quad \forall (x,t) \in \Omega_T,\ \xi \in \mathbb{R}^n.$$

Equivalently, the matrix $A(x,t) = (a_{ij}(x,t))_{1\leq i,j\leq n}$ satisfies

$$\langle A(x,t)\xi, \xi\rangle_{\mathbb{R}^n} \geq \lambda \quad \forall (x,t) \in \Omega_T,\ \xi \in \mathbb{R}^n,\ |\xi| = 1.$$

We also say that L is uniformly elliptic.

The simplest example of a parabolic operator is the heat operator. Indeed take

$$a_{ij} := \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and $b \equiv c \equiv 0$. Then $L = +\Delta$. Indeed, parabolic operators have many features similar to $\partial_t - \Delta$.

Definition VII.1.2. Let $X \subset \mathbb{R}^{n+1}$ be an $n+1$-dimensional domain. The *parabolic boundary* $\mathcal{P}X$ of X is defined as follows. For $\rho > 0$, $(x_0, t_0) \in \mathbb{R}^{n+1}$ define the (backwards-in-time) cylinder $Q_\rho(x_0, t_0)$ as

$$Q_\rho(x_0, t_0) = \left\{(x,t) \in \mathbb{R}^{n+1} : \ |x - x_0| < \rho, \ t \in (t_0 - \rho^2, t_0), \right\}.$$

Then the parabolic boundary $\mathcal{P}X$ of X is defined as

$$\mathcal{P}X := \{(x_0, t_0) \in \partial X \quad \text{so that } Q_\rho(x_0, t_0) \cap X^c \neq \emptyset \quad \forall \rho > 0\}$$

Exercise VII.1.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain and $\Omega_T = \Omega \times (0, T]$. Show that $\mathcal{P}\Omega_T = \Gamma_T$.*

VII.2. Maximum principles

VII.2.1. Weak maximum principle. We will always assume that the operators $\partial_t + L$ are uniformly parabolic and the coefficients a_{ij}, b^i, c are continuous. Moreover we assume symmetry,

$$a_{ij} = a_{ji} \quad 1 \leq i, j \leq n.$$

Also $X \subset \mathbb{R}^{n+1}$ bounded.

THEOREM VII.2.1 (Weak maximum principle, $c \equiv 0$). *Let $X \subset \mathbb{R}^{n+1}$ be open and bounded and let L be an elliptic operator with*

$$c = 0. \tag{VII.2.1}$$

Let $u \in C^2_1(X) \cap C^0(\bar{X})$.

(1) If u is a subsolution of $\partial_t - L$, i.e.

$$(\partial_t - L)u \leq 0, \tag{VII.2.2}$$

then

$$\sup_{\bar{X}} u = \sup_{\partial_P X} u. \tag{VII.2.3}$$

(2) If u is a supersolution of $\partial_t - L$, i.e.

$$(\partial_t - L)u \geq 0, \tag{VII.2.4}$$

then

$$\inf_{\bar{X}} u = \inf_{\partial_P X} u. \tag{VII.2.5}$$

PROOF. We only prove the first claim, the second one follows by replacing u with $-u$. Also we will assume that $X = \Omega_T$

For now assume that we have a *strict subsolution*. That is,

$$(\partial_t - L)u < 0 \quad \text{in } \Omega_T. \tag{VII.2.6}$$

Assume that there exists a point $(x_0, t_0) \in \Omega_T$ with $u(x_0, t_0) = \max_{\overline{\Omega_T}} u$. Then $x_0 \in \Omega$ and $t_0 \in (0, T]$, so the maximality condition tells us

$$\partial_t u(x_0, t_0) \geq 0, \quad Du(x_0, t_0) = 0, \quad D^2 u(x_0, t_0) \leq 0.$$

In particular, observing (VII.2.1),

$$\partial_t u(x_0, t_0) - Lu(x_0, t_0) \geq a_{ij}(x_0, t_0)\, \partial_{ij} u(x_0, t_0).$$

In view of Exercise VII.2.2 this implies

$$\partial_t u(x_0, t_0) - Lu(x_0, t_0) \geq 0,$$

a contradiction to (VII.2.6). So what do we do if we had only (VII.2.2)? We consider a subsolution slightly below u. Let $u^\varepsilon(x, t) := u(x, t) - \varepsilon t$. Then, again with (VII.2.1),

$$\partial_t u^\varepsilon - Lu^\varepsilon = \partial_t u - Lu - \varepsilon < 0 \quad \text{in } \Omega_T.$$

The above argument implies that

$$\max_{\overline{\Omega_T}} u_\varepsilon = \max_{\Gamma_T} u_\varepsilon \quad \forall \varepsilon > 0.$$

In particular we have

$$\max_{\overline{\Omega_T}} u \leq \varepsilon T + \max_{\overline{\Omega_T}} u_\varepsilon \leq \varepsilon T + \max_{\Gamma_T} u_\varepsilon \leq \varepsilon T + \max_{\Gamma_T} u.$$

Letting $\varepsilon \to 0$ we have

$$\max_{\overline{\Omega_T}} u \leq \max_{\Gamma_T} u.$$

The inverse estimate is always true, so the claim is proven. □

Exercise VII.2.2. *A matrix $A \in \mathbb{R}^{n \times n}$ is nonnegative, $A \geq 0$, if*

$$\langle Av, v \rangle \geq 0 \quad \forall v \in \mathbb{R}^n.$$

A matrix A is symmetric, if $A^T = A$.

Show that

(1) $A \geq 0$ implies $P^T A P \geq 0$ for any matrix $P \in \mathbb{R}^{n \times n}$.
(2) $A \geq 0$ implies that the diagonal entries $A_{ii} \geq 0$ for any $i \in \{1, \ldots, n\}$.
(3) $A \geq 0$ and $B \geq 0$ and B is symmetric then

$$A : B := \sum_{i,j=1}^{n} A_{ij} B_{ij} \geq 0.$$

If $c \geq 0$, then we have to adapt the claim. For a function f let $f_+ := \max\{f, 0\}$ and $f_- := \max\{-f, 0\}$.

Exercise VII.2.3. *Complete the above proof for general domain X.*

THEOREM VII.2.4 (Weak maximum principle, $c \le 0$). *Let u and X as in Theorem VII.2.1 and $\partial_t - L$ parabolic with $c \le 0$. Then if $u_t - Lu \le 0$ then*

$$\sup_{\bar{X}} u \le \sup_{\partial_P X} u_+. \tag{VII.2.7}$$

For $u_t - Lu \ge 0$, then

$$\inf_{\bar{X}} u \ge -\sup_{\partial_P X} u_-, \tag{VII.2.8}$$

where $u_+ = \max(0, u)$ and $u_- = -\min(u, 0)$. If $u_t = Lu$, then

$$\sup_{\bar{X}} |u| = \sup_{\partial_P X} |u| \tag{VII.2.9}$$

PROOF. We just prove the first claim, the second and third are simple corollaries.

Again, we assume Ω_T, general X is an exercise. we first simplify the equation, and assume that

$$(\partial_t - L)u < 0 \quad \text{in } \Omega_T.$$

The only situation we have to exclude is that there exists $(x_0, t_0) \in \Omega_T$ at which there is a *positive* maximum value $u(x_0, t_0) > 0$. With the arguments as in Theorem VII.2.1,

$$\partial_t u(x_0, t_0) - Lu(x_0, t_0) \ge -c(x_0, t_0)\, u(x_0, t_0) \ge 0,$$

and we have our contradiction. The full claim is obtained if we consider again $u^\varepsilon(x, t) := u(x, t) - \varepsilon t$. Then

$$\max_{\overline{\Omega_T}} u_\varepsilon \le \max_{\Gamma_T} (u_\varepsilon)_+ \le \max_{\Gamma_T} (u)_+.$$

We let $\varepsilon \to 0$ to conclude. □

A consequence of the weak maximum principle is uniqueness of solutions and the comparison principle.

Corollary VII.2.5 (Uniqueness)**.** *Let $X \subset \mathbb{R}^{n+1}$ and L as above with $c \le 0$. Let $u, v \in C_1^2(X) \cap C^0(\bar{X})$ satisfy*

$$u_t - Lu = v_t - Lv. \tag{VII.2.10}$$

Then if $u = v$ on $\partial_P X$, we have $u = v$ in X.

Corollary VII.2.6 (Comparison Principle)**.** *Let X and L as above and $u, v \in C_1^2(X) \cap C^0(\bar{X})$ with*

$$u_t - Lu \le v_t - Lv \tag{VII.2.11}$$

in X with $u \le v$ on $\partial_P X$, then we have $u \le v$ in X.

We leave the proofs as exercises, Exercise VII.2.7.

Exercise VII.2.7. *Prove Corollaries VII.2.5 and VII.2.6. Hint: What equation does $u - v$ satisfy?*

VII.2.2. Strong Maximum principle. Let

$$\partial_t u - Lu = 0 \quad \text{in } \Omega_T$$

We want to understand better the relation between u at different times. We have the following very important "propagation of positivity" property. See [**Lieberman, 1996**, II, Lemma 2.6]

Lemma VII.2.8. [Propagation of positivity] *For $R > 0$ and $\alpha > 0$ let $B_R(0) \subset \mathbb{R}^n$. Let $Q(R) = B_R \times (0, \alpha R^2)$. Let $0 \le u \in C_1^2(Q(R))$ satisfy*

$$u_t - Lu \ge 0, \tag{VII.2.12}$$

where L is elliptic with $b = c = 0$. If

$$u(x, 0) \ge h \quad \forall |x| < \varepsilon R \tag{VII.2.13}$$

for some $h > 0$ and $0 < \varepsilon < 1$, then

$$u(x, \alpha R^2) \ge c(\varepsilon, \lambda, R, \|a_{ij}\|_\infty) h \quad \forall |x| \le \frac{R}{2} \tag{VII.2.14}$$

for some positive c.

Proof. Let $\tilde{Q} \subset \mathbb{R}^{n+1}$ be a cone so that at time $t = 0$, $\tilde{Q} \cap (\mathbb{R}^n \times \{t = 0\})$ is the ball $\{|x| < \varepsilon R\}$ and at time $t = \alpha R^2$, $\tilde{Q} \cap (\mathbb{R}^n \times \{t = \alpha R^2\})$ is the ball $\{|x| < R\}$. See Figure VII.2.1. In formulas, $\tilde{Q}$ can be written

$$\tilde{Q} = \left\{ (x, t) \in \mathbb{R}^{n+1} : \quad |x|^2 < \psi(t), 0 < t < \alpha R^2 \right\}$$

for

$$\psi(t) := \frac{(1 - \varepsilon^2)}{\alpha} t + \varepsilon^2 R^2.$$

On $\tilde{Q}$ we will construct a comparison ("barrier") function v with the following properties:

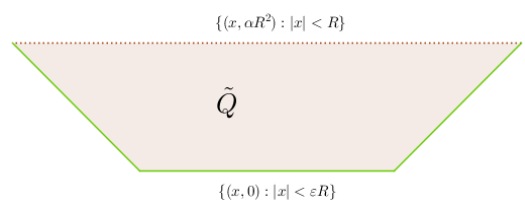


Figure VII.2.1. $\tilde{Q}$ and its parabolic boundary $\mathcal{P}\tilde{Q}$ (green)

$$\begin{cases} v_t - Lv \le 0 & \text{in } \tilde{Q} \\ v \le u & \text{on } \mathcal{P}\tilde{Q} \end{cases} \tag{VII.2.15}$$

and moreover

$$v(x, \alpha R^2) \ge c\, h \quad \text{whenever } |x| \le \frac{R}{2} \tag{VII.2.16}$$

If we have such a v, then by Corollary VII.2.6 (the general domain version)

$$u(x, \alpha R^2) \geq v(x, \alpha R^2) \geq ch \quad \text{whenever } |x| \leq \frac{R}{2}$$

So how do we construct such a v? We essentially rescale (in time) the map $(1-|x|^2)^2$. Choose the Ansatz

$$v(x,t) := \mu(t)\,(\nu(t) - |x|^2)^2.$$

For μ, ν nonnegative functions. In general, away from $t = 0$, we only know that $u \geq 0$, so to make v as large as possible, it seems reasonable to set $v(x,t) \equiv 0$ on the positive part of the parabolic boundary $\mathcal{P}\tilde{Q} \cap \{t > 0\}$. That is,

$$\nu(t) := \psi(t).$$

Now we compute the equation. Firstly

$$\partial_{x^i x^j} v(x,t) = 8\mu(t)\, x^j\, x^i - 4\mu(t)\,(\psi(t) - |x|^2)\delta_{ij}$$

Consequently, by ellipticity

$$-a_{ij}(x,t)\,\partial_{x^i x^j} v(x,t) \leq \mu(t)\left(-8\,\psi(t)\,\lambda + 8\,(\psi(t) - |x|^2)\,\lambda + 4(\psi(t) - |x|^2)\,\mathrm{tr}(A)\right).$$

Also,

$$v_t(x,t) = \mu'(t)\,(\psi(t) - |x|^2)^2 + 2\mu(t)\,(\psi(t) - |x|^2)\psi'(t).$$

This v_t has to be the positive guy, so we would like to be able to compare $\mu'(t)$ and $\nu'(t)$. We thus choose (note that $\psi(t) > 0$) for some constant $\eta > 0$,

$$\mu(t) := \eta\psi(t)^{-q}.$$

Then

$$-a_{ij}(x,t)\,\partial_{x^i x^j} v(x,t) \leq \eta\psi^{1-q}(t)\left(-8\,\lambda + 8\left(\frac{(\psi(t) - |x|^2)}{\psi(t)}\right)\lambda + 4\left(\frac{(\psi(t) - |x|^2)}{\psi(t)}\right)\mathrm{tr}(A)\right).$$

and (observe that $\psi'(t) = \frac{1-\varepsilon^2}{\alpha}R$,

$$\begin{aligned} v_t(x,t) =& \eta\left(-q\psi^{-q-1}(t)\,(\psi(t) - |x|^2)^2 + 2\psi(t)^{-q}\,(\psi(t) - |x|^2)\right)\,\frac{1-\varepsilon^2}{\alpha}R \\ =& \eta\psi(t)^{1-q}\left(-q\left(\frac{(\psi(t) - |x|^2)}{\psi(t)}\right)^2 + 2\psi(t)\left(\frac{(\psi(t) - |x|^2)}{\psi(t)}\right)\right)\,\frac{1-\varepsilon^2}{\alpha}R. \end{aligned}$$

We see a quadratic structure in

$$\xi(t) := \left(\frac{(\psi(t) - |x|^2)}{\psi(t)}\right),$$

namely

$$\begin{aligned} & v_t(x,t) - a_{ij}(x,t)\partial_{x^i x^j} v(x,t) \\ \leq & \eta\psi^{1-q}(t)\left(-\left(q\,\frac{1-\varepsilon^2}{\alpha}R\right)\,\xi(t)^2 + \left(2\frac{1-\varepsilon^2}{\alpha}R\,\psi(t)^2 + 8\lambda + 4\,\mathrm{tr}(A)\right)\xi(t) - 8\,\lambda\right). \end{aligned}$$

Observe that the leading order term and the zero-order term are negative, hence (see Exercise VII.2.9) there exists a large $q > 0$ so that

$$v_t(x,t) - a_{ij}(x,t)\,\partial_{x^i x^j} v(x,t) \le 0 \quad \text{in } \tilde{Q}.$$

On the other hand, for $t = 0$, in view of (VII.2.13),

$$v(x,0) = \eta\varepsilon^{-2q} R^{-2q}\,(\varepsilon^2 R^2 - |x|^2)^2 \le \eta\ (\varepsilon R)^{4-2q} \le \frac{1}{h}\eta\ (\varepsilon R)^{4-2q}\ u(x,0).$$

So we choose

$$\eta := h\,(\varepsilon R)^{2q-4}.$$

Then v satisfies (VII.2.15). It remains to check (VII.2.16). For $|x| \le \frac{R}{2}$,

$$v(x,\alpha R) = h\,(\varepsilon R)^{2q-4}\ R^{-2q}\,(R^2 - |x|^2)^2 \ge h\varepsilon^{2q-4}\ \frac{9}{16}.$$

This finishes the proof of Lemma VII.2.8. It is worth noting that we actually get an estimate of the form ε^κ, where κ is a uniform constant depending on R, λ, etc. For this assume w.l.o.g. that $\varepsilon < \frac{1}{2}$, for any $\varepsilon > \frac{1}{2}$ the claim follows from the $\varepsilon < \frac{1}{2}$ case since the positivity set is larger than required. □

Exercise VII.2.9. *Assume that $a, b, c \in \mathbb{R}$ be fixed. To any $\lambda \in \mathbb{R}$ we associate the polynomial*

$$p_\lambda(x) := \lambda a x^2 + bx + c \quad x \in \mathbb{R}.$$

Show that if $a < 0$ and $c < 0$ then there exists a $\lambda > 0$ so that

$$\sup_{x\in\mathbb{R}} p_\lambda(x) < 0.$$

Hint: p-q formula

THEOREM VII.2.10 (Strong Maximum Principle). *Let $b, c = 0$, L elliptic, $X \subset \mathbb{R}^{n+1}$ open and bounded, $u \in C^2_1(X) \cap C^0(\bar{X})$ and assume in X:*

$$(\partial_t - L)u \le 0. \tag{VII.2.17}$$

Assume there is $(x_0, t_0) \in X$, such that

$$u(x_0,t_0) = \sup_X u, \tag{VII.2.18}$$

then

$$u(x,t) = u(x_0,t_0) \quad \forall (x,t) \in S(x_0,t_0), \tag{VII.2.19}$$

where

$$\begin{aligned} S(x_0,t_0) = \{(x,t) \colon \exists\ g \in & C^0\left([0,1], X\backslash\partial_p X\right), g(0) = (x_0,t_0), \\ & g(1) = (x,t), g \text{ decreasing in } t\}. \end{aligned} \tag{VII.2.20}$$

PROOF. Set

$$M := \max_{\bar{X}} u. \tag{VII.2.21}$$

Claim: Assume a maximal point $(y_0, t_0) \in X$, $r > 0$, such that

$$Q(y_0, t_0, 3r) \subset X \tag{VII.2.22}$$

and such that there is $(y_1, t_1) \in Q(y_0, t_0, r)$ with

$$u(y_1, t_1) < M. \tag{VII.2.23}$$

Set $v = M - u$ and

$$R = 2|y_1 - y_0| < 2r, \quad \alpha := \frac{t_0 - t_1}{R^2}. \tag{VII.2.24}$$

By continuity there exists $\varepsilon > 0$ and $h > 0$ such that

$$v(x, t_1) > h, \quad |y| < \varepsilon R. \tag{VII.2.25}$$

By Lemma VII.2.8 there exists $c > 0$, such that $v(y, t_0) > ch > 0$ for all $|y - y_1| < R/2$, a contradiction. Hence if $u(x_0, t_0) = M$, then $u(y, t) = M$ for all $(y, t) \in Q(x_0, t_0; r)$, whenenver $Q(x_0, t_0; 3r) \subset X$. Hence $\{u = M\} \cap S(x_0, t_0)$ is (parabolically) open and closed and hence all of $S(x_0, t_0)$. □

Without proof, now we state:

THEOREM VII.2.11 (Parabolic Harnack inequality). *Assume $u \in C_1^2(U_T)$ and solves*

$$(\partial_t - L)u = 0 \quad \text{in } U_T$$

and

$$u \geq 0 \quad \text{in } U_T$$

Assume moreover that $b \equiv 0$ and $c \equiv 0$ and a is smooth.

If $V \Subset U$ is connected, then for each time $0 < t_1 < t_2 \leq T$ there is a constant C such that

$$\sup_{x \in V} u(x, t_1) \leq C \inf_{x \in V} u(x, t_2).$$

PROOF. See [**Evans, 2010**, Theorem 10, p.391]. □

CHAPTER 8

Principles of Semi-group theory

As references we refer to [**Evans, 2010**, §7.4] and [**Cazenave and Haraux, 1998**].

In Section VI.1 we looked at $(\partial_t - \Delta)\, u = 0$ and naïvely we should have

$$u = e^{t\Delta} u(0). \tag{VIII.0.1}$$

We made this precise with the help of the Fourier Transform.

Is there a similar relation if we look at an elliptic operator L instead of Δ?

Generally: Let X be a real Banach space and a linear map A,

$$A\colon D(A) \subset X \to X, \tag{VIII.0.2}$$

where $D(A)$ is the domain of A, a linear (usually dense) subset of X. We are looking for solutions $u \in C^1((0,T), X)$ of

$$\begin{aligned} \dot{u} &= Au, \quad t \in (0,T), \\ u(0) &= \varphi. \end{aligned} \tag{VIII.0.3}$$

A is in general not bounded, but closed. Assume there exists a solution to (VIII.0.3), then

$$T(t)\varphi := u(t) \tag{VIII.0.4}$$

defines an operator. Resonable properties of $T\colon$ are

- $T(t)\colon X \to X$ is linear.
- $T\colon [0,T) \to L(X)$. (hopefully)
- $T(0) = \mathrm{id}\,,$
- $T(t+s) = T(t) \circ T(s)$, (from uniqueness hopefully)
- $t \mapsto T(t)\varphi$ is continuous.

The latter three properties are characteristic for a *semigroup*.

Assume now that we have a semigroup

$$T\colon [0,\infty) \times X \to X. \tag{VIII.0.5}$$

Then we find some A such that T is the semigroup of A. A will then be called the *generator* of T.

$$\begin{aligned} \dot{u}(t) &= \lim_{s\to 0} \frac{u(t+s)-u(t)}{s} = \lim_{s\to 0} \frac{T(t+s)\varphi - T(t)\varphi}{s} \\ &= \lim_{s\to 0} \frac{T(s)-T(0)}{s} u(t) \\ &\equiv Au(t). \end{aligned} \tag{VIII.0.6}$$

Hence let

$$Au = \lim_{s\to 0} \frac{T(s)-T(0)}{s} u, \tag{VIII.0.7}$$

whenever the limit exists. Call $D(A)$ the set of $u \in X$ where this limit exists.

One might conjecture there is some sort of equivalence between generators A and semigroups T.

Questions: Which generators A allow semigroups? Which generators are implies by semigroups?

The main theorem which gives us an answer to this question is the *Hille-Yoshida Theorem*, Theorem VIII.2.6 and VIII.2.8. For Schrödinger equations this is (a generalization of) *Stones' theorem*, it also appears under the name *Lumer–Phillips theorem*.

VIII.1. m-dissipative operators

We want to solve

$$\begin{aligned} u'(t) &= Au, \quad t > 0 \\ u(0) &= \varphi \end{aligned} \tag{VIII.1.1}$$

with some operator

$$D(A) \subset X \to X, \tag{VIII.1.2}$$

where X is a Banach space and $D(A)$ a linear subspace, e.g. $X = L^2$ and $D(A) = H^2$ and $A = \Delta$. The norm of our space X is the L^2-norm, and then A is not a bounded operator. On the other hand, since C_c^∞ is dense in L^2, $D(A)$ is dense in L^2 (everything with respect to the L^2-norm).

VIII.1.1. linear bounded operators. (i) Let $X = \mathbb{R}^n$ or $\mathcal{C}^n$, $A\colon X \to X$ linear (and thus bounded), then

$$u(t) = e^{tA}\varphi \tag{VIII.1.3}$$

is the unique solution to (VIII.1.1), where

$$e^{tA} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k. \tag{VIII.1.4}$$

(ii) Let X be a general Banach space and $A \in L(X)$, where $L(X)$ is the space of bounded linear operators. Here e^{tA} also makes sense.

Lemma VIII.1.1. *Let $A, B \in L(X)$. Then*

(i) e^A converges absolutely,

(ii) $e^0 = \mathrm{id}$,

(iii) $AB = BA \quad \Rightarrow \quad e^{A+B} = e^A e^B$,

(iv) $e^{-A} = \left(e^A\right)^{-1}$.

Theorem VIII.1.2. *Let $A \in L(X)$, $\varphi \in X$, $T > 0$. Then there exists a unique solution $u \in C^1((0,T), X)$ of*

$$\begin{aligned} u'(t) &= Au(t) \\ u(0) &= \varphi. \end{aligned} \tag{VIII.1.5}$$

Proof. Put

$$u(t) = e^{tA}\varphi. \tag{VIII.1.6}$$

Then

$$u'(t) = e^{tA} A\varphi = Au(t). \tag{VIII.1.7}$$

For a second solution v set

$$w(t) = e^{-tA} v(t), \tag{VIII.1.8}$$

then $w'(t) = 0$ and hence $w(t) = w(0) = \varphi$. □

VIII.1.2. unbounded operators. Let X be a real or complex Banach space. An operator

$$A\colon D(A) \subset X \to X \tag{VIII.1.9}$$

is called linear, if and only if $D(A)$ is a linear subspace and A ist linear on $D(A)$. We say A is *densely defined*, if

$$\overline{D(A)} = X. \tag{VIII.1.10}$$

A is *bounded*, if and only if[1]

$$\|A\| := \sup_{\|x\|\le 1} \|Ax\| < \infty. \tag{VIII.1.11}$$

Otherwise it is called unbounded.

examples

(1) $X = L^2(\mathbb{R}^n)$, $A = \Delta$, $D(A) = H^2(\mathbb{R}^n)$ or $D(A) = C^\infty$.
(2) $X = C^0([0,1])$, $D(A) = X$, $K \in C^0([0,1]\times[0,1])$

$$Au(x) = \int_0^1 K(x,y)u(y)\ dy \tag{VIII.1.12}$$

is bounded.

We use the following notation.

$$G(A) = \{(u, Au) \subset X \times X \colon u \in D(A)\} \tag{VIII.1.13}$$

is the *graph of A*,

$$R(A) = \{Au \colon u \in D(A)\} \tag{VIII.1.14}$$

the *range of A*. An *extension of A* is

$$\tilde{A} \colon D(\tilde{A}) \subset X \to X, \tag{VIII.1.15}$$

such that

$$D(A) \subset D(\tilde{A}) \quad \text{and} \quad Au = \tilde{A}u \quad \forall u \in D(A). \tag{VIII.1.16}$$

A is called *closed*, if $G(A)$ is closed in $X \times X$. That is, whenever $(u_k)_{k\in\mathbb{N}} \subset D(A)$ and $u, g \in X$ with $u_k \xrightarrow{k\to\infty} u$ in X, $Au_k \xrightarrow{k\to\infty} g \in X$ we have $u \in D(A)$ and $g = Au$.

A is called *closable*, if there exists a closed extension $\tilde{A}$.

THEOREM VIII.1.3 (Closed Graph Theorem). *Let $A \colon X \to X$ be linear. Then A is continuous (i.e. bounded) if and only if A is closed.*

Observe that this assumes that A is defined on all of X (i.e. $D(A) = X$).

VIII.1.3. Notion of m-dissipative operators. $(X, \|\cdot\|)$ Banach space, $D(A) \subset X$ dense subspace. Let $A \colon D(A) \to X$ linear (not necessarily bounded, closed).

Definition VIII.1.4. A is *dissipative*, if

$$\|u - \lambda Au\| \ge \|u\| \quad \forall u \in D(A), \lambda > 0. \tag{VIII.1.17}$$

A is called *accretive*, if $-A$ is dissipative.

[1] this notion is equivalent with A being continuous, see any functional analysis book

Lemma VIII.1.5. *Let X be a Hilbert*[2] *space,*

$$A\colon D(A) \subset X \to X \tag{VIII.1.18}$$

linear, then A is dissipative if and only if

$$\operatorname{Re}\langle u, Au\rangle \le 0 \quad \forall u \in D(A). \tag{VIII.1.19}$$

Proof of Lemma VIII.1.5. Assume A dissipative, then:

$$\|u\|^2 + \lambda^2\|Au\|^2 - 2\lambda \operatorname{Re}\langle u, Au\rangle - \|u\|^2 = \|u - \lambda Au\|^2 - \|u\|^2 \ge 0. \tag{VIII.1.20}$$

Dividing by λ and letting $\lambda \to 0$ gives

$$\operatorname{Re}\langle u, Au\rangle \le 0. \tag{VIII.1.21}$$

For the converse, assume that

$$\operatorname{Re}\langle Au, u\rangle \le 0, \tag{VIII.1.22}$$

then

$$\|u - \lambda Au\|^2 = \|u\|^2 + \lambda^2\|Au\|^2 - 2\lambda \operatorname{Re}\langle u, Au\rangle \ge \|u\|^2. \tag{VIII.1.23}$$

□

Example VIII.1.6.

- Heat equation $(\partial_t - \Delta)u = 0$: $A = \Delta$, $X = L^2(\mathbb{R}^n)$, $D(A) = H^2(\mathbb{R}^n)$, then
$$\langle u, \Delta u\rangle = -\int_{\mathbb{R}^n} |\nabla u|^2 \le 0.$$
- Heat equation $(\partial_t - \Delta)u = 0$: $A = \Delta$, $X = L^2(\Omega)$, $D(A) = H^2 \cap H^1_0(\Omega)$, then
$$\langle u, \Delta u\rangle = -\int_{\mathbb{R}^n} |\nabla u|^2 \le 0.$$
- Schrödinger equation $(\partial_t - i\Delta)u = 0$. $A = i\Delta$, $D(A) = H^2(\mathbb{R}^n)$,
$$\langle u, \pm i\Delta u\rangle = \mp i \int_{\mathbb{R}^n} |\nabla u|^2$$
and hence the real part is 0 and both $i\Delta$ and $-i\Delta$ are dissipative.

Definition VIII.1.7 (m-dissipative)**.** A linear operator $A\colon D(A) \subset X \to X$ is called *m-dissipative*, if A is dissipative and $I - \lambda A$ is surjective for all $\lambda > 0$. (hence $I - \lambda A$ is continuously invertible as a map from $D(A)$ to X.)

[2] i.e. there exists a scalar product $\langle , \rangle : X \times X \to \mathcal{C}$ such that $\|v\|^2 = \langle v, v\rangle$. For $L^2(\Omega)$ the scalar product $\langle f, g\rangle := \int_\Omega fg$ (real) or $\langle f, g\rangle := \int_\Omega f\overline{g}$ (complex)

Observe that for the notion of m-dissipative the right choice of $D(A)$ becomes relevant. I.e. if we choose $D(\Delta) = C_c^\infty(\Omega)$ then Δ is dissipative, but not m-dissipative; but if we choose $D(\Delta) = H^2(\Omega)$ then Δ is

Our aim is to show that for any m-dissipative A we can define (some sort of) e^A. We also call A m-accretive, if $-A$ is m-dissipative. Set

$$J_\lambda = (I - \lambda A)^{-1} \colon X \to D(A). \tag{VIII.1.24}$$

Then (VIII.1.17) implies for m-dissipative A

$$\|J_\lambda v\| \le \|v\| \quad \forall v \in X. \tag{VIII.1.25}$$

Lemma VIII.1.8. *Let A be dissipative, then A is m-dissipative if and only if there exists $\lambda_0 > 0$ such that $I - \lambda_0 A$ is surjective.*

Proof. Let $\lambda \in (0, \infty)$ and $v \in X$. Our goal is to find $u \in D(A)$ such that $u - \lambda A u = v$. Observe that it is equivalent to find u such that

$$u - \lambda_0 A u = \frac{\lambda_0}{\lambda} v + \left(1 - \frac{\lambda_0}{\lambda}\right) u, \tag{VIII.1.26}$$

or equivalently (recall $J_{\lambda_0} = (I - \lambda_0 A)^{-1} : X \to D(A)$)

$$u = J_{\lambda_0}\left(\frac{\lambda_0}{\lambda} v + \left(1 - \frac{\lambda_0}{\lambda}\right) u\right) =: F(u). \tag{VIII.1.27}$$

That is, u is a fixed point of $u = F(u)$. We show that $F : D(A) \to D(A)$ is a contraction, to apply Banach Fixed Point theorem.

Observe, by (VIII.1.25),

$$\|F(u) - F(w)\| = \left\| J_{\lambda_0}\left(\left(1 - \frac{\lambda_0}{\lambda}\right)(u - w)\right)\right\| \le \left|1 - \frac{\lambda_0}{\lambda}\right| \|u - w\|. \tag{VIII.1.28}$$

Hence F is a contraction, whenever $\lambda > \lambda_0/2$.

Then we apply Banach Fixed Point theorem , on $D(A)$ (which is not closed in general)! The argument of Banach Fixed point theorem shows there $u_k := F^k(0)$ converges in X to some $u \in X$. To see that $u \in D(A)$, observe that $u_k \to u$ in X implies $J_{\lambda_0}(u_k) \to J_{\lambda_0} u \in D(A)$ in X by continuity. By affine linearity of F we conclude that $F(u_k) \xrightarrow{k\to\infty} F(u) \in D(A)$. Since $u_{k+1} = F(u_k)$ we see that $u = F(u) \in D(A)$.

That is, there is a unique $u \in D(A)$ with $F(u) = u$, and $I - \lambda A$ is surjective, whenever $\lambda > \frac{\lambda_0}{2}$.

Iterating this argument, e.g setting $\lambda_{i+1} := \frac{2}{3}\lambda_i$, we find that for any $\lambda > \lambda_i$, $i \in \mathbb{N}$, $I - \lambda A$ is surjective, and letting $i \to \infty$ we see that for any $\lambda > 0$ we have $I - \lambda A$ is surjective. $\square$

Proposition VIII.1.9. *All m-dissipative operators $(A, D(A))$ are closed.*

Proof. Let $u_k \to u$ in X and $Au_k \to g$ in X. We need to show that $u \in D(A)$ and $Au = g$.

Observe that $(I - A)u_k \to u + g$. Since A is m-dissipative, $J_1 = (I - A)^{-1}$ exists and is a continuous map $X \to D(A)$. Thus

$$u \xleftarrow{k\to\infty} u_k = J_1((I - A)u_k) \xrightarrow{k\to\infty} J_1(u - g) = J_1 u - J_1 g \in D(A)$$

That is, $u = J_1 u - J_1 g \in D(A)$, and applying $I - A$ we obtain $(I - A)u = u - g$, i.e. $Au = g$. □

Example VIII.1.10. $X = L^2(\mathbb{R}^n)$, $A = \Delta$, $D(A) = H^2(\mathbb{R}^n)$. Then A is m-dissipative.

We already know that A is dissipative (Example VIII.1.6), so by Lemma VIII.1.8 we only need to show that

$$\forall v \in L^2\ \exists u \in H^2 \colon u - \Delta u = v. \tag{VIII.1.29}$$

Here we see that the choice of $D(A)$ is important (the above will not work for $D(A) = C^\infty$.) We solve this by Fourier-transform (on domains: exercise!)

$$\hat{u}(\xi) + |\xi|^2 \hat{u}(\xi) = \hat{v}(\xi) \tag{VIII.1.30}$$

and hence we conjecture

$$\hat{u}(\xi) := \frac{1}{1 + |\xi|^2} \hat{v}(\xi). \tag{VIII.1.31}$$

Hence $\hat{u} \in L^2$ and

$$\frac{\xi^1 \xi^2}{1 + |\xi|^2} \hat{v}(\xi) \in L^2 \tag{VIII.1.32}$$

implies that $u, \nabla^2 u \in L^2$.

Proposition VIII.1.11. *Let A be m-dissipative, then*

$$\forall u \in \overline{D(A)} \colon \quad \|J_\lambda u - u\| \xrightarrow{\lambda\to 0} 0. \tag{VIII.1.33}$$

Proof. Observe that by (VIII.1.25), $J_\lambda - I : X \to X$ is bounded linear operator, i.e.

$$\|J_\lambda - I\| \le \|J_\lambda\| + \|I\| \le 2. \tag{VIII.1.34}$$

By density $D(A) \subset \overline{D(A)}$, it thus suffices to prove the result for $u \in D(A)$. Since $J_\lambda = (I - \lambda A)^{-1}$ using again (VIII.1.25)

$$\|J_\lambda u - u\| = \|J_\lambda (u - (I - \lambda A)u)\| \le \lambda \|Au\| \to 0, \quad \lambda \to 0. \tag{VIII.1.35}$$

□

Set

$$A_\lambda := AJ_\lambda = \frac{1}{\lambda}(J_\lambda - I). \tag{VIII.1.36}$$

This $A_\lambda \in L(X)$ will serve as an "approximation" for A, so that we can make (certain) sense of an operator e^{tA} in terms of $\lim_{\lambda\to 0} e^{tA_\lambda}$. This is justified by the following

Proposition VIII.1.12. *Let A be m-dissipative and $\overline{D(A)} = X$. Then*

$$A_\lambda u \xrightarrow{\lambda\to 0} Au, \quad \forall u \in D(A). \tag{VIII.1.37}$$

Proof. We have

$$(I - \lambda A)A = A(I - \lambda A). \tag{VIII.1.38}$$

Thus, multiplying both sides with J_λ from the left and also from the right, we have $A_\lambda = AJ_\lambda = J_\lambda A$.

Now observe that by Proposition VIII.1.11,

$$J_\lambda Au \xrightarrow{\lambda\to 0} Au, \tag{VIII.1.39}$$

since $D(A)$ is dense in X □

VIII.2. Semigroup Theory

Let X be a Banach space. A *semigroup* is an operator

$$T\colon [0,\infty) \to L(X), \tag{VIII.2.1}$$

such that

(i) $T(0) = I$,
(ii) $T(t+s) = T(t)T(s)$.

T is called *C^0-semigroup* (strongly continuous semigroup), if

(iii) $\lim_{t\to 0} \|T(t)u - u\| = 0 \quad \forall u \in X$.

Note, that by (ii) we necessarily have $T(s)T(t) = T(t)T(s)$.

Example VIII.2.1. (1) $A \in L(X)$, $T(t) = e^{tA}$.
(2) $X = L^p(\mathbb{R})$, $p \in [1,\infty]$.

$$T(t)u(x) = u(t+x). \tag{VIII.2.2}$$

If $p < \infty$, then T is a continuous semigroup, since C_c^∞ is dense and hence for $u \in L^p$ and $\varepsilon > 0$ there exists $f \in C_c^\infty$ with

$$\|f - u\|_p < \varepsilon/3. \tag{VIII.2.3}$$

We have for all small t,

$$\sup_x |f(x-t) - f(x)| < t\|\nabla f\|_\infty < \varepsilon/3, \tag{VIII.2.4}$$

and thus for $t \ll 1$

$$\left(\int_{\mathbb{R}} |T(t)f - f|^p\right)^{\frac{1}{p}} < \frac{\varepsilon}{3}\left(\operatorname{diam}(\operatorname{supp} f) + 1\right). \tag{VIII.2.5}$$

Moreover, by the definition of $T(t)$

$$\|T(t)(u-f)\|_{p,\mathbb{R}^n} = \|u-f\|_{p,\mathbb{R}^n} \leq \frac{\varepsilon}{3}.$$

Thus,

$$\begin{aligned}\|T(t)u - u\|_p &\leq \|T(t)f - f\|_p + \|T(t)(u-f)\|_p + \|u-f\|_p \\ &\leq \varepsilon\end{aligned}$$

The situation is different for $p = \infty$: let $u = \chi_{[0,1]}$, then

$$\|u - T(t)u\|_\infty = \sup_x |u(x) - u(x+t)| \geq 1 \quad \forall t > 0. \tag{VIII.2.6}$$

Thus T is no C^0-semigroup for $p = \infty$.

Proposition VIII.2.2. *Let $T(t)$ be a C^0-semigroup. Then $\exists M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$\|T(t)\| \leq Me^{\omega t}. \tag{VIII.2.7}$$

Proof. We show that there exists $\delta > 0$ such that

$$M := \sup_{0<t<\delta} \|T(t)\| < \infty. \tag{VIII.2.8}$$

If this was not the case, then there exists a sequence $t_n \to 0$ with

$$\|T(t_n)\| \to \infty. \tag{VIII.2.9}$$

Recall Banach-Steinhaus Theorem: If for a sequence $A_n \in L(X)$ we have

$$\forall u \in X\colon \sup_n \|A_n u\| < \infty, \tag{VIII.2.10}$$

then $\sup_n \|A_n\| < \infty$.

Hence, if (VIII.2.9) holds, then there must be $u \in X$ such that $\|T(t_n)u\| \to \infty$. But this is in contradiction to the C^0-property. Hence (VIII.2.9) cannot hold, i.e. (VIII.2.8) must be true.

Now let $t > 0$, then there exists $n \in \mathbb{N}$ and $s \in (0, \delta)$, such that

$$t = n\delta + s. \tag{VIII.2.11}$$

Then by the semigroup property

$$T(t) = \underbrace{T(\delta) \circ \cdots \circ T(\delta)}_{\text{n times}} \circ T(s). \tag{VIII.2.12}$$

That is, with (VIII.2.8),

$$\|T(t)\| \leq \|T(\delta)\|^n \|T(s)\| \leq M^{n+1} \leq M M^{\frac{t}{\delta}} = M e^{t \log \frac{M}{\delta}}. \tag{VIII.2.13}$$

□

Proposition VIII.2.3. *Let $T(t)$ be a C^0-semigroup. Then the map*

$$(t, u) \mapsto T(t)u \tag{VIII.2.14}$$

is continuous.

Proof. Exercise. □

Definition VIII.2.4. Let $T(t)$ be a C^0-semigroup. Then

$$\omega_0 = \inf\{w \in \mathbb{R} \colon \exists M \geq 1, \|T(t)\| \leq M e^{\omega t}\} \tag{VIII.2.15}$$

ist called the *growth bound* of the semigroup.

Definition VIII.2.5. A C^0-semigroup is called *contraction semigroup*, if

$$\forall t > 0 \colon \|T(t)\| \leq 1. \tag{VIII.2.16}$$

Recall that

$$\|J_\lambda\| \leq 1, \quad \|A_\lambda\| \leq \frac{2}{\lambda}, \tag{VIII.2.17}$$

where

$$A_\lambda := A J_\lambda = \frac{1}{\lambda}(J_\lambda - I). \tag{VIII.2.18}$$

We define

$$T_\lambda(t) = e^{t A_\lambda}. \tag{VIII.2.19}$$

For any $\lambda > 0$, T_λ is a C^0-semigroup (because $A_\lambda \in L(X)$). Moreover, for any $\lambda > 0$ we have that T_λ it is a contraction semigroup:

$$\|T_\lambda(t)\|_{L(X)} = \|e^{t J_\lambda \frac{1}{\lambda}} e^{-\frac{t}{\lambda} I}\| = e^{-\frac{t}{\lambda}} \|e^{\frac{t}{\lambda} J_\lambda}\| \leq e^{-\frac{t}{\lambda}} e^{\frac{t}{\lambda}} = 1. \tag{VIII.2.20}$$

Observe that in contrast, $e^{t\|A_\lambda\|}$ is in generally not uniformly bounded w.r.t $\lambda \to 0$.

The semigroup for an m-dissipative operator A will be constructed out of $T_\lambda \xrightarrow{\lambda \to 0} T(t)$. The following is the (first part) of the main theorem of the semigroup theory:

Theorem VIII.2.6 (Hille Yoshida (Part I)). *Let $A \colon D(A) \subset X \to X$ m-dissipative and densely defined. Then for all $u \in X$ the limit*

$$T(t)u = \lim_{\lambda \to 0} T_\lambda(t) u \tag{VIII.2.21}$$

exists and the convergence is uniform (w.r.t. t) on time-intervals of the form $[0, T]$.

Furthermore $(T(t))_{t\geq 0}$ *is a contraction semigroup,* $T(t)(D(A)) \subset D(A)$*, and for all* $u \in D(A)$*,*

$$u(t) := T(t)u \tag{VIII.2.22}$$

is the unique solution $u \in C^0([0,\infty), D(A)) \cap C^1((0,\infty), X)$ *to*

$$\begin{cases} \dot{u}(t) &= Au(t) \quad t > 0 \\ u(0) &= u \end{cases} \tag{VIII.2.23}$$

Proof. **Step (1): On the contraction semigroup property**

First we show that for $\mu, \lambda, t > 0$

$$\|T_\lambda(t)u - T_\mu(t)u\|_X \leq t\,\|A_\mu u - A_\lambda u\|_X. \tag{VIII.2.24}$$

Indeed, observe that (A_λ and A_μ commute!)

$$\begin{aligned} T_\lambda(t) - T_\mu(t) =& e^{tA_\lambda} - e^{sA_\lambda} = \int_0^t \frac{d}{ds}\left(e^{(t-s)A_\mu} e^{sA_\lambda}\right) ds \\ =& \int_0^t e^{(t-s)A_\mu} e^{sA_\lambda}\, ds\ (A_\lambda - A_\mu) \end{aligned}$$

Consequently, (recall (VIII.2.20))

$$\begin{aligned} \|T_\lambda(t)u - T_\mu(t)u\|_X \leq& \int_0^t \underbrace{\|e^{(t-s)A_\mu}\|_{L(X)}}_{\leq 1} \underbrace{\|e^{sA_\lambda}\|_{L(X)}}_{\leq 1} ds\, \|(A_\lambda - A_\mu)u\|_X \\ \leq& t\|(A_\lambda - A_\mu)u\|_X. \end{aligned}$$

Thus (VIII.2.24) is established.

From (VIII.2.24) we conclude for any fixed $u \in D(A)$, using also Proposition VIII.1.12, that $(T_\lambda(t)u)_{\lambda>0}$ is a Cauchy-sequence as $\lambda \to 0$ in X w.r.t λ, and this Cauchy sequence is uniform in $t \in [0,T]$ for any fixed $T < \infty$.

Hence the proposed limit $T(t)u := \lim_{\lambda\to 0} T_\lambda(t)u$ exists for any $u \in D(A)$, and Precisely,

$$\|T_\lambda(t)u - T(t)u\|_X \leq t\|(A_\lambda - A)u\|_X \quad \forall u \in D(A).$$

Clearly, $T(t)$ a linear operator, and from the above estimate together with (VIII.2.20) we find

$$\begin{aligned} \|T(t)u\|_X \leq& \|T_\lambda(t)u - T(t)u\|_X + \|T_\lambda(t)u\|_X \\ \leq& \|T_\lambda(t)u - T(t)u\|_X + \|u\|_X \\ \leq& t\|(A_\lambda - A)u\|_X + \|u\|_X. \end{aligned}$$

Letting $\lambda \to 0$, we find

$$\|T(t)u\|_X \leq \|u\|_X \quad \forall u \in D(A).$$

That is $T(t) : D(A) \to X$ is a linear bounded operator; since $D(A)$ is a dense set in X, $T(t)$ can be extended to a linear bounded operator on all of X.

Now let $u \in X$ with approximating sequence $u_n \in D(A)$.

$$\begin{aligned} \|T_\lambda(t)u - T(t)u\| &\le \|T_\lambda(t)u - T_\lambda(t)u_n\| + \|T_\lambda(t)u_n - T(t)u_n\| \\ &\quad + \|T(t)(u_n - u)\| \\ &\le 2\|u_n - u\| + \|T_\lambda(t)u_n - T(t)u_n\| \\ &\le 2\|u_n - u\| + t\|(A_\lambda - A)u_n\|_X \end{aligned} \tag{VIII.2.25}$$

Hence by (VIII.2.24) we find $T_\lambda(t)u \to T(t)u$ for all $u \in X$, and indeed the above estimate implies even uniformity in t: namely, for any $t_0 > 0$,

$$\sup_{t\in[0,t_0)} \|T_\lambda(t)u - T(t)u\| \xrightarrow{\lambda\to 0} 0 \quad \forall u \in X. \tag{VIII.2.26}$$

Furthermore we have the semigroup property

$$\begin{aligned} \|T(t)T(s)u - T(t+s)u\| &\le \|T(t)T(s)u - T(t)T_\lambda(s)u\| \\ &\quad + \|T(t)T_\lambda(s)u - T_\lambda(t)T_\lambda(s)u\| \\ &\quad + \|T_\lambda(t+s)u - T(t+s)u\| \\ &\to 0. \end{aligned} \tag{VIII.2.27}$$

As for the C^0-continuity, we have

$$\|T(t)u - u\|_X \le \|T_\lambda(t)u - u\|_X + \|T_\lambda(t)u - T(t)u\|_X.$$

By (VIII.2.26), for any $\varepsilon > 0$ and any $t_0 > 0$ there exists $\lambda > 0$ such that

$$\sup_{t\in t_0} \|T_\lambda(t)u - T(t)u\|_X < \frac{1}{2}\varepsilon.$$

On the other hand for this fixed $\lambda > 0$ there exists $t_1 \in (0, t_0)$ such that

$$\sup_{t<t_1} \|T_\lambda(t)u - u\|_X < \frac{\varepsilon}{2}.$$

In particular,

$$\sup_{t<t_1} \|T(t)u - u\|_X \le \varepsilon.$$

That is $T(t)$ is a contractive C^0-semigroup.

Next we show that <u>$T(t)$ maps $D(A)$ to $D(A)$</u>. First we observe that T_λ maps $D(A)$ to $D(A)$. Indeed, since we can write $e^{J_\lambda} = I + J_\lambda B$ (with B convergent since $J_\lambda \in L(X)$) and since J_λ maps X into $D(A)$ we have that e^{J_λ} maps $D(A)$ into $D(A)$. Moreover, by (VIII.2.18) we have $T_\lambda(t) = e^{tA_\lambda} = e^{t\frac{1}{\lambda}(J_\lambda - I)} = e^{-\frac{t}{\lambda}}e^{J_\lambda}$, so $T_\lambda(t)$ maps $D(A)$ into $D(A)$. Now let $u \in D(A)$, then $T_\lambda(t)u \xrightarrow{\lambda\to 0} T(t)u$, and $AT_\lambda(t)u = T_\lambda(t)Au \to T(t)Au$. By Proposition VIII.1.9, A is a closed operator, which implies that $T(t)u \in D(A)$ and $\lim_{\lambda\to 0} AT_\lambda(t)u = AT(t)u$, and in particular,

$$AT(t)u = T(t)Au \quad \forall u \in D(A). \tag{VIII.2.28}$$

Step (2): On the equation (VIII.2.23)

Let $u \in D(A)$ and set

$$u_\lambda(t) = e^{tA_\lambda} u. \tag{VIII.2.29}$$

Then

$$\frac{d}{dt} u_\lambda = e^{tA_\lambda} A_\lambda u = T_\lambda(t) A_\lambda u. \tag{VIII.2.30}$$

Equivalently, for $u \in D(A)$ using $A_\lambda u \to Au$ and $T_\lambda \to T$,

$$u(t) \leftarrow u_\lambda(t) = u + \int_0^t T_\lambda(s) A_\lambda u \, ds \to u + \int_0^t T(s) \, Au \, ds. \tag{VIII.2.31}$$

Thus $u(\cdot) \in C^1$ and (cf. (VIII.2.28))

$$\dot{u}(t) = T(t)Au = Au(t). \tag{VIII.2.32}$$

Uniqueness proceeds as in Theorem VIII.1.2. □

VIII.2.1. Generators of semigroups. Let $T(t)$ be a contraction semigroup. Define

$$D(A) := \left\{ u \in X : \lim_{h \to 0^+} \frac{T(h)u - u}{h} \text{ exists} \right\}. \tag{VIII.2.33}$$

For $u \in D(A)$ set

$$Au = \lim_{h \to 0^+} \frac{T(h)u - u}{h}. \tag{VIII.2.34}$$

Example VIII.2.7. $X = C_{ub}(\mathbb{R})$ be the set of uniformly continuous, bounded functions with the A^∞-norm.

$$T(t)u(x) := u(x + t). \tag{VIII.2.35}$$

Then $T(t)$ is a contraction semigroup. Then

$$Au = u', \quad D(A) = \{u, u' \in C_{ub}(\mathbb{R})\}. \tag{VIII.2.36}$$

Proof. It is clear that $u, u' \in C_{ub}(\mathbb{R})$ implies

$$\left\| \frac{u(x+h) - u(x)}{h} - u'(x) \right\|_\infty \to 0. \tag{VIII.2.37}$$

Now let $u \in D(A)$, then $u'_+ \in C_{ub}(\mathbb{R})$ and hence $u'_+ = u' \in C_{ub}(\mathbb{R})$. □

Theorem VIII.2.8 (Hille Yoshida Part II). *Let $T(t)$ be a contraction semigroup with generator A. Then A is m-dissipative and densely defined.*

PROOF. (i) A is dissipative, i.e. for all $\lambda > 0$, $\|u - \lambda Au\| \geq 0$. Indeed,

$$\begin{aligned} \left\| u - \lambda \frac{T(h)u - u}{h} \right\| &\geq \left\| \left(1 + \frac{\lambda}{h}\right) u \right\| - \left\| \frac{\lambda}{h} T(h)u \right\| \\ &= \left(1 + \frac{\lambda}{h}\right) \|u\| - \frac{\lambda}{h} \|T(h)u\| \\ &\geq \left(1 + \frac{\lambda}{h}\|u\| - \frac{\lambda}{h}\|u\|\right) = \|u\|. \end{aligned} \tag{VIII.2.38}$$

In the last step we used that $T(h)$ is contracting, i.e. $\|T(h)u\|_X \leq \|u\|_X$. Letting $h \to 0$ on the left hand side shows A is dissipative.

(ii) A is m-dissipative. It suffices to show that $(I - A)$ is surjective. Thus we want to find Ju, such that

$$(I - A)Ju = u. \tag{VIII.2.39}$$

Ansatz:

$$Ju = \int_0^\infty e^{-t} T(t)\ dt. \tag{VIII.2.40}$$

Why? Because (formally!) we know

$$(I - A)T(t)u = T(t)u - \partial_t(T(t)u)$$

This is equivalent to

$$(I - A)e^{-t}T(t)u = -\partial_t(e^{-t}T(t)u)$$

Integrating on both sides on $t \in (0, \infty)$ we then should get

$$(I - A)Ju = T(0)u = u.$$

To make this more precise, first observe

$$\|Ju\| \leq \int_0^\infty e^{-t} \|T(t)u\|\ dt \leq \|u\| \tag{VIII.2.41}$$

and hence $\|J\| \leq 1$. Now let us compute (with the semigroup property $T(h)T(t) = T(h+t)$ for $h, t > 0$),

$$\begin{aligned} (T(h) - I)\, Ju &= \int_0^\infty e^{-t} T(t+h)u\ dt - \int_0^\infty e^{-t} T(t)u\ dt \\ &= \int_h^\infty e^{-t+h} T(t)u\ dt - \int_0^\infty e^{-t} T(t)u\ dt \\ &= \int_0^\infty \left(e^{-t+h} - e^{-t}\right) T(t)u - \int_0^h e^{-t+h} T(t)u\ dt \\ &= (e^h - 1) \int_0^\infty e^{-t} T(t)u\ dt - e^h \int_0^h e^{-t} T(t)u\ dt \\ &= (e^h - 1)Ju - e^h \int_0^h e^{-t} T(t)u\ dt. \end{aligned} \tag{VIII.2.42}$$

Hence

$$\frac{T(h)-I}{h}Ju = \frac{e^h-1}{h}Ju - \frac{e^h}{h}\int_0^h e^{-t}T(t)u\,dt. \tag{VIII.2.43}$$

The right-hand side converges as $h \to 0$ (we use that T is continuous at 0), consequently so does the left-hand side. Thus $Ju \in D(A)$ and

$$AJu = Ju - u, \tag{VIII.2.44}$$

which is the claim.

(iii) $D(A)$ is dense. Let $u \in X$, then we claim that the following $u_h \in D(A)$ and $u_h \xrightarrow{h\to 0} u$ in X. Namely,

$$u_h := \frac{1}{h}\int_0^h T(s)u\,ds. \tag{VIII.2.45}$$

First we show that $u_h \xrightarrow{h\to 0} u$ holds. Indeed, we use again that T is continuous at 0,

$$\begin{aligned} \|u_h - u\| &= \left\| \frac{1}{h}\int_0^h (T(s)-I)\,u\,ds \right\| \\ &\le \frac{1}{h}\int_0^h \|(T(s)-I)\,u\| \xrightarrow{h\to 0} 0. \end{aligned} \tag{VIII.2.46}$$

Now show that $u_h \in D(A)$ for all $h > 0$. Let $t << h$. We calculate

$$\begin{aligned} \frac{T(t)-I}{t}u_h &= \frac{1}{ht}\int_0^h T(t)\circ T(s)u\,ds - \frac{1}{ht}\int_0^h T(s)u\,ds \\ &= \frac{1}{ht}\int_t^{t+h} T(s)u\,ds - \frac{1}{ht}\int_0^h T(s)u\,ds \\ &= \frac{1}{ht}\int_h^{t+h} T(s)u\,ds + \frac{1}{ht}\int_t^h T(s)u\,ds \\ &\quad - \frac{1}{ht}\int_0^t T(s)u\,ds - \frac{1}{ht}\int_t^h T(s)u\,ds \\ &\xrightarrow{t\to 0} \frac{1}{h}T(h)u - \frac{1}{h}T(0)u \in X. \end{aligned} \tag{VIII.2.47}$$

Hence the left hand side converges in X, that is $u_h \in D(A)$. □

VIII.3. An example application of Hille-Yoshida

Example VIII.3.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. We want to find solutions to the equation

$$\begin{cases} (\partial_t - L)u = 0 & \text{in } \Omega \times (0,\infty) \\ u = 0 & \text{in } (\partial\Omega)\times(0,\infty) \\ u = u_0 & \text{in } \Omega \times \{0\} \end{cases} \tag{VIII.3.1}$$

where the operator L given as

$$Lu(x) := \partial_i(a_{ij}(x)\partial_j u) + b_i(x)\partial_i u(x) + c(x)u(x)$$

has smooth (*time-independent*) coefficients $a, b, c \in C^\infty(\overline{\Omega})$, and is uniformly elliptic, i.e. for some $\lambda > 0$

$$a_{ij}(x)\xi_i\,\xi_j \geq \lambda|\xi|^2 \quad \forall x \in \Omega, \xi \in \mathbb{R}^n.$$

We claim that there exists a solution to (VIII.3.1) (in the sense of a semigroup) such that

$$\|u(t)\|_{L^2(\Omega)} \leq e^{ct}\|u_0\|_{L^2(\Omega)} \quad \forall t \in (1, \infty).$$

Proof. To see this let $X = L^2(\Omega)$, $D(A) := H^2(\Omega) \cap H^1_0(\Omega)$, and $A := L - \omega$ for an $\omega \geq 0$ yet to be chosen. Observe that if we show that A is m-dissipative then by Hille-Yoshida, Theorem VIII.2.6, we find a contraction semigroup $T(t)$ such that for $\tilde{u}(t) := T(t)u_0$

$$\partial_t(\tilde{u}(t)) = A\tilde{u}(t)$$

with the contractive property

$$\|\tilde{u}(t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)}.$$

Setting $u(t) := e^{\omega t}\tilde{u}(t)$ then $u(0) = \tilde{u}(0) = u_0$ and the requested equation is satisfied

$$\partial_t(u(t)) = \omega e^{\omega t}\tilde{u}(t) + Ae^{\omega t}\tilde{u}(t) = Lu(t).$$

Moreover, we obtain

$$\|u(t)\|_{L^2(\Omega)} \leq e^{\omega t}\|u_0\|_{L^2(\Omega)}.$$

That is, we <u>need to show that A is m-dissipative</u> (clearly $D(A)$ is dense in L^2). □

First we would like to show that A is dissipative.

Lemma VIII.3.2. *There exist $\omega_0 > 0$ such that for any $\omega \geq \omega_0$, A as above is dissipative.*

Proof. In view of Lemma VIII.1.5 we need to show that

$$\langle Au, u\rangle_{L^2(\Omega)} \leq 0 \quad \forall u \in D(A).$$

Here is where we need to make use of ω:

$$\langle Au, u\rangle_{L^2(\Omega)} \leq -\int_{\mathbb{R}^n} a_{ij}\partial_i u\partial_j u + C(b)\left(\|\nabla u\|_{L^2}\|u\|_{L^2} + \|u\|^2_{L^2}\right) - \omega\|u\|_{L^2(\Omega)}$$

By ellipticity,

$$-\int_{\mathbb{R}^n} a_{ij}\partial_i u\partial_j u \leq -\lambda\|\nabla u\|_{L^2(\Omega)}$$

Moroever by Young's inequality, for any $\varepsilon > 0$,

$$\|\nabla u\|_{L^2}\|u\|_{L^2} \leq \varepsilon\|\nabla u\|^2_{L^2(\Omega)} + C\frac{1}{\varepsilon}\|u\|^2_{L^2(\Omega)},$$

so that for small enough $\varepsilon > 0$ we arrive at

$$\langle Au, u\rangle_{L^2(\Omega)} \leq -\frac{\lambda}{2}\|\nabla u\|^2_{L^2} + C\|u\|^2_{L^2} - \omega\|u\|_{L^2(\Omega)}$$

Choosing $\omega := C$ we conclude

$$\langle Au, u\rangle_{L^2(\Omega)} \leq -\frac{\lambda}{2}\|\nabla u\|^2_{L^2} \leq 0. \tag{VIII.3.2}$$

that is, by Lemma VIII.1.5, A is dissipative. □

The next step is to prove that A is actually m-dissipative.

In view of Lemma VIII.1.8 it suffices to show that $I - \lambda_0 A$ is surjective for some $\lambda_0 > 0$.

That is, for some fixed $\lambda_0 > 0$ and $\omega > \omega_0$ we need to show that for any $f \in L^2(\Omega)$ there exists $u \in H^1_0(\Omega) \cap H^2(\Omega)$ with $(I - \lambda_0 A)u = f$, that is we need to find u such that

$$\partial_i(a_{ij}\partial_j u) = -\frac{1}{\lambda_0} f + (\frac{1}{\lambda_0} + \omega)u - b_j\partial_j u - cu \tag{VIII.3.3}$$

The $\partial_i(a_{ij}\partial_j u)$ is the leading order term, the remaining terms are lower order terms that can be dealt with by a fixed point argument. First we treat the leading order term:

Lemma VIII.3.3. *Let $g \in L^2(\Omega)$[3] then there exists exactly one $u \in H^1_0(\Omega)$ with*

$$\partial_i(a_{ij}\partial_j u) = g \tag{VIII.3.4}$$

in distributional sense. Moreover,

$$\|u\|_{H^1(\Omega)} \leq C\,\|g\|_{L^2(\Omega)} \tag{VIII.3.5}$$

Proof. This can be done, e.g. by a variational argument, the direct method (cf. Theorem IV.0.2)

Let $\mathcal{E} : H^1_0(\Omega) \to \mathbb{R}$ be an energy defined as

$$\mathcal{E}(u) := \frac{1}{2}\int_\Omega a_{ij}\,\partial_i u\,\partial_j u + \int_\Omega g\,u.$$

- Assume that $\mathcal{E}(\cdot)$ has a minimizer $u \in H^1_0(\Omega)$, i.e. $\mathcal{E}(u) \leq \mathcal{E}(v)$ for all $v \in H^1_0(\Omega)$. Then $\mathcal{E}(u) \leq \mathcal{E}(u + t\varphi)$ for any $\varphi \in C^\infty_c(\Omega)$, $t \in \mathbb{R}$. Thus $t \mapsto \mathcal{E}(u + t\varphi)$ as a minimum at $t = 0$, and consequently, by Fermat's theorem (a_{ij} is symmetric!)

  $$0 = \frac{d}{dt}\Big|_{t=0} \mathcal{E}(u + t\varphi) = \int_\Omega a_{ij}\,\partial_i u\,\partial_j\varphi + \int_\Omega g\,\varphi.$$

  This holds for any $\varphi \in C^\infty_c(\Omega)$; That is, (VIII.3.4) is satisfied (in distributional sense). Indeed, we say that (VIII.3.4) is the *Euler-Lagrange equation* of the energy $\mathcal{E}$.
- $\mathcal{E}(\cdot)$ is coercive (it controls $\|\nabla u\|_{L^2(\Omega)}$.

  $$\mathcal{E}(u) \geq \lambda\|\nabla u\|^2_{L^2(\Omega)} - \|g\|_{L^2(\Omega)}\,\|u\|_{L^2(\Omega)}.$$

[3] $g \in H^{-1}(\Omega)$ suffices

By Poincare inequality, and Youngs inequality (C changes from line to line)

$$\mathcal{E}(u) \geq \lambda \|\nabla u\|^2_{L^2(\Omega)} - C\|g\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} \geq \lambda \|\nabla u\|^2_{L^2(\Omega)} - C\|g\|^2_{L^2(\Omega)} - \frac{\lambda}{2}\|\nabla u\|^2_{L^2(\Omega)}.$$

That is,

$$\frac{\lambda}{2}\|\nabla u\|^2_{L^2(\Omega)} \leq \mathcal{E}(u) + C\|g\|^2_{L^2}.$$

Again, by Poincaré inequality, this implies

$$\|u\|^2_{H^1(\Omega)} \lesssim \mathcal{E}(u) + C\|g\|^2_{L^2}. \tag{VIII.3.6}$$

- Now let $u_k \in H^1_0(\Omega)$ be a sequence such that $\mathcal{E}(u_k) \xrightarrow{k\to\infty} \inf_{H^1_0(\Omega)} \mathcal{E}$. Since $0 \in H^1_0(\Omega)$ we may assume, w.l.o.g., that $\mathcal{E}(u_k) \leq \mathcal{E}(0) = 0$. By coercivity, (VIII.3.6),

$$\|u_k\|^2_{H^1(\Omega)} \lesssim \mathcal{E}(u_k) + C\|g\|^2_{L^2} \leq 0 + C\|g\|^2_{L^2},$$

that is

$$\sup_{k\in\mathbb{N}} \|u_k\|^2_{H^1(\Omega)} < \infty.$$

By weak compactness, Rellich, Theorem III.3.18 and Theorem III.3.24 we may assume w.l.o.g. (otherwise taking a subsequence) that there is $u \in H^1_0(\Omega)$ and
  - $u_k \xrightarrow{k\to\infty} u$ strongly in $L^2(\Omega)$ and a.e.
  - ∇u_k weakly converges to ∇u in $L^2(\Omega)$, that is for any $G \in L^2(\Omega, \mathbb{R}^n)$,

$$\int_\Omega \nabla u_k G \xrightarrow{k\to\infty} \int_\Omega \nabla u G.$$

Consequently,

$$\int_\Omega g u_k \xrightarrow{k\to\infty} \int_\Omega g u.$$

Moreover,

$$\begin{aligned}\int_\Omega a_{ij}\partial_i u_k \partial_j u_k &= \int_\Omega a_{ij}\partial_i u_k \partial_j u + \int_\Omega a_{ij}\partial_i u_k \partial_j (u_k - u) \\ &= \int_\Omega a_{ij}\partial_i u_k \partial_j u + \underbrace{\int_\Omega a_{ij}\partial_i u \partial_j (u_k - u)}_{\xrightarrow{k\to\infty} 0} + \underbrace{\int_\Omega a_{ij}\partial_i (u_k - u)\partial_j (u_k - u)}_{\geq 0}\end{aligned}$$

so

$$\liminf_{k\to\infty} \int_\Omega a_{ij}\partial_i u_k \partial_j u_k \geq \liminf_{k\to\infty} \int_\Omega a_{ij}\partial_i u_k \partial_j u = \int_\Omega a_{ij}\partial_i u \partial_j u \tag{VIII.3.7}$$

Together we have shown

$$\liminf_{k\to\infty} \mathcal{E}(u_k) \geq \mathcal{E}(u).$$

But since $u \in H^1_0(\Omega)$ we have that

$$\mathcal{E}(u) \geq \inf_{H^1_0(\Omega)} \mathcal{E},$$

and by construction

$$\lim_{k\to\infty} \mathcal{E}(u_k) = \inf_{H_0^1(\Omega)} \mathcal{E}$$

So (VIII.3.7) implies that

$$\mathcal{E}(u) = \inf_{H_0^1(\Omega)} \mathcal{E},$$

that is u minimizes $\mathcal{E}$ in $H_0^1(\Omega)$.

- we have found a minimizer of $\mathcal{E}$ and this minimizer satisfies the equation.

We still need to show uniqueness and the estimate (VIII.3.5) of the solution. Observe that uniqueness follows once we show that (VIII.3.5) holds for any solution $u \in H_0^1(\Omega)$ of (VIII.3.4). Assume this is true, then if we have two solutions u and v for the same g, then $w := u - v \in H_0^1(\Omega)$ satisfies

$$\partial_i(a_{ij}\partial_j w) = 0.$$

By (VIII.3.5) we have $\|w\|_{H^1} = 0$ that is $w \equiv 0$.

It remains to show (VIII.3.5) for an arbitrary solution $u \in H_0^1(\Omega)$ of (VIII.3.4). This follwos from multiplying (VIII.3.4) by u and integrating. By ellipticity we then find,

$$\|\nabla u\|^2_{L^2(\Omega)} \lesssim \int_\Omega a_{ij}\partial_i u\, \partial_j u = |\int_\Omega gu| \lesssim \|g\|_{L^2(\Omega)}\, \|u\|_{L^2(\Omega)}.$$

By Poincare on the left-hand side we find

$$\|u\|^2_{H^1(\Omega)} \leq |g\|_{L^2(\Omega)}\, \|u\|_{H^1(\Omega)}.$$

Dividing both sides by $\|u\|_{H^1(\Omega)}$ we have established (VIII.3.4). □

Now we need to take care of the lower order guys in (VIII.3.3), and for this we use a fixed point argument. Let f be fixed and for $u \in H_0^1(\Omega)$ set $Tu \in H_0^1(\Omega)$ be the solution of

$$\partial_i(a_{ij}\partial_j(T_f u)) = -\frac{1}{\lambda_0} f + (\frac{1}{\lambda_0} + \omega)u - b_j\partial_j u - cu. \tag{VIII.3.8}$$

By Lemma VIII.3.3 is well-defined, linear, and bounded (observe if $u \in H^1$ then the right-hand side of the equation above belongs to $L^2(\Omega)$). But more is true: for any $f \in L^2(\Omega)$, T_f is compact, which will be a consequence of the following estimate

Lemma VIII.3.4. *For $f \in L^2(\Omega)$ let $T_f : H_0^1(\Omega) \to H_0^1(\Omega)$ be defined as above. Then*

$$\|T_f(u)\|_{H^1(\Omega)} \leq C\left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right)$$

and

$$\|T_f(u) - T_f(v)\|_{H^1(\Omega)} \leq C\, \|u - v\|_{L^2(\Omega)}$$

Proof. Testing (VIII.3.8) we have by ellipticity and Poincaré inequality

$$\|T_f(u)\|^2_{H^1(\Omega)} \lesssim C\left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right)\, \|T_f(u)\|_{L^2(\Omega)} + \left|\int_\Omega b_j\partial_j u\, T_f(u)\right|$$

By another integration by parts, (recall: $b \in C^\infty$)

$$\left| \int_\Omega b_j \partial_j u \, T_f(u) \right| \leq C(b, \nabla b) \, \|u\|_{L^2(\Omega)} \|T_f(u)\|_{H^1(\Omega)}.$$

Consequently, we find

$$\|T_f(u)\|^2_{H^1(\Omega)} \lesssim C \left(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right) \|T_f(u)\|_{H^1(\Omega)},$$

and dividing both sides by $\|T_f(u)\|_{H^1(\Omega)}$ we find the first estimate.

The second estimate follows by the same argument: observe that $T_f(u) - T_f(v) = T_0(u-v)$. Then, by the above argument,

$$\|T_0(u-v)\|^2_{H^1(\Omega)} \lesssim C \, \|u-v\|_{L^2(\Omega)}$$

□

From Lemma VIII.3.4 we readily conclude that T_f is actually compact for any $f \in L^2(\Omega)$

Corollary VIII.3.5. *For $f \in L^2(\Omega)$, let T_f be defined as above. Then $T_f : H^1_0(\Omega) \to H^1_0(\Omega)$ is a continuous, compact operator. That is, for any bounded sequence $(u_k)_k \in H^1_0(\Omega)$,*

$$\sup_k \|u_k\|_{H^1(\Omega)} < \infty$$

there exists a subsequence u_{k_i} such that $(T_f u_{k_i})_i$ is convergent in $H^1_0(\Omega)$.

Proof. Clearly from Lemma VIII.3.4 we obtain that T_f is Lipschitz continuous.

Now let u_k be a bounded H^1_0-sequence. Up to a subsequence we can assume that u_k weakly converge to some $u \in H^1_0(\Omega)$ and by Rellich, Theorem III.3.24, the convergence is strong in $L^2(\Omega)$. By Lemma VIII.3.4,

$$\|T_f u_k - T_f u\|_{H^1(\Omega)} \lesssim \|u_k - u\|_{L^2(\Omega)} \xrightarrow{k \to 0} 0.$$

Thus, T_f is compact. □

Observe that this does not mean that T_f is contracting, so we cannot apply e.g. Banach Fixed Point argument. Instead we use the *Schauder fixed point theorem* in the form of *Schaefer's fixed point theorem* also known as the *Leray-Schauder theorem*. For a proof see [**Gilbarg and Trudinger, 2001**, Theorem 11.3].

Theorem VIII.3.6 (Leray-Schauder). *Let $T : X \to X$ be a continuous and compact mapping of a Banach space X into itself, such that the set*

$$\{x \in X : \quad x = \mu T x \text{ for some } 0 \leq \mu \leq 1\}$$

is bounded, i.e. there exists $M > 0$ such that whenever for some $\mu \in [0,1]$ and some $x \in X$ we have $x = \mu T x$ then

$$\|x\|_X \leq M.$$

Then T has a fixed point.

We can apply this theorem to the compact operator T essentially since we dissipativity:

Lemma VIII.3.7. *There exist $\omega_0 > 0$ such that for any $\omega > \omega_0$ and any $\lambda_0 > 0$ we have the following:*

Let $\mu \in [0,1]$ and assume that $u \in H_0^1(\Omega)$ satisfies

$$u = \mu T u$$

then with a constant independent of μ,

$$\|u\|_{H^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

Lemma VIII.3.8. *From $u = \mu T u$ we conclude in view of the definition of T, (VIII.3.8),*

$$-\partial_i(a_{ij}\partial_j u) = -\mu\left(-\frac{1}{\lambda_0}f + (\frac{1}{\lambda_0} + \omega)u - b_j\partial_j u - cu\right).$$

Similar to the proof of A being dissipativ, we test this equation with u, and obtain

$$\|\nabla u\|^2_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)} + \int_\Omega \left(c - \omega - \frac{1}{\lambda_0}\right)|u|^2 + \int_\Omega b_j\partial_j u\, u.$$

Observe that $\partial_j uu = \frac{1}{2}\partial_j|u|^2$ and integrating by parts we find

$$\|\nabla u\|^2_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)} - \left(\omega + \frac{1}{\lambda_0}\right)\int_\Omega |u|^2 + C(c,b)\,\|u\|^2_{L^2(\Omega)}$$

If ω is large enough, namely $\omega > C(c,b)$ this implies

$$\|\nabla u\|^2_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

By Poincaré inequality we obtain the claim.

No we can conclude:

A IS m-DISSIPATIVE. We argued above that A is m-dissipative if for any $f \in L^2(\Omega)$ there exists $u \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $Tu = u$. Since T is compact by Corollary VIII.3.5 and in view of Lemma VIII.3.7, Leray-Schauder in form of Theorem VIII.3.6 is applicable, and implies that there exists a fixed point $u \in H_0^1(\Omega)$ such that $Tu = u$.

Observe that this implies that u solves an equation of the form

$$\begin{cases} \partial_i(a_{ij}\partial_j u) = g & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

for $g \in L^2(\Omega)$.

Now we argue as in Theorem IV.0.1, to obtain interior regularity, indeed we find

$$\|u\|_{W^{2,2}_{loc}(\Omega)} \leq C(\Omega)\,\|g\|_{L^2(\Omega)}.$$

Close to the boundary one needs to do a reflection argument to obtain the same estimate up to the boundary.

Up to doing this, we have shown that there is $u \in D(A)$ and $(I - \lambda_0 A)u = f$;

That is, A is m-dissipative, and thus generates a contractive semigroup as claimed. □

Exercise VIII.3.9. *Find a solution to the inhomogeneous equation*

$$\begin{cases} \partial_t u(x,t) - Lu(x,t) = f(x,t) & \text{in } \Omega \times (0,\infty) \\ u(x,0) = u_0(x). \end{cases}$$

Here L is the usual elliptic operator with smooth coefficients.

Find an estimate of $\|u(t)\|_{L^2(\Omega)}$ in terms of $\|f\|_{L^2}, \|u_0\|_{L^2}$ and t.

hint: Use the Duhamel Ansatz from the heat equation in $\mathbb{R}^n$, i.e. set

$$u(t) = \int_0^t v_s(t)\, ds$$

where $v_s(t)$ is the solution associated to

$$\begin{cases} \partial_t v(x,t) - Lv(x,t) = 0 & \text{in } \Omega \times (s,\infty) \\ v(x,s) = f(x,s). \end{cases}$$

Observe that $v_s(t)$ can be written as semigroup!

VIII.4. Formal regularity theory (a priori estimates)

Here we show estimates that hold for the heat equation (and which can be easily extended to linear parabolic equations).

For simplicity we prove only *a priori* estimates, i.e. estimates under the assumption that u is already sufficiently regular. It is further work to show that these estimates hold also without this a priori regularity assumption (we did something similar for the Laplace equation, cf. Lemma I.2.17).

Cf. [**Evans, 2010**, 7.1.3].

Assume that $\Omega \subset\subset \mathbb{R}^n$ with smooth boundary, and we have

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \Omega \times (0,\infty) \\ u = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u(0,x) = u_0(x) & \text{on } \Omega \times \{0\} \end{cases}$$

We first test the equation with u, (i.e. we multiply with u and integrate by parts).

$$\int_\Omega \partial_t u \cdot u - \int_\Omega \Delta u \cdot u = \int_\Omega f u \tag{VIII.4.1}$$

Observe that

$$\partial_t u \cdot u = \frac{1}{2}\partial_t |u|^2,$$

$$\int_\Omega \Delta u \cdot u = -\int_\Omega |\nabla u|^2$$

and by Hölder, then Poincaré inequality and then Young's inequality,

$$\int_\Omega fu \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \leq C \leq \|f\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} \leq C(\varepsilon) \|f\|^2_{L^2(\Omega)} + \varepsilon \|Du\|^2_{L^2(\Omega)}$$

Consequently (VIII.4.1) becomes (for $\varepsilon = \frac{1}{2}$)

$$\partial_t \int_\Omega |u|^2 + \frac{1}{2} \int_\Omega |Du|^2 \leq C(\varepsilon) \int_\Omega |f|^2$$

We integrate in t from 0 to some $s > 0$ and find

$$\int_\Omega |u(s)|^2 - \int_\Omega |u_0|^2 + \frac{1}{2} \int_{\Omega\times(0,s)} |Du|^2 \leq C(\varepsilon) \int_{\Omega\times(0,s)} |f|^2$$

From which we obtain

$$\int_{\Omega\times(0,s)} |Du|^2 + \int_\Omega |u(s)|^2 \leq C \int_{\Omega\times(0,s)} |f|^2 + \int_\Omega |u_0|^2.$$

From this we conclude

Lemma VIII.4.1.

$$\int_{\Omega\times(0,T)} |Du|^2 + \sup_{t\in(0,T)} \int_\Omega |u(t)|^2 \leq C \int_{\Omega\times(0,T)} |f|^2 + \int_\Omega |u_0|^2.$$

That is, we get an $L^\infty_t(L^2_x)$-bound on u and an $L^2_t L^2_x$-bound on u. in terms of the L^2-bound of u_0 and the $L^2_t L^2_x$-bound of f. (These are the very simple versions of what is for Schrödinger equations referred as Strichartz estimates).

We obtain another estimate from testing with f, i.e. integrating

$$(\partial_t u - \Delta u)^2 = f^2 \tag{VIII.4.2}$$

Observe that

$$(\partial_t u - \Delta u)^2 = |\partial_t u|^2 + |\Delta u|^2 - 2\partial_t u \Delta u,$$

and (using that $\partial_t u = 0$ on $\partial\Omega$ since $u = 0$ on $\partial\Omega$ for all times)

$$\int_\Omega \partial_t u \Delta u = -\int_\Omega (\partial_t \nabla u) \cdot \nabla u = -\frac{1}{2} \partial_t \int_\Omega |\nabla u|^2$$

Thus, after integrating in x and t, (VIII.4.2) becomes

$$\int_{\Omega\times(0,s)} \left(|\partial_t u|^2 + |\Delta u|^2\right) + \int_\Omega |\nabla u(s)|^2 - \int_\Omega |\nabla u(0)| = \int_{\Omega\times(0,s)} f^2$$

which leads to

Lemma VIII.4.2.

$$\int_{\Omega\times(0,T)} \left(|\partial_t u|^2 + |\Delta u|^2\right) + \sup_{t\in(0,T)} \int_\Omega |\nabla u(t)|^2 \leq \int_{\Omega\times(0,s)} f^2 + \int_\Omega |\nabla u(0)|^2$$

As in the elliptic case, cf. Chapter 4: higher order estimates in t are obtained by differentiating the equation in t, i.e. considering that $\tilde{u} := \partial_t u$ is a solution to

$$\begin{cases} \partial_t \tilde{u} - \Delta \tilde{u} = \partial_t f & \text{in } \Omega \times (0,\infty) \\ \tilde{u} = 0 & \text{on } \partial\Omega \times (0,\infty) \\ \tilde{u}(0,x) = \partial_t u(0,x) = \Delta u_0 + f & \text{on } \Omega \times \{0\} \end{cases}$$

In this way we obtain from Lemma VIII.4.1

Lemma VIII.4.3.

$$\int_{\Omega\times(0,T)} |D\partial_t u|^2 + \sup_{t\in(0,T)} \int_\Omega |\partial_t u(t)|^2 \le C \int_{\Omega\times(0,T)} |\partial_t f|^2 + \int_\Omega |\Delta u_0|^2 + |f(0)|^2.$$

Remark VIII.4.4. Observe that these estimate do not work if the sign in front of Δ changes! Parabolic equations (in contrast to, e.g., Schrödinger equations) have control only in one time direction!

$$\begin{cases} \partial_t u + \Delta u = f \quad \text{in } \Omega \times (0,\infty) & \\ u = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u(0,x) = u_0(x) & \text{on } \Omega \times \{0\} \end{cases}$$

CHAPTER 9

Linear Hyperbolic equations: waves

IX.1. Halfwave-decomposition

The model case for hyperbolic equations is the wave equation

$$\begin{cases} \partial_{tt}u - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{in } \mathbb{R}^n \times \{0\} \\ \partial_t u = h & \text{on } \mathbb{R}^n \times \{0\} \end{cases}$$

For the elliptic (Laplace equation) and parabolic (heat equation) it was useful to use the Fourier transform to get a feeling of the equation.

Taking the Fourier transform in x, we find that

$$\partial_{tt}\hat{u}(\xi,t) + c|\xi|^2\hat{u}(\xi,t) = 0.$$

Solutions of this second order ODE are of the form

$$\hat{u}(\xi,t) = e^{it|\xi|}a(\xi) + e^{-it|\xi|}b(\xi).$$

where i is the complex unit! a and b have to satisfy

$$\begin{cases} a(\xi) + b(\xi) = \hat{g}(\xi) \\ i|\xi|a(\xi) - i|\xi|b(\xi) = \hat{h}(\xi) \end{cases}$$

that is

$$\begin{cases} a(\xi) + b(\xi) = \hat{g}(\xi) \\ -a(\xi) + b(\xi) = i|\xi|^{-1}\hat{h}(\xi) \end{cases}$$

which leads to

$$\begin{cases} a(\xi) = \frac{1}{2}\left(\hat{g}(\xi) - i|\xi|^{-1}\hat{h}(\xi)\right) \\ b(\xi) = \frac{1}{2}\left(\hat{g}(\xi) + i|\xi|^{-1}\hat{h}(\xi)\right) \end{cases}$$

In the semigroup approach we interpreted $e^{-t|\xi|^2}$ as the Fourier symbol of $e^{-t\Delta}$ (since $|\xi|^2$ is the Fourier symbol of Δ). Since $|\xi| = \sqrt{|\xi|^2}$ we denote the operator with Fourier symbol $|\xi|$ as *half-Laplacian* $(-\Delta)^{\frac{1}{2}}$ – and the solution $\hat{u}$ as

$$u = e^{it(-\Delta)^{\frac{1}{2}}}\frac{1}{2}\left(g - i(-\Delta)^{-\frac{1}{2}}h\right) + e^{-it(-\Delta)^{\frac{1}{2}}}\frac{1}{2}\left(g + i(-\Delta)^{-\frac{1}{2}}h\right)$$

If we call $u_- := e^{it(-\Delta)^{\frac{1}{2}}}\frac{1}{2}\left(g - i(-\Delta)^{-\frac{1}{2}}h\right)$ and $u_+ := e^{-it(-\Delta)^{\frac{1}{2}}}\frac{1}{2}\left(g + i(-\Delta)^{-\frac{1}{2}}h\right)$ then we observe that $u_\pm$ satisfies

$$\begin{cases} \partial_t u_\pm - i(-\Delta)^{\frac{1}{2}}u_\pm = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u_\pm = \frac{1}{2}\left(g \pm i(-\Delta)^{-\frac{1}{2}}h\right) & \text{in } \mathbb{R}^n \times \{0\} \end{cases}$$

Observe that the differential operator $\partial_t - i(-\Delta)^{\frac{1}{2}}$ looks formally similar to the Schrödinger equation $\partial_t + i\Delta$. This decomposition of the wave-equation is called *half-wave decomposition* of the wave equation.

IX.2. D'Alambert's formula in one dimension

In one dimension we consider

$$\begin{cases} \partial_{tt}u - \partial_{xx}u = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = g & \text{in } \mathbb{R} \times \{0\} \\ \partial_t u = h & \text{on } \mathbb{R} \times \{0\} \end{cases}$$

This equation can be rewritten as

$$(\partial_t + \partial_x)(\partial_t - \partial_x)u = 0.$$

So if we set

$$v(x,t) := (\partial_t - \partial_x)u, \tag{IX.2.1}$$

then v satisfies

$$v_t + \partial_x v = 0.$$

This is a transport equation, cf. (I.1.1), which has the solution

$$v(x,t) = v(x-t,0)$$

Observe that

$$v(z,0) = \partial_t u(z,0) - \partial_x u(z,0) = h(z) - g'(z).$$

Plugging this into (IX.2.1)

$$h(x-t) - g'(x-t) = (\partial_t - \partial_x)u,$$

i.e. an inhomogeneous transport equation. This, in turn, has the solution

$$\begin{aligned} u(x,t) =& \int_0^t v(x+(t-s)-s,0)ds + u(x+t,0) \\ =& \frac{1}{2}\int_{x-t}^{x+t} h(s) - g'(s)ds + g(x+t) \\ =& \frac{1}{2}\int_{x-t}^{x+t} h(s)ds + \frac{1}{2}(g(x+t) - g(x-t)) \end{aligned}$$

This formula is called the *D'Alambert formula*.

As we have seen for the transport equation, this solution is not smoothing: If $g \in C^k$, $h \in C^{k-1}$ then u might as well only be in C^k as well for all $t > 0$ (this is contrast to the heat equation which is immediately smooth).

Another observation is finite speed of propagation: $u(x,t)$ takes into account h and g only in a neighborhood of x – for the heat equation if we change the boundary data in one point it changes everywhere. This is *finite speed of propagation* vs *infinite speed of propagation*. In particular no such thing as a maximum principle will hold.

In higher dimensions, a solution representation can be computed with a method called spherical means, see [**Evans, 2010**, 2.4.1].

IX.3. Second-order hyperbolic equations

Let L be the usual elliptic operator, then the operator $\partial_{tt} - L$ is called *hyperbolic* (the model case being the wave equation).

We need to define weak solutions of $(\partial_{tt} - L)u$.

As usual we would like to have $u(\cdot, t) \in H^1(\Omega)$ for any fixed time t. Since $\partial_{tt}u = Lu$ is in general not a function, but a distribution, which we have to test with a H^1-Sobolev function. This is denoted as $H^{-1}(\Omega)$.

Definition IX.3.1. A distribution f belongs to $H^{-1}(\Omega)$ if for all $\varphi \in H^1(\Omega)$,

$$|f[\varphi]| \leq \Lambda \|\varphi\|_{H^1(\Omega)}$$

The minimal value of Λ is denoted by $\|f\|_{H^{-1}(\Omega)}$.

We have $H^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$.

Definition IX.3.2. A function

$$u \in L^2(0,T; H_0^1(\Omega))$$
$$u' \in L^2(0,T; L^2(\Omega))$$
$$u'' \in L^2(0,T; H^{-1}(\Omega))$$

[1] is a weak solution to

$$\begin{cases} (\partial_{tt} - L)u = f & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u = g,\ \partial_t u = h & \text{on } \Omega \times \{0\} \end{cases} \tag{IX.3.1}$$

if for almost every $t \in (0,T)$ and any $\varphi \in C_c^\infty(\Omega)$

$$\partial_{tt}u(t)[\varphi] - \int_\Omega Lu(t)\,\varphi = \int_\Omega f(t)\varphi$$

[1] in particular $u \in H^2(0,T; H^{-1}(\Omega))$ and as such u and u' are continuous in t as mappings into L^2 or H^{-1}, respectively.

and if the boundary data is attained in the trace sense.

While there is a method based on semigroups to show existence, here we use another argument (that could be used also for parabolic equations), that is less algebraic and more numerical:

IX.4. Existence via Galerkin approximation

The basic idea of Galerkin method is to approximate the infinite dimensional problem $(\partial_{tt}-L)u = f$ (plus boundary data) by finite dimensional problems: we project the problem into finite dimensional subspaces (of $H^1_0(\Omega)$), solve it there, and pray for convergence as the dimension of the subspaces goes to infinity.

So assume[2] that we have $(w_k)_{k=1}^\infty \subset H^1_0(\Omega)$ that is an basis of $H^1_0(\Omega)$ and of $L^2(\Omega)$ such that its orthogonal in $H^1_0(\Omega)$ and orthonormal in $L^2(\Omega)$, i.e.

$$\int \nabla w_k \cdot \nabla w_\ell = \int w_k \cdot w_\ell = 0 \quad k \neq \ell,$$

and $\|w_k\|_{L^2} = 1$.

For $m \in \mathbb{N}$ we "project the equation" onto $\operatorname{span}\{w_1, \dots, w_m\}$, i.e. we set the "approximating solution"

$$u_m(t) := \sum_{k=1}^m d_m^k(t)\, w_k$$

[2]One can take an eigenvalue basis for the Laplacian. One can show, [**Evans, 2010**, 6.5.1], that there exist an infinite sequence of eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \dots$ with $\lambda_k \xrightarrow{k\to\infty} \infty$ such the solutions $w_k \in H^1_0(\Omega)$

$$\begin{cases} -\Delta w_k = \lambda_k w_k & \text{in } \Omega \\ w_k = 0 & \text{on } \partial\Omega \end{cases}$$

form an orthonormal basis of $L^2(\Omega)$, i.e. $\|w_k\|_{L^2(\Omega)} = 1$, $\int_\Omega w_k w_j = 0$ whenever $k \neq j$, and for any $f \in L^2(\Omega)$ there is a unique sequence $\mu_k \in \ell^2$ such that

$$f = \sum_{k=1}^\infty \mu_k w_k$$

(with convergence in $L^2(\Omega)$ if $f \in L^2(\Omega)$ and convergence in $H^1_0(\Omega)$ if $f \in H^1_0(\Omega)$).

Observe that then also (integrating by parts)

$$\int_\Omega \nabla w_k \nabla w_j = \begin{cases} 0 & k \neq j \\ \mu_k & k = j \end{cases}$$

in particular w_k forms an orthonormal basis of $H^1_0(\Omega)$ with respect to the scalar product

$$\langle f, g\rangle_{H^1(\Omega)} = \int \nabla f \cdot \nabla g.$$

where the coefficients $d_m^k(t)$ have to be chosen such that for all $t \in (0,T)$ and all $k = 1,\dots,m$,

$$\int_\Omega \partial_{tt} u_m w_k - L u_m w_k = \int_\Omega f w_k. \tag{IX.4.1}$$

with initial value for $k = 1,\dots,m$

$$d_m^k(0) = \int g w_k, \quad \partial_t d_m^k(0) = \int h w_k.$$

Observe that by orthonormality,

$$\int_\Omega \partial_{tt} u_m w_k = \partial_{tt} d_m^k(t).$$

Since L is linear and the derivatives are spacial only, we can write

$$\int_\Omega L u_m w_k = \sum_{\ell=1}^m e_{k\ell}(t)\, d_m^\ell(t).$$

for some coefficients $e_{k\ell}(t)$. That is, setting $f_k := \int_\Omega f w_k$ (IX.4.1) becomes a linear second order initial value ODE,

$$\partial_{tt} d_m^k(t) + \sum_{\ell=1}^m e_{k\ell}(t)\, d_m^\ell(t) = f_k(t). \tag{IX.4.2}$$

ODE theory tells us there is a unique C^2-solution to this equation.

The hope is to find a weak solution u to (IX.3.1) as limit $m \to \infty$ of u_m, and for this we need uniform estimates. First we use

Proposition IX.4.1.

$$\begin{aligned}&\sup_m \left(\sup_{t\in(0,T)} \left(\|u_m(t)\|_{H_0^1(\Omega)} + \|\partial_t u_m(t)\|_{H_0^1(\Omega)} \right) + \|\partial_{tt} u_m\|_{L^2(0,T;H^{-1}(\Omega)} \right) \\ \le & C \left(\|f\|_{L^2(0,T;L^2(\Omega)} + \|g\|_{H_0^1(\Omega)} + \|h\|_{L^2(\Omega)} \right).\end{aligned}$$

Here $C = C(T,\Omega,L)$.

An important ingredient is

Lemma IX.4.2 (Grönwall's inequality)**.** *Let $f : [a,b] \to \mathbb{R}$ be differentiable on (a,b) and continuous on $[a,b]$. If f satisfies the differential inequality*

$$f'(t) \le \alpha(t)\, f(t) \quad \forall t \in (a,b),$$

for some integrable function α, then for any $t \in (a,b)$,

$$f(t) \le f(a) \exp\left(\int_a^t \alpha(s) ds \right).$$

whenever the right-hand side makes sense.

Proof of Lemma IX.4.2. Set $g(t) := \exp(\int_a^t \alpha(s)ds)$.

Then

$$g'(t) = \alpha(t)\, g(t).$$

Observe moreover that $g(0) = 1$ and $g \geq 0$. Then for

$$h(t) := \frac{f(t)}{g(t)}$$

we have

$$h'(t) = \frac{f'(t)\, g(t) - f(t)g'(t)}{(g(t))^2} \leq \frac{\alpha(t)\, f(t)g(t) - f(t)\alpha(t)g(t)}{(g(t))^2} = 0.$$

That is, h is monotonically decreasing, so

$$h(t) \leq h(a) \quad \forall t \in (a, t),$$

that is

$$\frac{f(t)}{g(t)} \leq f(a),$$

which gives the claim. □

Proof of Proposition IX.4.1. Multiply (IX.4.1) with $\partial_t d_k^m$ and sum over k. Then

$$\int_\Omega \partial_{tt} u_m \partial_t u_m - L u_m \partial_t u_m = \int_\Omega f \partial_t u_m. \tag{IX.4.3}$$

We observe

$$\int_\Omega \partial_{tt} u_m \partial_t u_m = \frac{1}{2}\partial_t \|\partial_t u_m\|^2_{L^2(\Omega)}.$$

Also

$$\begin{aligned}&\int_\Omega a_{ij}(x,t)\, \partial_{x_i} u_m\, \partial_{x_j} \partial_t u_m \\ =&\frac{1}{2}\partial_t \left(\int_\Omega a_{ij}(x,t)\, \partial_{x_i} u_m\, \partial_{x_j} u_m\right) - \frac{1}{2}\int_\Omega \partial_t a_{ij}(x,t)\, \partial_{x_i} u_m\, \partial_{x_j} u_m\end{aligned}$$

So from (IX.4.3) we obtain

$$\begin{aligned}&\partial_t \left(\|\partial_t u_m\|^2_{L^2(\Omega)} + \frac{1}{2}\left(\int_\Omega a_{ij}(x,t)\, \partial_{x_i} u_m\, \partial_{x_j} u_m\right)\right) \\ \leq& C\left(\|\partial_t u_m\|^2_{L^2(\Omega)} + \|u_m\|^2_{H^1_0(\Omega)} + \|f\|^2_{L^2(\Omega)}\right).\end{aligned}$$

Since by ellipticity

$$\|u_m\|^2_{H^1_0(\Omega)} \lesssim \int_\Omega a_{ij}(x,t)\, \partial_{x_i} u_m\, \partial_{x_j} u_m$$

we can set

$$\eta(t) := \|\partial_t u_m\|^2_{L^2(\Omega)} + \frac{1}{2}\left(\int_\Omega a_{ij}(x,t)\, \partial_{x_i} u_m\, \partial_{x_j} u_m\right)$$

and have for all $t \in (0, T)$

$$\eta'(t) \leq C\, \left(\eta(t) + \|f(t)\|^2_{L^2(\Omega)}\right)$$

This is an ODE inequality, and *Grönwall's inequality*, Lemma IX.4.2,[3] implies

$$\sup_{t\in(0,T)} \|\partial_t u_m\|^2_{L^2(\Omega)} + \|u_m\|^2_{H^1_0(\Omega)} \leq C\, e^{Ct} \left(\eta(0) + \int_0^T \|f(t)\|^2_{L^2(\Omega)}.\right)$$

By the initial values for u_m we find the first part of the claim, namely

$$\sup_m \sup_{t\in(0,T)} \left(\|u_m(t)\|_{H^1_0(\Omega)} + \|\partial_t u_m(t)\|_{H^1_0(\Omega)}\right) \leq C\, \left(\|f\|_{L^2(0,T;L^2(\Omega)} + \|g\|_{H^1_0(\Omega)} + \|h\|_{L^2(\Omega)}\right).$$

To obtain the $H^{-1}(\Omega)$ estimate of $\partial_{tt} u_m$, we need to estimate

$$\|\partial_{tt} u_m\|_{H^{-1}(\Omega)} \equiv \sup_{\|v\|_{H^1_0(\Omega)}\leq 1} \int_\Omega \partial_{tt} u_m v.$$

Fix $v \in H^1_0(\Omega)$. Since w_k is an L^2-orthonormal basis, there exists a unique $\lambda \in \ell^2$ with

$$v = \sum_{k=1}^\infty \lambda_k w_k.$$

Set

$$v_1 := \sum_{k=1}^m \lambda_k w_k,$$

and $v_2 := v - v_1$. Since w_k are still orthogonal as $H^1_0(\Omega)$-maps, we have

$$\|v_1\|^2_{H^1_0(\Omega)} + \|v_2\|^2_{H^1_0(\Omega)} = \|v\|^2_{H^1_0(\Omega)} \leq 1.$$

Moreover, orthogonality and the definition of u_m also implies

$$\int_\Omega \partial_{tt} u_m v = \sum_{k=1}^m \lambda_k \int_\Omega \partial_{tt} u_m \omega_k.$$

Thus, (IX.4.1) gives

$$\int_\Omega \partial_{tt} u_m v = \int_\Omega L u_m v_1 + \int_\Omega f v_1,$$

that is

$$\|\partial_{tt} u_m\|_{H^{-1}(\Omega)} \lesssim \|u_m\|_{H^1_0(\Omega)} + \|f\|_{L^2(\Omega)}.$$

Squaring this estimate and integrating in t we obtain

$$\|\partial_{tt} u_m\|^2_{L^2(0,T)H^{-1}(\Omega)} \leq T \sup_{t\in(0,T)} \|u_m\|_{H^1_0(\Omega)} + \|f\|^2_{L^2(0,T;L^2(\Omega)},$$

which by the estimates we already proved leads to the claim. □

[3]Here in the following version: let η be *nonnegative*, absolutely continuous function on $[0,T]$ such that for almost every t,

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t),$$

for $\phi(t)$ and $\psi(t)$ nonnegative, integrable functions on $[0,T]$. Then

$$\eta(t) \leq exp(\int_0^t \phi(s)\, ds) \left(\eta(0) + \int_0^t \psi(s)\, ds\right).$$

See [**Evans, 2010**, §B.2]. (exercise!)

THEOREM IX.4.3. *The Galerkin method above converges to a weak solution of* (IX.3.1) *which satisfies the same estimates as Proposition IX.4.1.*

PROOF. By Proposition IX.4.1 the sequence u_m is bounded in $L^2(0,1;H_0^1(\Omega))$, $\partial_t u_m$ is bounded in $L^2(0,1;L^2(\Omega))$, and $\partial_{tt}u_m$ is bounded in $L^2(0,1;H^{-1}(\Omega))$. These are all Hilbert spaces, and by weak compactness we find

$u \in L^2(0,1;H_0^1(\Omega))$, with $\partial_t u$ in $L^2(0,1;L^2(\Omega))$, and $\partial_{tt}u$ in $L^2(0,1;H^{-1}(\Omega))$ such that (up to taking a subsequence of u_m which by an abuse of notation we call again u_m)

$$u_m \hookrightarrow u \quad \text{weakly in } L^2(0,1;H_0^1(\Omega))$$
$$\partial_t u_m \hookrightarrow \partial_t u \quad \text{weakly in } L^2(0,1;L^2(\Omega))$$
$$\partial_{tt} u_m \hookrightarrow \partial_{tt} u \quad \text{weakly in } L^2(0,1;H^{-1}(\Omega))$$

In particular we have for any $v \in C^1(0,1;H_0^1(\Omega))$

$$\int_0^T \int_\Omega \partial_{tt} u_m \, v \xrightarrow{m\to\infty} \int_0^T \int_\Omega \partial_{tt} u v.$$

as well as

$$\int_0^T \int_\Omega L u_m \, v \xrightarrow{m\to\infty} \int_0^T \int_\Omega L u v.$$

Now let for $K \le m$ the projection onto $\operatorname{span}(w_1,\dots,w_K)$ of w be v_K, i.e.

$$v_K := \sum_{k=1}^K w_k \int_\Omega w_k v.$$

Then from (since $K \le m$) the equation (IX.4.2)

$$\int_0^T \int_\Omega \partial_{tt} u_m \, v_K - \int_0^T \int_\Omega L u_m \, v_K = \int_0^T \int_\Omega f v_K$$

Letting $m \to \infty$ we obtain for any K

$$\int_0^T \int_\Omega \partial_{tt} u \, v_K - \int_0^T \int_\Omega L u \, v_K = \int_0^T \int_\Omega f v_K$$

Since $(w_k)_{k\in\mathbb{N}}$ is a basis of $H_0^1(\Omega)$ we can take the limit as $K \to \infty$ to obtain

$$\int_0^T \int_\Omega \partial_{tt} u \, v - \int_0^T \int_\Omega L u \, v = \int_0^T \int_\Omega f v.$$

We still need to verify $u = g$ and $\partial_t u = h$ – this follows along the arguments above by choosing testfunctions v that vanish at time T but not at time 0. We refer to [**Evans, 2010**, 7.2.2., Theorem 3] for the details.

The estimates of Proposition IX.4.1 survive the convergence. □

Let us remark that the solution is actually unique, see [**Evans, 2010**, 7.2.2, Theorem 4]. Regularity estimates are also possible depending on the data, see [**Evans, 2010**, 7.2.2, Theorem 5,6].

IX.5. Propagation of Disturbances

Parabolic equations have infinite speed of propagation. For example, if we consider a solution to

$$\Big\{\partial_t u - \Delta u = 0 \qquad \text{in } \mathbb{R}^n \times (0,\infty)$$

then if $u(x,0) \geq 0$ everywhere but we know that $u(x_0,0) > 0$ for some $x_0 \in \mathbb{R}^n$ then (by maximum principle) $u(x,t) > 0$ for all $x \in \mathbb{R}^n$ and $t > 0$. I.e., the positivity information has travelled from the point x_0 with infinite speed to every point $x \in \mathbb{R}^n$ in space. This is closely related to the maximum principles. For hyperbolic equations this is not true anymore, indeed we have the opposite, which is called propagation of disturbances.

$$\Big\{\partial_{tt} u - \Delta u = 0 \qquad \text{in } \mathbb{R}^n \times (0,\infty) \tag{IX.5.1}$$

In this case we have

Theorem IX.5.1 (Finite propagation of speed). *Assume that u solves (IX.5.1) and $u(x,0) \equiv 0$ and $\partial_t u(x,0) \equiv 0$ in $B(x,r)$ then $u(x,t) = 0$ in the cone $C(x_0,r)$,*

$$C(x_0,r) := \{(x,t) \in \mathbb{R}^n \times (0,\infty) : \quad 0 \leq t \leq r,\ |x - x_0| < r - t\}$$

Proof. W.l.o.g. $x_0 = 0$. For $0 < t < r$ we set

$$e(t) := \frac{1}{2}\int_{B(r-t)} |\partial_t u(x,t)|^2 + |Du(x,t)|^2\, dx$$

We take the derivative

$$e'(t) = -\frac{1}{2}\int_{\partial B(r-t)} |\partial_t u|^2 d\mathcal{H}^{n-1} - \frac{1}{2}\int_{\partial B(r-t)} |Du(x,t)|^2 d\mathcal{H}^{n-1}\, dx + \int_{B(r-t)} \partial_t u \partial_{tt} u + Du \cdot \partial_t Du\, dx$$

Integrating by parts we obtain

$$\begin{aligned}\int_{B(r-t)} Du \cdot \partial_t Du\, dx &= \int_{\partial B(r-t)} \partial_\nu u\, \partial_t u\, d\mathcal{H}^{n-1} - \int_{B(r-t)} \Delta u\, \partial_t u\, dx \\ &= \int_{\partial B(r-t)} \partial_\nu u\, \partial_t u\, d\mathcal{H}^{n-1} - \int_{B(r-t)} \partial_{tt} u\, \partial_t u\, dx\end{aligned}$$

that is

$$e'(t) = -\frac{1}{2}\int_{\partial B(r-t)} |\partial_t u|^2 d\mathcal{H}^{n-1} - \frac{1}{2}\int_{\partial B(r-t)} |Du|^2 d\mathcal{H}^{n-1}\, dx + \int_{\partial B(r-t)} \partial_\nu u\, \partial_t u\, dx$$

Now by Young's inequality (in this case simply because $2ab \leq a^2 + b^2$)

$$\int_{\partial B(r-t)} \partial_\nu u\, \partial_t u\, dx \leq \frac{1}{2}\int_{\partial B(r-t)} |Du|^2 + |\partial_t u|^2\, dx.$$

That is for all $t \in (0,r)$,

$$e'(t) \leq 0.$$

This implies for all $t \in (0,r)$

$$0 \leq e(t) \leq e(0).$$

Since $u(x,0) \equiv const$ in $B(r)$, $|Du| \equiv 0$ in $B(r)$. Since moreover $\partial_t u(x,0) = 0$ in $B(r)$ we have $e(0) = 0$, and consequently $e(t) \equiv 0$ for all $t \in (0,r)$. This in turn means that $|Du| = |\partial_t u| = 0$ in $C(r)$, i.e u is constant in the cone $C(0,r)$. Thus $u \equiv 0$. $\square$

CHAPTER 10

Introduction to Navier–Stokes

For a thorough exposition we refer to [**Galdi, 2011**, **Robinson et al., 2016**].

X.1. The Navier–Stokes equations

The incompressible Navier-Stokes system is supposed to represent the evolution of the dynamics of an incompressible viscous fluid.

For $u = (u^1, u^2, u^3) : \mathbb{R}^3 \to \mathbb{R}$ (represents the velocity field of a fluid) and $p : \mathbb{R}^3 \to \mathbb{R}$ (the pressure) the Navier-Stokes equation is

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{for } x \in \mathbb{R}^3. \end{cases}$$

In more physical version, there are factors of viscosity and density of the fluid, which for analytical simplicity we set here equal to 1. We also assume that no external forces are acting on the fluid. Also let us remark that there is (often: implicitely) assumed a boundedness for $|x| \to \infty$ of u.

We do not have a good understanding of this equation, indeed it is one of the Millenium Problems formulated by Charles Fefferman for the Clay Math Institute.

Conjecture X.1.1. Let $\nu > 0$ and let $u_0 : \mathbb{R}^3 \to \mathbb{R}^3$ be a divergence-free vector field in the Schwartz class. Then there exist a smooth vector field $u : [0, \infty] \times \mathbb{R}^3 \to \mathbb{R}^3$ (the velocity field) and a smooth function $p : \mathbb{R}^3 \to \mathbb{R}$ (the pressure field) obeying the equations

$$\partial_t u + (u \cdot \nabla)u = \nu \Delta u - \nabla p \forall (0, \infty) \times \mathbb{R}^3$$

$$\nabla \cdot u(t, x) = 0 \quad \forall t \in (0, \infty),\, x \in \mathbb{R}^3$$

$$u(0, \cdot) = u_0(\cdot)$$

as well as the finite energy condition $u \in L^\infty_t(L^2_x([0, T] \times \mathbb{R}^3))$ for every $0 < T < \infty$.

We will look at two-dimensional and three-dimensional versions of this equation (in two dimensions things are a bit easier), i.e. for (nice!) domains $\Omega \subseteq \mathbb{R}^n$, $n = 2, 3$ we consider

solutions $u : \Omega \to \mathbb{R}^n$, $p : \Omega \to \mathbb{R}$ of

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p = 0 & \text{in } \Omega \times (0,T) \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0,T) \end{cases} \tag{X.1.1}$$

We need to define some operators that appear here. The operator $\nabla\cdot$ denotes the divergence, in two dimensions:

$$\nabla \cdot (u^1, u^2) \equiv \operatorname{div}((u^1, u^2)) = \partial_1 u_1 + \partial_2 u_2.$$

and in three dimensions

$$\nabla \cdot (u^1, u^2, u^3) \equiv \operatorname{div}((u^1, u^2, u^3)) = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3.$$

The notion $u \cdot \nabla$ is to be understood as a vector product of u (a vector field) and the Gradient, i.e. in twod dimensions,

$$u \cdot \nabla := u_1 \partial_1 + u_2 \partial_2$$

and thus

$$u \cdot \nabla u := u_1 \partial_1 \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} + u_2 \partial_2 \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}.$$

In three dimensions,

$$u \cdot \nabla := u_1 \partial_1 + u_2 \partial_2 + u_3 \partial_3$$

and thus

$$u \cdot \nabla u := u_1 \partial_1 \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} + u_2 \partial_2 \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix}$$

The pressure term is never explicitely specified because it follows intrinsically from u: Since $\operatorname{div} u = 0$ we can apply divergence to the equation (X.1.1) and find

$$\operatorname{div}((u \cdot \nabla) u) + \operatorname{div}(\nabla p) = 0.$$

Since $\operatorname{div}(\nabla p) = \Delta p$ we have that p must solve the following equation,

$$\Delta p = -\operatorname{div}((u \cdot \nabla) u) \quad \text{in } \Omega.$$

Observe that this equation is uniquely solvable if boundary data on $\partial\Omega$ is given (and u is sufficiently regular in the right Sobolev space).

In the following we will always assume that Ω is a smoothly bounded set or the whole space, in particular we assume that $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$.

X.2. Weak formulation

For $n = 2, 3$ and $\Omega \subset \mathbb{R}^n$ open denote by

$$C^\infty_{c,\sigma}(\Omega) := \Big\{\varphi \in [C^\infty_c(\Omega)]^3 : \quad \operatorname{div}\varphi = 0\Big\}$$

Set

$$L^2_{\sigma,n} \equiv L^2_{\sigma,n}(\Omega) := \overline{C^\infty_{c,\sigma}(\Omega)}^{\|\cdot\|_{L^2}}$$

equipped with the L^2-norm.

Lemma X.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain with (if existing) smooth boundary. For any $f \in L^2_{\sigma,n}(\Omega)$ and any $g \in H^1(\Omega)$ (without any boundary assumption) we have*

$$\int_\Omega f \nabla g = 0$$

Proof. Since $f \in L^2_{\sigma,n}(\Omega)$ there are $C^\infty_c(\Omega, \mathbb{R}^3) \ni f_k \to f \in L^2(\Omega)$, with $\nabla \cdot f_k = 0$. Thus,

$$\int_\Omega f \nabla g = \lim_{k\to\infty} \int_\Omega f_k \nabla g = \lim_{k\to\infty} \int_\Omega \nabla \cdot f_k g = 0$$

□

Remark X.2.2. Observe that formally (if f was nicer; ν denotes the outer unit normal of $\partial\Omega$)

$$\int_\Omega f \nabla g = \int_{\partial\Omega} f \cdot \nu g - \int_\Omega \nabla \cdot f\, g.$$

So if $\nabla \cdot f = 0$ this implies that

$$\int_{\partial\Omega} f \cdot \nu g = 0 \quad \forall g \in H^1(\Omega).$$

Since the trace space of $H^1(\Omega)$ is $H^{\frac{1}{2}}(\partial\Omega)$, this implies

$$\int_{\partial\Omega} f \cdot \nu g = 0 \quad \forall g \in H^{\frac{1}{2}}(\partial\Omega).$$

Of course, if $f \in L^2(\Omega)$, f is not well-defined on $\partial\Omega$ – but this argument shows that $f \cdot \nu$ has a distributional meaning.

Indeed, one can show [**Galdi, 2011**, Theorem III.2.3] that whenever Ω is a smooth bounded domain, then

$$L^2_{\sigma,n}(\Omega) = \Big\{u \in L^2(\Omega, \mathbb{R}^3) : \ \nabla \cdot u = 0, \ u\Big|_{\partial\Omega} \cdot \nu = 0\Big\} \tag{X.2.1}$$

Here, $\nabla \cdot u = 0$ is to be understood in the distributional sense and $u \cdot \nu$ has again a distributional meaning (in the dual space of trace spaces).

An important theorem is the *Helmoltz-Weyl decomposition* (in more geometric contexts known as *Hodge decomposition*).

THEOREM X.2.3. *Let $\Omega \subset \mathbb{R}^3$ be an open smoothly bounded set in $\mathbb{R}^3$. Then for any $f \in L^2(\Omega)$ there are $h \in L^2_{\sigma,n}(\Omega)$ and $g \in H^1(\Omega)$ such that*

$$f = h + \nabla g,$$

with

$$\|h\|_{L^2(\Omega)} + \|\nabla g\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}$$

h in the decomposition above is sometimes called Helmholtz-Leray projection of f.

SKETCH OF THE PROOF. First we solve for $g_1 \in H^1_0(\Omega)$,

$$\begin{cases} \Delta g_1 = \nabla \cdot f & \text{in } \Omega, \\ g_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

This can be done with the estimate

$$\|\nabla g_1\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}.$$

We set $h_1 := f - \nabla g_1$, which satisfies

$$\|h_1\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}.$$

Observe that $\operatorname{div} h_1 = 0$. However this does not mean that $h_1 \in L^2_{\sigma,n}(\Omega)$ because $h_1 \cdot \nu$ is not necessarily zero (however, as in Remark X.2.2, the notion of $h_1 \cdot \nu\Big|_{\partial\Omega}$ can be defined in a distributional sense). So we solve

$$\begin{cases} \Delta g_2 = \nabla \cdot h_1 = 0 & \text{in } \Omega, \\ \partial_\nu g_2 = h_1 \cdot \nu & \text{on } \partial\Omega. \end{cases}$$

Again (this needs to be proven),

$$\|\nabla g_2\|_{L^2(\Omega)} \lesssim \|h_1\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}.$$

Now,

$$h := f - \nabla g_1 - \nabla g_2 \equiv h_1 - \nabla g_2$$

satisfies $\operatorname{div}(h) = 0$ and $h \cdot \nu = 0$ on $\partial\Omega$, so by (X.2.1) we have that $h \in L^2_{\sigma,n}(\Omega)$. □

Now we set

$$V := L^2_{\sigma,n}(\Omega, \mathbb{R}^n) \cap H^1_0(\Omega, \mathbb{R}^n).$$

In particular,

$$V \subset \Big\{ f \in H^1_0(\Omega, \mathbb{R}^n) : \quad \operatorname{div} f = 0 \Big\}$$

but these two spaces are not the same. Also observe that $H^1_0(\mathbb{R}^n, \mathbb{R}^n) = H^1(\mathbb{R}^n, \mathbb{R}^n)$.

Lemma X.2.4. *For $n = 1, 2, 3, 4$ and $\Omega \subset \mathbb{R}^n$ open with smooth boundary the map*

$$(u, v, w) \mapsto \int_\Omega u^\alpha \partial_\alpha v^\beta w^\beta$$

is a trilinear continuous map on $H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)$.

Proof. It is clearly tri-linear, all that is needed to show is boundedness. By Hölder's inequality

$$\int_\Omega u^\alpha \partial_\alpha v^\beta w^\beta \leq \|v\|_{H^1(\Omega)} \|u\|_{L^4(\Omega)} \|w\|_{L^4(\Omega)}$$

By Sobolev embedding, Corollary III.3.31, $H^1(\Omega) \subset L^p(\Omega)$ for any $p \in [2, p^*]$, $p < \infty$ for $p^* = \frac{2n}{n-2}$ (for $n \geq 3$), $p \in [2, \infty)$ for $n \leq 2$. As long as $n \leq 4$ this implies that

$$\|u\|_{L^4(\Omega)} \leq \|u\|_{H^1(\Omega)},$$

$$\|w\|_{L^4(\Omega)} \leq \|w\|_{H^1(\Omega)}.$$

□

What makes Navier-Stokes equation a "nonlinear" equation (this notion is debatable) is the term $(u \cdot \nabla)u$. One important observation is that it vanishes when testing the equation with u. For this we introduce the notion of the L^2-scalar product,

$$\langle f, g \rangle = \int_\Omega fg.$$

and the L^2-norm

$$\|f\| := \sqrt{\langle f, f \rangle}.$$

Lemma X.2.5. *Let $u \in V$ and $v, w \in H^1(\Omega, \mathbb{R}^n)$ then*

$$\langle u \cdot \nabla v,\, w \rangle = -\langle u \cdot \nabla w,\, v \rangle$$

and in particular

$$\langle u \cdot \nabla v, v \rangle = 0.$$

Proof. By Lemma X.2.4,

$$(u, v, w) \mapsto \langle u \cdot \nabla v, w \rangle$$

is a bounded trilinear form on the spaces involved (in particular it makes sense as an integral). By approximation (here we use that Ω is a nice set) we can assume that $v, w \in C^\infty(\overline{\Omega}, \mathbb{R}^3) \cap W^{1,\infty}(\Omega)$.

Observe now that

$$\begin{aligned} &\langle u \cdot \nabla v, w \rangle + \langle u \cdot \nabla w, v \rangle \\ =&\langle u^\alpha \cdot \partial_\alpha v^\beta, w^\beta \rangle + \langle u^\alpha \cdot \partial_\alpha w^\beta, v^\beta \rangle \end{aligned}$$

Observe that $\partial_\alpha v^\beta w^\beta \in L^\infty(\Omega)$ and since $u^\alpha \in L^2_{\sigma,n}(\Omega)$ we assume w.l.o.g. (by approximation) that $u^\alpha \in C_c^\infty$ with $\operatorname{div}(u) = 0$. Then

$$\begin{aligned}
&= \int_\Omega u^\alpha \cdot \partial_\alpha (v^\beta w^\beta) \\
&= -\int_\Omega \partial_\alpha u^\alpha \cdot v^\beta w^\beta \\
&= -\int_\Omega \operatorname{div}(u) \cdot (v^\beta w^\beta) \\
&= 0.
\end{aligned}$$

In the third to fourth line we used integration by parts and that u^α is zero in trace sense on $\partial\Omega$. In the last line we use that $\operatorname{div} u = 0$.

This proves the claims. □

Observe that we also have

Lemma X.2.6. *Let $u \in V$, and $p \in H^1(\Omega)$ then*

$$\langle u, \nabla p \rangle = 0$$

Proof. Again by approximation we may assume that $p \in C^\infty(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$.

So, integration by parts again, since $u = 0$ on $\partial\Omega$ and $\operatorname{div} u = 0$,

$$\langle u, \nabla p \rangle = -\langle \operatorname{div} u, p \rangle = 0.$$

□

So if we have a smooth solution $u(t) \in V$ of the Navier-Stokes equation, (X.1.1), then we (formally) should have

$$\langle \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p, u \rangle = 0.$$

By Lemma X.2.4, Lemma X.2.5, Lemma X.2.6 this implies

$$\langle \partial_t u(t), u(t) \rangle - \langle \Delta u(t), u(t) \rangle = 0.$$

Integration by parts ($u(t) \in V$ so no boundary terms)

$$\langle \Delta u(t), u(t) \rangle = -\langle \nabla u(t), \nabla u(t) \rangle = -\int_\Omega |\nabla u(t)|^2 dx.$$

Also (formally!) interchanging derivative ∂_t and the integral we have

$$\langle \partial_t u(t), u(t) \rangle = \frac{1}{2} \partial_t \int_\Omega |u(t)|^2.$$

Integrating formally in $t = 0$ to some s we then find that

$$\int_\Omega |u(s)|^2 + \int_0^s \int_\Omega |\nabla u(t)|^2 dx\, ds = \int_\Omega |u_0|^2,$$

that is we find that any (nice enough) solution on $(0,T)$ of the Navier stokes equation with $u(0)=u_0$ satisfies immediately $u \in L^\infty(0,T;L^2_{\sigma,n}) \cap L^2(0,T;V)$. We like this so much that we put this into the definition of solution.

Definition X.2.7. We say that $u : [0,T) \to V$ is a *weak solution of the Navier-Stokes equation* (X.1.1) with initial value $u_0 \in H^1_0(\Omega,\mathbb{R}^3)$ if

- $u \in L^\infty(0,T;L^2_{\sigma,n}) \cap L^2(0,T;V)$ for all $T>0$ and
- u satisfies the equation

$$\int_0^s -\langle u, \partial_t \varphi\rangle + \int_0^s \langle \nabla u, \nabla \varphi\rangle \, ds + \int_0^s \langle (u\cdot\nabla)u, \varphi\rangle \, ds = \langle u_0, \varphi(0)\rangle - \langle u(s), \varphi(s)\rangle$$

for almost all $s \in (0,T)$ and all test functions $\varphi \in \mathcal{D}_\sigma$, where

$$\mathcal{D}_\sigma = \{\varphi \in C_c^\infty(\Omega \times [0,\infty)) : \operatorname{div}(\varphi(t)) = 0 \quad \forall t \in [0,\infty)\}.$$

Remark X.2.8 (A problem in 3D)**.** Observe that in Definition X.2.7 one can not just test with $\varphi = u$ (even with a density argument). It is true that by Lemma X.2.5 we can make sense (by approximation) of

$$\langle (u(t)\cdot\nabla)u(t), u(t)\rangle,$$

if $u(t) \in V$; and this expression is zero a.e.; however

$$\int_0^s \langle (u(t)\cdot\nabla)v(t), w(t)\rangle \, dt$$

is *not* well defined if $u,v,w \in L^\infty(0,T;L^2_{\sigma,n}) \cap L^2(0,T;V)$ if $\Omega \subset \mathbb{R}^3$. This holds, however, for $\Omega \subset \mathbb{R}^2$, which is why 2D Navier-Stokes is easier than 3D. See Proposition X.2.9 below.

Proposition X.2.9. *For $n=2$ and $\Omega \subset \mathbb{R}^2$ open with smooth boundary the map*

$$(u,v,w) \mapsto \int_0^T \int_\Omega u^\alpha \partial_\alpha v^\beta w^\beta$$

is a trilinear continuous map on

$$u \in L^\infty(0,T;L^2_{\sigma,n}).$$
$$v,w \in L^2(0,T;V).$$

The proof is based on a technique called *compensated compactness* or *commutator estimates*. The version we use here is due to [**Coifman et al., 1993**].

THEOREM X.2.10 (Coifman-Lions-Meyer-Semmes). *Let $A \in L^2_{\sigma,n}(\mathbb{R}^n,\mathbb{R}^n)$, $b \in H^1(\mathbb{R}^n)$. Then for any $\varphi \in C_c^\infty(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} A\cdot\nabla b \; \varphi \le C\|A\|_{L^2(\mathbb{R}^n)} \|\nabla b\|_{L^2(\mathbb{R}^n)} [\varphi]_{BMO},$$

where BMO denotes the space of bounded mean oszillation, defined via the seminorm

$$[\varphi]_{BMO} := \sup_B \fint_B |\varphi - (\varphi)_B|,$$

the supremum being over all balls with finite radius in $\mathbb{R}^n$.

Proof of Proposition X.2.9. We assume $\Omega = \mathbb{R}^n$, the other cases follow by approximation (observe that u has zero boundary data). By Theorem X.2.10,

$$\int_{\mathbb{R}^2} u^\alpha \partial_\alpha v^\beta w^\beta \leq \|u\|_{L^2(\mathbb{R}^2)} \|Dv\|_{L^2(\mathbb{R}^2)} [w]_{BMO}$$

From Poincaré inequality it is elementary to observe (here we use the dimension $n = 2$)

$$[w]_{BMO} \lesssim \|\nabla w\|_{L^2(\mathbb{R}^n)} \leq \|w\|_{H^1(\mathbb{R}^n)}.$$

So we have (by approximation) the estimate for almost all s,

$$\int_{\mathbb{R}^2} u^\alpha \partial_\alpha v^\beta w^\beta \leq \|u\|_{L^2(\mathbb{R}^2)} \|\nabla v\|_{L^2(\mathbb{R}^2)} \|\nabla w\|_{L^2(\mathbb{R}^2)}$$

Thus,

$$\int_0^T \int_{\mathbb{R}^2} u^\alpha \partial_\alpha v^\beta w^\beta \leq \|u\|_{L^\infty((0,T);L^2(\mathbb{R}^2))} \|\nabla v\|_{L^2(0,T;L^2(\mathbb{R}^2)} \|\nabla w\|_{L^2(0,T;L^2(\mathbb{R}^2))}.$$

□

Sketch of the proof of Theorem X.2.10. Theorem X.2.10 is a deep theorem of harmonic analysis.

We present here a sketch of the proof of the following (weaker, but for our purposes sufficient) estimate (same conditions on A, b, φ):

$$\int_{\mathbb{R}^2} A \cdot \nabla b \, \varphi \leq C\|A\|_{L^2(\mathbb{R}^2)} \|\nabla b\|_{L^2(\mathbb{R}^2)} \|\nabla \varphi\|_{L^2(\mathbb{R}^2)}.$$

We follow essentially a proof by Brezis-Nguyen [**Brezis and Nguyen, 2011**], sharp estimates can also be obtained with these arguments [**Lenzmann and Schikorra, 2019**].

- A can be written as $\nabla^\perp a := \begin{pmatrix} -\partial_y a \\ \partial_x a \end{pmatrix}$. This is called Hodge decomposition, Helmholtz decomposition, or in this case the Poincaré Lemma. The idea is that we can solve
$$\Delta a = \partial_2 A^1 - \partial_1 A^2$$
then $B := A - \nabla^\perp a$ is divergence free, $\operatorname{div} B = 0$ but also curl-free, $\partial_1 B^2 - \partial_2 B^1 = 0$. One compute that this implies in particular that B is harmonic, $\Delta B^1 = \Delta B^2 = 0$. If we choose the right boundary data for a above then we can ensure that $B = 0$ on its boundary (i.e. decay at infinity), but then $\Delta B = 0$ implies $B \equiv 0$ by uniqueness of solutions. That is, $A = \nabla^\perp a$.
- Observe that $\nabla^\perp a \cdot \nabla b = \det(\nabla a, \nabla b)$.
- Use the harmonic extension to $\mathbb{R}^3_+$, solve
$$\begin{cases} \Delta \alpha = 0 & \text{in } \mathbb{R}^3_+ \\ \alpha = a & \text{in } \mathbb{R}^2 \times \{0\} \end{cases}$$

$$\begin{cases} \Delta\beta = 0 & \text{in } \mathbb{R}^3_+ \\ \beta = b & \text{in } \mathbb{R}^2 \times \{0\} \end{cases}$$

$$\begin{cases} \Delta\Phi = 0 & \text{in } \mathbb{R}^3_+ \\ \Phi = \varphi & \text{in } \mathbb{R}^2 \times \{0\} \end{cases}$$

By Stokes' theorem we then have (and this is a cancellation property)

$$\int_{\mathbb{R}^2} \det(\nabla a, \nabla b)\varphi = \int_{\mathbb{R}^3_+} \det(\nabla_{\mathbb{R}^3}\alpha, \nabla_{\mathbb{R}^3}\beta, \nabla_{\mathbb{R}^3}\Phi).$$

Thus, by Hölder's inequality, we find the estimate

$$\int_{\mathbb{R}^2} A \cdot \nabla b \; \varphi \leq C\|\nabla\alpha\|_{L^3(\mathbb{R}^3_+)} \|\nabla\beta\|_{L^3(\mathbb{R}^3_+)} \|\nabla\Phi\|_{L^3(\mathbb{R}^3_+)}.$$

Now harmonic extensions attain the trace inequality, i.e. one can show

$$\|\nabla\alpha\|_{L^3(\mathbb{R}^3_+)} \approx [\alpha]_{W^{1,3}(\mathbb{R}^3_+)}[a]_{W^{1-\frac{1}{3},3}(\mathbb{R}^2)},$$

and by Sobolev inequality for fractional Sobolev spaces

$$[a]_{W^{1-\frac{1}{3},3}(\mathbb{R}^2)} \lesssim \|\nabla a\|_{L^2(\mathbb{R}^2)} = \|A\|_{L^2(\mathbb{R}^2)}.$$

We do this for α, β, φ and obtain the claim. □

The following is the time-independent version:

Corollary X.2.11. *Let $\Omega \subseteq \mathbb{R}^2$. Let $u \in L^2_{\sigma,n}(\Omega)$, $v, w \in H^1_0(\Omega)$, then*

$$\langle (u \cdot \nabla)v, w\rangle \lesssim \|u\|_{L^2} \|\nabla v\|_{L^2} \|\nabla w\|_{L^2}.$$

Remark X.2.12. The *div − curl* argument is actually not necessary, one can base this on Gagliardo-Nirenberg inequality (in the context of Navier-Stokes equation also called Ladyzhenskaya inequality).

The following is essentially the equation of Definition X.2.7 for φ independent of time.

Lemma X.2.13. *Let u be a weak solution of the Navier-Stokes equation as in Definition X.2.7, then for all $\varphi \in C^\infty_{c,\sigma}(\Omega)$ (i.e. constant in time) and almost every $t_2 \geq t_1$, for almost every $t_1 \geq 0$ we have*

$$\int_{t_1}^{t_2} \langle \nabla u, \nabla\varphi\rangle + \int_{t_1}^{t_2} \langle u \cdot \nabla u, \varphi\rangle = \langle u(t_1), \varphi\rangle - \langle u(t_2), \varphi\rangle. \tag{X.2.2}$$

Secondly, set

$$X := \left(\overline{C^\infty_{c,\sigma}(\Omega)}^{\|\cdot\|_{H^1_0(\Omega)}}\right)^*.$$

Then for almost every time $t > 0$ there exists an operator $\partial_t u \in X^*$ such that for every $\varphi \in C^\infty_{c,\sigma}(\Omega)$

$$\langle \partial_t u(t), \varphi\rangle = \lim_{s\to 0}\langle \frac{u(t+s)-u(t)}{s}, \varphi\rangle,$$

and moreover

$$\langle \partial_t u, \varphi\rangle + \langle \nabla u, \nabla\varphi\rangle = \langle u\cdot\nabla u, \varphi\rangle \tag{X.2.3}$$

Proof. Recall from Definition X.2.7, u is a weak solution if

$$\int_0^s -\langle u, \partial_t\varphi\rangle + \int_0^s \langle \nabla u, \nabla\varphi\rangle\, ds + \int_0^s \langle (u\cdot\nabla)u, \varphi\rangle\, ds = \langle u_0, \varphi(0)\rangle - \langle u(s), \varphi(s)\rangle$$

for almost all $s \in (0,T)$ and all test functions $\varphi \in \mathcal{D}_\sigma$, where

$$\mathcal{D}_\sigma = \{\varphi \in C^\infty_c(\Omega\times[0,\infty)) : \operatorname{div}(\varphi(t)) = 0 \quad \forall t \in [0,\infty)\}.$$

Applying this definition for $s = t_1$ and $s = t_2$ and subtracting we obtain

$$\int_{t_1}^{t_2} -\langle u, \partial_t\varphi\rangle + \int_{t_1}^{t_2} \langle \nabla u, \nabla\varphi\rangle\, ds + \int_{t_1}^{t_2} \langle (u\cdot\nabla)u, \varphi\rangle\, ds = \langle u(t_1), \varphi(t_1)\rangle - \langle u(t_2), \varphi(t_2)\rangle.$$

This holds in particular if we take $\varphi(x,t) := \eta(t)\psi(x)$ for some $\eta \equiv 1$ in (t_1,t_2). In that case, $\partial_t\varphi \equiv 0$ in (t_1,t_2), consequently we have

$$\int_{t_1}^{t_2} \langle \nabla u, \nabla\psi\rangle\, ds + \int_{t_1}^{t_2} \langle (u\cdot\nabla)u, \psi\rangle\, ds = \langle u(t_1), \psi\rangle - \langle u(t_2), \psi\rangle.$$

This establishes (X.2.2).

For the (X.2.3) we argue as follows: let now $\varphi \in C^\infty_c(\Omega)$, $\operatorname{div}\varphi = 0$. We set

$$G(s)[\varphi] := \langle \nabla u(s), \nabla\varphi\rangle + \langle (u(s)\cdot\nabla)u(s), \varphi\rangle$$

then from (X.2.2) we have for almost every t and s,

$$\left\langle \frac{u(t+s)-u(t)}{s}, \varphi\right\rangle = \frac{1}{s}\int_t^{t+s} G(\sigma)[\varphi]d\sigma.$$

Observe that by Lemma X.2.4,

$$G(\sigma)[\varphi] \lesssim \left(\|\nabla u(\sigma)\|_{L^2(\Omega)} + \|\nabla u(\sigma)\|^2_{L^2(\Omega)}\right)\|\nabla\varphi\|_{L^2(\Omega)}$$
$$\left(1 + \|u(\sigma)\|^2_{H^1(\Omega)}\right)\|\varphi\|_{H^1(\Omega)}$$

So we have

$$\left|\left\langle \frac{u(t+s)-u(t)}{s}, \varphi\right\rangle\right| \lesssim \left(\frac{1}{s}\int_t^s \left(1 + \|u(\sigma)\|^2_{H^1(\Omega)}\right) d\sigma\right)\|\varphi\|_{H^1(\Omega)}.$$

Recall that this holds whenever $\varphi \in C^\infty_c(\Omega,\mathbb{R}^3)$ with $\operatorname{div}(\varphi) = 0$. So if we set

$$X := \left(\overline{C^\infty_{c,\sigma}(\Omega)}^{\|\cdot\|_{H^1_0(\Omega)}}\right)^*$$

then

$$T_{t,s}\varphi := \varphi \mapsto \left\langle \frac{u(t+s)-u(t)}{s}, \varphi \right\rangle$$

belongs to X^*.

On the other hand, $u \in L^2((0,T), H_0^1(\Omega))$, and thus by Lebesgue's Theorem for almost any $t > 0$ we have

$$\lim_{s\to 0} \frac{1}{s} \int_t^s \left(1 + \|u(\sigma)\|^2_{H^1(\Omega)}\right) d\sigma = \left(1 + \|u(t)\|^2_{H^1(\Omega)}\right) < \infty.$$

That is, for any such t the sequence $(T_{t,s})_{s\in(0,T-t)}$ is a bounded sequence of elements in X^*, which is a reflexive Banach space. By reflexivity, $X^* = X^{**}$ and $X = X^{**}$, there exists a weak limit operator $\partial_t u := T_{t,0} \in X^*$ such that for any $\varphi \in X$,

$$\langle \partial_t u, \varphi \rangle := T_{t,0}[\varphi] = \lim_{s\to 0} \left\langle \frac{u(t+s)-u(t)}{s}, \varphi \right\rangle.$$

and on the other hand (by Lebesgue's theorem)

$$\langle \partial_t u, \varphi \rangle = \lim_{s\to 0} \frac{1}{s} \int_t^{t+s} G(\sigma)[\varphi] d\sigma = G(t)[\varphi].$$

This establishes the second part of the claim. □

X.3. Existence of weak solutions

See [**Robinson et al., 2016**, Theorem 4.4]

THEOREM X.3.1. *For each $u \in L^2_{\sigma,n}(\Omega)$, for $\Omega \subseteq \mathbb{R}^n$, $n = 2, 3$ either a smoothly bounded domain or all of $\mathbb{R}^n$, there exists a weak, global in time, solution u which for any $T > 0$ belongs to$L^\infty(0,T; L^2_{\sigma,n}(\Omega)) \cap L^2(0,\infty; V)$ in the sense of Definition X.2.7.*

X.4. Uniqueness of weak solutions in dimension two

Cf. [**Robinson et al., 2016**, Theorem 3.15]

THEOREM X.4.1. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and assume that $u_0 \in L^2_{\sigma,n}(\Omega)$. If u and v both solve (X.1.1) in $\Omega \times (0,T)$ in the weak sense defined in Definition X.2.7, then $u(t) \equiv v(t)$ for almost every $t \in (0,T)$.*

PROOF. Set $w := u - v \in L^\infty(0,T; L^2_{\sigma,n}(\Omega)) \cap L^2(0,T;V)$.

By Lemma X.2.13 we have

$$-\langle \partial_t w, \varphi \rangle = \langle \nabla w, \nabla \varphi \rangle + \langle u \cdot \nabla u, \varphi \rangle - \langle v \cdot \nabla v, \varphi \rangle.$$

Plugging $\varphi = w$ into Lemma X.2.13 (here we need a suitable approximation argument which we drop for now) we obtain

$$\frac{1}{2}\partial_t \|w\|_{L^2}^2 = -\|\nabla w\|_{L^2}^2 + \langle u \cdot \nabla u, w\rangle - \langle v \cdot \nabla v, w\rangle.$$

Observe that

$$\begin{aligned}&\langle u \cdot \nabla u, w\rangle - \langle v \cdot \nabla v, w\rangle\\ =&\langle u \cdot \nabla w, w\rangle + \langle w \cdot \nabla v, w\rangle.\end{aligned}$$

By Lemma X.2.5, $\langle u \cdot \nabla w, w\rangle = 0$. So, by Corollary X.2.11,

$$|\langle u \cdot \nabla u, w\rangle - \langle v \cdot \nabla v, w\rangle| \lesssim \|w\|_{L^2} \|\nabla u\|_{L^2} \|\nabla w\|_{L^2}.$$

By Young's inequality $ab \leq \varepsilon a^2 + C(\varepsilon) b^2$ we find that

$$\partial_t \|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \lesssim \frac{1}{2} \|\nabla w\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \|w\|_{L^2}^2.$$

Absorbing to the left-hand side

$$\partial_t \|w\|_{L^2}^2 \lesssim \|\nabla u\|_{L^2}^2 \|w\|_{L^2}^2.$$

By Grönwall's inequality, Lemma IX.4.2,

$$\|w(t)\|_{L^2}^2 \lesssim \|w(0)\|_{L^2}^2 \exp\left(\int_0^t \|\nabla u(s)\|_{L^2}^2 ds\right) \tag{X.4.1}$$

Observe that (here is where we use $2D$, so that $\|\nabla u\|_{L^2}$ has the right exponent!),

$$\exp\left(\int_0^t \|\nabla u(s)\|_{L^2}^2 ds\right) \leq \exp \|\nabla u\|_{L^2(0,T;L^2(\Omega)} < \infty.$$

Moreover $\|w(0)\|_{L^2} = 0$, so (X.4.1) implies $w \equiv 0$, i.e. $u \equiv v$. □

Instead of using above the Theorem X.2.10 one can also use the Gagliardo-Nirenberg-inequality (also referred to as Ladyzhenskaya-inequality in special cases):

Proposition X.4.2. *For $n < 4$,*

$$\|w\|_{L^4(\mathbb{R}^n)} \leq C \|w\|_{L^2(\mathbb{R}^n)}^{\frac{4-n}{4}} \|\nabla w\|_{L^2(\mathbb{R}^n)}^{\frac{n}{4}}.$$

Proof. By Sobolev inequality, for all $w \in W^{1,2}(\mathbb{R}^n)$,

$$\|w\|_{L^4(\mathbb{R}^n)} \leq C \left(\|w\|_{L^2(\mathbb{R}^n)} + \|\nabla w\|_{L^2(\mathbb{R}^n)}\right).$$

The claim now follows from scaling: Fix $w \in W^{1,2}(\mathbb{R}^n)$ and set for $r > 0$

$$w_r(x) := w(rx).$$

Apply the Sobolev inequality to this w_r

$$\|w_r\|_{L^4(\mathbb{R}^n)} \leq \|w_r\|_{L^2(\mathbb{R}^n)} + \|\nabla w_r\|_{L^2(\mathbb{R}^n)}.$$

Now

$$\|w_r\|_{L^4(\mathbb{R}^n)} = r^{-\frac{n}{4}} \|w\|_{L^4(\mathbb{R}^n)}$$

$$\|w_r\|_{L^2(\mathbb{R}^n)} = r^{-\frac{n}{2}}\,\|w\|_{L^4(\mathbb{R}^n)}$$

and since $(\nabla w_r)(x) = r(\nabla w)(rx)$,

$$\|\nabla w_r\|_{L^2(\mathbb{R}^n)} = r\,r^{-\frac{n}{2}}\,\|\nabla w\|_{L^2(\mathbb{R}^n)},$$

so that

$$r^{-\frac{n}{4}}\|w\|_{L^4(\mathbb{R}^n)} \le r^{-\frac{n}{2}}\|w\|_{L^2(\mathbb{R}^n)} + r^{1-\frac{n}{2}}\|\nabla w\|_{L^2(\mathbb{R}^n)}.$$

i.e.

$$\|w\|_{L^4(\mathbb{R}^n)} \le C\left(r^{-\frac{n}{4}}\|w\|_{L^2(\mathbb{R}^n)} + r^{1-\frac{n}{4}}\|\nabla w\|_{L^2(\mathbb{R}^n)}\right).$$

This holds for any $r > 0$, so let us choose

$$r := \frac{\|w\|_{L^2(\mathbb{R}^n)}}{\|\nabla w\|_{L^2(\mathbb{R}^n)}}$$

then

$$\|w\|_{L^4(\mathbb{R}^n)} \le 2C\|w\|_{L^2(\mathbb{R}^n)}^{\frac{4-n}{4}}\,\|\nabla w\|_{L^2(\mathbb{R}^n)}^{\frac{n}{4}}.$$

□

Remark X.4.3. If we apply Proposition X.4.2 in the proof of Theorem X.4.1 for estimating the crucial term

$$\langle w\cdot\nabla v, w\rangle$$

we find for $n = 2, 3$

$$\begin{aligned}|\langle w\cdot\nabla v, w\rangle| &\lesssim \|w\|_{L^4}^2\,\|\nabla v\|_{L^2}\\ &\lesssim \left(\|w\|_{L^2(\mathbb{R}^n)}^{\frac{4-n}{4}}\,\|\nabla w\|_{L^2(\mathbb{R}^n)}^{\frac{n}{4}}\right)^2\,\|\nabla v\|_{L^2}\\ &= \|w\|_{L^2(\mathbb{R}^n)}^{\frac{4-n}{2}}\,\|\nabla w\|_{L^2(\mathbb{R}^n)}^{\frac{n}{2}}\,\|\nabla v\|_{L^2}\end{aligned}$$

Since we can absorb an $\|\nabla w\|_{L^2(\mathbb{R}^n)}^2$-term to the left-hand side, Young's inequality $ab \le \varepsilon a^{\frac{4}{n}} + C(\varepsilon)b^{\frac{4}{4-n}}$

$$\begin{aligned}|\langle w\cdot\nabla v, w\rangle| &\lesssim \|w\|_{L^4}^2\,\|\nabla v\|_{L^2}^2\\ &\le \frac{1}{2}\|\nabla w\|_{L^2(\mathbb{R}^n)}^2 + C\left(\|w\|_{L^2(\mathbb{R}^n)}^{\frac{4-n}{2}}\,\|\nabla v\|_{L^2}\right)^{\frac{4}{4-n}}\\ &= \frac{1}{2}\|\nabla w\|_{L^2(\mathbb{R}^n)}^2 + C\,\|w\|_{L^2(\mathbb{R}^n)}^2\,\|\nabla v\|_{L^2}^{\frac{4}{4-n}}.\end{aligned}$$

The argument in the proof of Proposition X.4.2 thus will lead to an estimate

$$\partial_t\|w\|_{L^2}^2 + c\|\nabla w\|_{L^2}^2 \lesssim C\,\|w\|_{L^2(\mathbb{R}^n)}^2\,\|\nabla v\|_{L^2}^{\frac{4}{4-n}}$$

And Grönwall's inequality, Lemma IX.4.2 will lead to the analogue of (X.4.1),

$$\|w(t)\|_{L^2}^2 \lesssim \underbrace{\|w(0)\|_{L^2}^2}_{=0}\exp\left(\int_0^t \|\nabla u(s)\|_{L^2}^{\frac{4}{4-n}}\,ds\right)$$

What is the problem? The problem is that in order to conclude that $w \equiv 0$ we need to ensure that

$$\int_0^t \|\nabla u(s)\|_{L^2}^{\frac{4}{4-n}} ds < \infty$$

For $n = 2$ this follows from $u \in L^2(0,T;H^1)$. For $n = 3$ we would need $u \in L^4(0,T;H^1)$ (which we don't have).

We see that for $n = 2$ the Navier-Stokes equation is *critical* because the "nonlinearity" is controlled by $L^2(0,T;H^1)$ which is what the $\partial_t - \Delta$-leading order term controls. For $n = 3$ the energy is *supercritical* because the nonlineary needs to be controlled $L^4(0,T;H^1)$ while the parabolic operator only provides $L^4(0,T;H^1)$.

Notice that essentially all our argument relied on scaling (scaling invariant Sobolev inequalities, Gagliardo-Nirenberg) etc. – One can flip through the spaces, i.e. replace H^1 with something else, but the scaling argument will always lead to a problem for $n = 3$.

[**Tao, 2016**] calls this the "super-criticality" barrier for the the Navier-Stokes problem in 3D, Conjecture X.1.1, – and he argued that in order to solve the Navier-Stokes problem in the positive one should need more than just sharp embedding theorems to rule out singularities – and if one wants to solve the Navier-Stokes problem in the negative, one can make use of that supercriticality.

We see in the above argument the difference between 2D and 3D. This might become more clearer via the (instead of the div-curl estimate):

X.5. Some Regularity-type theory for $n = 2$ (global), $n = 3$ (domains)

To illustrate the differences in two dimensions and three dimensions we show some improved regularity arguments for the following two cases

Definition X.5.1. For $n = 2, 3$ we treat the case

- $\Omega \subsetneq \mathbb{R}^n$ smoothly bounded set
- $\Omega = \mathbb{R}^2$.

First we observe (we essentially did this before in the motivation for weak solution) that Lemma X.2.13 implies some initial estimates.

Proposition X.5.2. *Let $n = 2, 3$, Ω as in Definition X.5.1.*

Assume for almost every $t \in (0,T)$ the map $u \in L^\infty((0,T), L^2_{\sigma,n}) \cap L^2((0,T), H^1_0(\Omega))$ satisfies

$$-\langle u(t), \partial_t \varphi\rangle + \langle \nabla u(t), \nabla \varphi\rangle = \langle u(t) \cdot \nabla u(t), \varphi\rangle$$

for all $\varphi \in C_c^\infty(\Omega, \mathbb{R}^n)$ with $\operatorname{div}(\varphi) = 0$.

Then

$$\sup_t \|u(t)\|_{L^2} + \int_0^t \|\nabla u(s)\|_{L^2}^2 \, ds \lesssim \|u(0)\|_{L^2(\Omega)}$$

Proof. If $\Omega \subsetneq \mathbb{R}^n$ is bounded (with smooth boundary), we observe that (in a distributional sense) $\partial_t u$ exists and belongs to $H_0^1(\Omega)^*$, because thats what is true for $\langle \nabla u(t), \nabla \varphi \rangle$ and $\langle u(t) \cdot \nabla u(t), \varphi \rangle$. If $\Omega = \mathbb{R}^2$ this is still true by Corollary X.2.11. See Lemma X.2.13.

Since moreover $u \in H_0^1(\Omega)$ we can approximate it by $u_k \in C_c^\infty(\Omega)$ with respect to the H^1-norm. By Helmholtz decomposition, Theorem X.2.3

$$u_k = \varphi_k + \nabla g_k$$

with $\varphi_k \in L^2_{\sigma,n}(\Omega)$ and $g_k \in H^1(\Omega)$.

By density we can use φ_k as a testfunction,

$$-\langle \partial_t u(t), \varphi_k \rangle + \langle \nabla u(t), \nabla \varphi_k \rangle = \langle u(t) \cdot \nabla u(t), \varphi_k \rangle$$

Observe that g_k satisfies

$$\begin{cases} \Delta g_k = \operatorname{div}(u_k) & \text{in } \Omega, \\ \partial_\nu g_k = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\operatorname{div}(u_k) \to 0$ in $L^2(\Omega)$, by a reflection argument $\nabla g_k \to 0$ in $W^{1,2}(\Omega)$. By density, thus

$$\int_\Omega u \cdot \nabla \Delta g_k = 0,$$

moreover

$$\int_\Omega \partial_t u \cdot \nabla g_k = 0.$$

and either by div-curl ($n = 2$, global) or boundedness (Poincare),

$$\langle u \cdot \nabla u, g_k \rangle \xrightarrow{k \to \infty} 0.$$

Thus, we can test with u_k, and in the limit with u, and find

$$\partial_t \|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 = 0.$$

Integrating this we have

$$\|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(s)\|_{L^2}^2 ds \leq \|u(0)\|_{L^2}.$$

□

When passing to higher order estimates the pressure comes into play. Observe that the equation

$$\partial_t u - \Delta u + u \cdot \nabla u = \nabla p$$

implies (together with $\operatorname{div}(u) = 0$) that

$$\Delta p = \operatorname{div}(u \cdot \nabla u) = \partial_{\alpha\beta}(u^\alpha u^\beta).$$

The problem is that we have no boundary data given for p.

In $\mathbb{R}^2$ we can argue as follows: by CLMS

$$\int_{\mathbb{R}^2} \nabla p \cdot \nabla \varphi = \int_{\mathbb{R}^2} \partial_\alpha u^\beta \partial_\beta u^\alpha \varphi \lesssim \|\nabla u\|_{L^2}^2 \|\nabla \varphi\|_{L^2}.$$

By duality we obtain (using implicitely that p decays at infinity which can be deduced essentially from the fact that $u \in L^2$)

$$\|\nabla p\|_{L^2(\mathbb{R}^2)} \lesssim \|\nabla u\|_{L^2(\mathbb{R}^2)}^2. \tag{X.5.1}$$

For $n = 3$, in a bounded domain, it is more complicated (see [**Robinson et al., 2016**, Theorem 5.7]), in this case (cf. [**Robinson et al., 2016**, Theorem 6.12]) we assume a priori that the following formal estimate holds:

$$\|\nabla p\|_{L^2(\Omega)} \lesssim \|u\|_{L^4\Omega} \|\nabla u\|_{L^4(\Omega)} \lesssim \|\nabla u\|_{L^4(\Omega)}. \tag{X.5.2}$$

Proposition X.5.3. *Let $n = 2, 3$, Ω as in Definition X.5.1.*

Assume for almost every $t \in (0, T)$ the map $u \in L^\infty((0, T), L^2_{\sigma,n}) \cap L^2((0, T), H^1_0(\Omega))$ satisfies

$$-\langle u(t), \partial_t \varphi\rangle + \langle \nabla u(t), \nabla \varphi\rangle = \langle u(t) \cdot \nabla u(t), \varphi\rangle + \langle \nabla p, \varphi\rangle$$

for all $\varphi \in C_c^\infty(\Omega, \mathbb{R}^n)$ (without the assumption $\operatorname{div}(\varphi) = 0$) for p satisfying (X.5.1) (for $n = 2$) or (X.5.2) for $n = 3$.

If moreover

$$\|u(0)\|_{L^2} + \|\nabla u(0)\|_{L^2} < \varepsilon$$

then

$$\sup_t \|\nabla u(t)\|_{L^2}^2 + \int_0^t \|\nabla^2 u(s)\|_{L^2}^2 \, ds \lesssim C(\|u\|_{H^2})$$

Proof. Test with Δu (here we loose that Δu is divergence free as in the previous arguments, thats why we deal with ∇p), then

$$\left|\partial_t \|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2\right| \leq |\langle (u \cdot \nabla) u, \Delta u\rangle| + |\langle \nabla u, \Delta u\rangle|$$

<u>In $n = 3$</u> but a bounded domain we get

$$|\langle (u \cdot \nabla) u, \Delta u\rangle| \leq \|u\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2}$$

Now we use the so-called Agmon's inequality (this clearly holds only in a bounded set), cf. Proposition X.4.2,

$$\begin{aligned}\|u\|_{L^\infty(\Omega)} \leq & C \|\nabla u\|_{L^2(\Omega)}^{\frac{1}{2}} \|u\|_{H^2(\Omega)}^{\frac{1}{2}} \lesssim \|u\|_{L^2} \|\nabla u\|_{L^2}^{\frac{1}{2}} + \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla^2 u\|_{L^2}^{\frac{1}{2}} \\ \lesssim & \|\nabla u\|_{L^2}^{\frac{3}{2}} + \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\end{aligned}$$

From this we find with the help of Youngs inequality etc. for $p > 4$

$$\partial_t \|\nabla u\|_{L^2}^2 \leq C \|\nabla u\|_{L^2}^p - c \|\Delta u\|_{L^2}^2. \tag{X.5.3}$$

For $n = 3$ and on bounded domains we can estimate by Sobolev (not Poincare)

$$\|\Delta u\|_{L^2}^2 \gtrsim \|\nabla u\|_{L^2},$$

so we have

$$\partial_t \|\nabla u\|_{L^2}^2 \leq C \|\nabla u\|_{L^2}^2 \left(\|\nabla u\|_{L^2}^{p-2} - c\right)$$

Since the initial data is small, $\partial_t \|\nabla u\|_{L^2} \leq 0$, thus

$$\sup_{t>0} \|\nabla u\|_{L^2}^2(t) \leq \|\nabla u(0)\|_{L^2}.$$

Using this estimate we also obtain by integrating (X.5.3) again using Proposition X.5.2

$$\int_0^T \|\Delta u\|_{L^2}^2 \leq C(u(0)).$$

<u>For $n = 2$</u> but on $\mathbb{R}^2$ we use CLMS and integration by parts to obtain

$$|\langle (u \cdot \nabla) u, \Delta u \rangle| \leq |\langle (u \cdot \nabla) \partial_\alpha u, \partial_\alpha u \rangle| + |\langle (\partial_\alpha u \cdot \nabla) u, \partial_\alpha u \rangle| \lesssim \|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2} + \|u\|_{L^2} \|\nabla^2 u\|_{L^2}^2.$$

In view of (X.5.1) we have the same estimate for

$$|\langle \nabla p, \Delta u \rangle| \lesssim \|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2}.$$

By Proposition X.5.2 and the smallness condition we have

$$\|u\|_{L^2} \leq \varepsilon,$$

so we can absorb as in the $3D$-case and find

$$\left|\partial_t \|\nabla u\|_{L^2}^2 + c\|\Delta u\|_{L^2}^2\right| \leq C\|\nabla u\|_{L^2}^4. \tag{X.5.4}$$

Now we can argue as in the 3D case to get the claim. □

CHAPTER 11

Short introduction to Calderon-Zygmund Theory

Calderon-Zygmund theory is the L^p-regularity theory for elliptic equations. For example assume that u solves

$$\Delta u = \partial_\beta f \quad \text{in } \mathbb{R}^n$$

and that $f \in L^p(\mathbb{R}^n)$, $p \in (1, \infty)$. We would like to conclude that $\nabla u \in L^p(\mathbb{R}^n)$ with the estimates

$$\|\nabla u\|_{L^p} \lesssim \|f\|_{L^p}.$$

Example: Assume that $u \in W^{1,2}$ solves

$$\Delta u = \partial_\alpha u$$

then we would like to conclude that u is smooth and a classical solution. Observe that $u \in L^{2^*}$ by Sobolev embedding so we would like to conclude that $\nabla u \in L^{2^*}$ and then bootstrap our way to smoothness of u.

This theory is closely connected with harmonic analysis and *Calderon-Zygmund operators*. Denote by $I^2 = (-\Delta)^{-1}$ the *Riesz potential* (we called this Newton potential before) (we assume for simplicity that $n \geq 3$), we have the formula (I.2.4)

$$I^2 g(x) = c \int_{\mathbb{R}^n} |x-y|^{2-n} \, g(y) \, dy.$$

Then,

$$\partial_\alpha u = \partial_\alpha \Delta^{-1} \Delta u = \partial_\alpha \Delta^{-1} \partial_\beta f.$$

Computing the derivative we find that

$$\partial_\alpha \Delta^{-1} \partial_\beta f(x) = c \int_{\mathbb{R}^n} \frac{\frac{(x-y)^\alpha}{|x-y|} \frac{(x-y)^\beta}{|x-y|}}{|x-y|^n} \, f(y) \, dy. \tag{XI.0.1}$$

We will see below that this operator is an *Calderon-Zygmund operator* which as such is a bounded linear operator from L^p to L^p, namely

$$\|\partial_\alpha \Delta^{-1} \partial_\beta f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \quad \forall f \in L^p(\mathbb{R}^n), p \in (1, \infty).$$

From this we obtain immmediately that

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

In the following we make these statements precise.

XI.1. Calderon-Zygmund operators

The typical Calderón-Zygmund operator is the *Riesz transform*

$$\mathcal{R}_\alpha f(x) := c\int_{\mathbb{R}^n} \frac{\frac{(x-y)^\alpha}{|x-y|}}{|x-y|^n} f(y)dy.$$

One can compute that the *Fourier symbol* of $\mathcal{R}_\alpha$ is $i\frac{\xi^\alpha}{|\xi|}$, $\alpha = 1, \dots, n$, i.e.

$$(\mathcal{R}_\alpha f)^\wedge(\xi) = i\frac{\xi^\alpha}{|\xi|} f^\wedge(\xi).$$

In particular we have that $\partial_\alpha\partial_\beta f = c\mathcal{R}_\alpha\mathcal{R}_\beta\Delta f$, which is what we used in (XI.0.1).

Observe that the symbol of $\mathcal{R}_\alpha$ belongs to $L^\infty(\mathbb{R}^n)$,

$$\left\| i\frac{\xi^\alpha}{|\xi|} \right\|_{L^\infty} \leq 1.$$

It is easy to show that such an operator is bounded on L^2:

Lemma XI.1.1. *Let $m \in L^\infty(\mathbb{R}^n)$ and define*

$$Tf := (m(\xi)f^\wedge(\xi))^\vee.$$

Then T is a linear bounded operator on $L^2(\mathbb{R}^n)$ with

$$\|Tf\|_{L^2(\mathbb{R}^n)} \lesssim \|m\|_{L^\infty}\|f\|_{L^2(\mathbb{R}^n)}.$$

Such a T is usually called a multiplier operator, and m is the symbol.

Proof. By Plancherel identity, $\|g\|_{L^2(\mathbb{R}^n)} = \|g^\wedge\|_{L^2(\mathbb{R}^n)}$. Thus,

$$\|Tf\|_{L^2(\mathbb{R}^n)} = \|(Tf)^\wedge\|_{L^2(\mathbb{R}^n)} = \|m\, f^\wedge\|_{L^2(\mathbb{R}^n)} \leq \|m\|_{L^\infty}\, \|f^\wedge\|_{L^2} = \|m\|_{L^\infty}\, \|f\|_{L^2}.$$

□

Observe that we cannot simply replace L^2 with L^p in Lemma XI.1.1, since there is no Plancherel identity on L^p for $p \neq 2$.

Theorem XI.1.2 (Boundedness of Calderon-Zygmund-Operators). *Let $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be a bounded linear operator, which for $f \in C_c^\infty(\mathbb{R}^n)$ can be written as*

$$Tf(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy$$

(in a principle value sense). If moreover, the kernel Ω satisfies

- *$\Omega : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$ is bounded, $\|\Omega\|_{L^\infty} < \infty$*
- *Ω is homogeneous of order 0, i.e. $\Omega(rz) = \Omega(z)$ for all $r > 0$, $z \in \mathbb{R}^n\backslash\{0\}$.*
- *$\Omega : \mathbb{R}^n\backslash\{0\}$ is Lipschitz with the bound $\sup_{z\in\mathbb{R}^n\backslash\{0\}} |z||\nabla\Omega(z)| < \infty$*

then T is[1] *a bounded linear operator from $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$.*

Lemma XI.1.3. *Let $f \in L^2(B(r))$ then for all $c \in \mathbb{R}$,*

$$\|f - (f)_{B(r)}\|_{L^2(B(r))} \leq \|f - c\|_{L^2(B(r))}.$$

Proof. Exercise! □

We are not proving Theorem XI.1.2 in its full generality, but only for the case we need (a relatively easy adaptation of the following does the job).

Proposition XI.1.4. . *For a monomial p of degree k let*

$$Tf := \int_{\mathbb{R}^n} \frac{p(x-y)}{|x-y|^{n+k}} f(y)\, dy$$

Then T is a linear bounded operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$, and for $f \in L^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,

$$[Tf]_{BMO} \lesssim \|f\|_{L^\infty(\mathbb{R}^n)}.$$

Here

$$[g]_{BMO} = \sup_{B(x,\rho)} \left(\fint_{B(x,\rho)} |g - (g)_{B(x,\rho)}|^2 \right)^{\frac{1}{2}}.$$

Proof. The L^2-boundedness follows from Lemma XI.1.1.

For $x_0 \in \mathbb{R}^n$ and $r > 0$ let $f_{r,x_0}(x) := f(x_0 + rx)$. Observe that by the structure of T,

$$T(f_{r,x_0})(x) = (Tf)(x_0 + rx).$$

This implies that

$$\fint_{B(0,1)} T(f_{r,x_0}) = \fint_{B(x_0,r)} Tf,$$

and

$$\fint_{B(0,1)} \left| T(f_{r,x_0}) - (T(f_{r,x_0}))_{B(0,1)} \right|^2 = \fint_{B(x_0,r)} \left| T(f) - (T(f)_{B(x_0,r)} \right|^2,$$

and

$$\|f_{r,x_0}\|_{L^\infty(\mathbb{R}^n)} = \|f\|_{L^\infty(\mathbb{R}^n)}.$$

Thus, if we can show that for any $f \in L^2 \cap L^\infty(\mathbb{R}^n)$

$$\fint_{B(0,1)} |Tf - (Tf)_{B(0,1)}|^2 \lesssim \|f\|^2_{L^\infty(\mathbb{R}^n)},$$

then the full claim follows via scaling and translation.

Now let

$$f := f_1 + f_2,$$

[1] more precisely: extends to

with $f_1 = \chi_{B(0,2)} f$ and $f_2 = \chi_{\mathbb{R}^n \setminus B(0,2)} f$. Then,

$$\begin{aligned}\fint_{B(0,1)} |Tf - (Tf)_{B(0,1)}|^2 \lesssim & \fint_{B(0,1)} |Tf_1 - (Tf_1)_{B(0,1)}|^2 + \fint_{B(0,1)} |Tf_2 - (Tf_2)_{B(0,1)}|^2 \\ \lesssim & 2\fint_{B(0,1)} |Tf_1|^2 + \fint_{B(0,1)} |Tf_2 - (Tf_2)_{B(0,1)}|^2.\end{aligned}$$

Observe that by the L^2-boundedness, Lemma XI.1.1,

$$\fint_{B(0,1)} |Tf_1|^2 \lesssim \|Tf_1\|^2_{L^2(\mathbb{R}^n)} \lesssim \|f_1\|^2_{L^2(B(0,2))} = \|f\|^2_{L^2(B(0,2))} \lesssim \|f\|^2_{L^\infty(\mathbb{R}^n)}.$$

Now in view of Lemma XI.1.3,

$$\fint_{B(0,1)} |Tf_2 - (Tf_2)_{B(0,1)}|^2 \lesssim \fint_{B(0,1)} |Tf_2 - Tf_2(0)|^2 \tag{XI.1.1}$$

Now,

$$Tf_2(x) - Tf_2(0) = \int_{\mathbb{R}^n} \left(\frac{p(x-y)}{|x-y|^{n+k}} - \frac{p(-y)}{|y|^{n+k}} \right) f_2(y)\, dy = \int_{\mathbb{R}^n \setminus B(0,2)} \left(\frac{p(x-y)}{|x-y|^{n+k}} - \frac{p(y)}{|y|^{n+k}} \right) f(y)\, dy$$

If $x \in B(0,1)$ and $y \notin B(0,2)$ then $|x-y| \approx |y| \gtrsim 1$. In this case we obtain from the fundamental theorem of calculus that

$$\left(\frac{p(x-y)}{|x-y|^{n+k}} - \frac{p(-y)}{|y|^{n+k}} \right) \lesssim \frac{|x|}{|x-y|^{n+1}}.$$

Consequently,

$$|Tf_2(x) - Tf_2(0)| \lesssim |x| \int_{\mathbb{R}^n \setminus B(0,2)} |x-y|^{-n-1} |f(y)|\, dy \lesssim \|f\|_{L^\infty(\mathbb{R}^n)} \int_{|x-y| \gtrsim 1} |x-y|^{-n-1}\, dy.$$

Observe that

$$\sup_x \int_{|x-y| \gtrsim 1} |x-y|^{-n-1}\, dy < \infty.$$

So we have shown that

$$\sup_{x \in B(0,1)} |Tf_2(x) - Tf_2(0)| \lesssim \|f\|_{L^\infty(\mathbb{R}^n)}.$$

which together with (XI.1.1) implies

$$\fint_{B(0,1)} |Tf_2 - (Tf_2)_{B(0,1)}|^2 \lesssim \|f\|_{L^\infty(\mathbb{R}^n)}.$$

Thus we have shown

$$\fint_{B(0,1)} |Tf - (Tf)_{B(0,1)}|^2 \lesssim \|f\|_{L^\infty(\mathbb{R}^n)},$$

which by the scaling argument leads to the claim. □

Why are we happy about Proposition XI.1.4? Because BMO represents "almost L^∞", and we have

THEOREM XI.1.5. *Let $1 \leq p < \infty$ and T be a linear operator of "strong (p,p)-type", meaning that*

$$\|Tf\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \quad \forall f \in L^p(\mathbb{R}^n),$$

and bounded from L^∞ to BMO, i.e

$$[Tf]_{BMO} \lesssim \|f\|_{L^\infty(\mathbb{R}^n)} \quad \forall f \in L^\infty(\mathbb{R}^n),$$

Then for any $q \in [p, \infty)$, T maps $L^q(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ with

$$\|Tf\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^q(\mathbb{R}^n)} \quad \forall f \in L^q(\mathbb{R}^n).$$

PROOF. [**Giaquinta and Martinazzi, 2012**, Theorem 6.29] □

PROOF OF THEOREM XI.1.2. Observe that T is bounded from L^2 to L^2, Lemma XI.1.1, and from L^∞ to BMO, Proposition XI.1.4, and thus by Theorem XI.1.5 for any $p \in [2, \infty)$ we have

$$\|Tf\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}.$$

For $p < 2$ we argue by duality. Observe that by Riesz Representation Theorem

$$\|Tf\|_{L^p(\mathbb{R}^n)} = \sup_{\|g\|_{L^{p'}(\mathbb{R}^n)}} \int_{\mathbb{R}^n} Tf\, g = \sup_{\|g\|_{L^{p'}(\mathbb{R}^n)}} \int_{\mathbb{R}^n} f\, T^* g \lesssim \|f\|_{L^p(\mathbb{R}^n)} \|T^* g\|_{L^{p'}(\mathbb{R}^n)}.$$

Now observe that T^* is of the same type of operator, so we have for any $q \in [2, \infty)$ we have

$$\|T^* g\|_{L^q(\mathbb{R}^n)} \leq C_p \|g\|_{L^q(\mathbb{R}^n)}.$$

Since for $p < 2$ we have that $q := p' > 2$ this concludes the proof. □

Remark XI.1.6. There is another, older, way, using the Calderon-Zygmund decomposition and an L^1-L^1-weak type estimate to obtain Theorem XI.1.2.

XI.2. $W^{1,p}$-theory for the Laplace equation

THEOREM XI.2.1. *Let $\Omega_1 \subset\subset \Omega \subset\subset \mathbb{R}^n$ be two smoothly bounded domains, and let $p \geq 2$. Assume that for some $f \in L^p(\Omega)$ there is $u \in W^{1,2}(\Omega)$ that satisfies in distributional sense*

$$\Delta u = \partial_\alpha f \quad \text{in } \Omega$$

Then

$$\|\nabla u\|_{L^p(\Omega_1)} \leq C(\Omega_1, \Omega, p) \left(\|f\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)} \right).$$

Remark XI.2.2.
- The L^2-norm for u on the right-hand side is necessary, since otherwise $f = 0$ would imply that u is constant (which is false without the assumption of appropriate boundary values).
- This statement holds for more general equations, e.g. $\partial_i(A_{ij}\partial_j u) = \partial_\alpha f$, if A is smooth enough (the sharp assumption being VMO, [**Iwaniec and Sbordone, 1998**])

- This is an interior statement, but it holds up to the boundary: for example if

$$\begin{cases} \Delta u = \partial_\alpha f & \text{in } \Omega_2 \\ u = 0 & \text{on } \partial\Omega_2 \end{cases}$$

then $\|\nabla u\|_{L^p(\Omega_2)} \leq \|f\|_{L^p(\Omega_2)}$; Cf. [**Giaquinta and Martinazzi, 2012**, Chapter 7].

The proof of Theorem XI.2.1 follows a sequence of cutoff arguments, such as the following

Lemma XI.2.3. *For $p, q \in [2, \infty)$ assume that $u \in W^{1,p}(\Omega)$ satisfies for some $f \in L^q(\Omega)$*

$$\Delta u = \partial_\alpha f \quad \text{in } \Omega$$

Let $\eta \in C_c^\infty(\Omega)$. Then for $v := \eta u$ we have

$$\Delta v = \tilde{g} + \partial_\alpha \tilde{f} \quad \text{in } \mathbb{R}^n$$

with

$$\|\tilde{f}\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^q(\Omega)},$$

and for $1 \leq r \leq \min\{p, q\}$ we have

$$\|\tilde{g}\|_{L^r(\mathbb{R}^n)} \lesssim \|f\|_{L^q(\Omega)} + \|u\|_{W^{1,p}(\Omega)}$$

Moreover $\tilde{f}$ and $\tilde{g}$ have compact support, so does v and we have

$$\|v\|_{W^{1,p}(\mathbb{R}^n)} \lesssim \|u\|_{W^{1,p}(\Omega)}.$$

All the constants depend on η.

Proof.

$$\Delta v = \underbrace{(\Delta\eta)u}_{L^p(\mathbb{R}^n)} + \underbrace{2\nabla\eta \cdot \nabla u}_{L^p(\mathbb{R}^n)} - \underbrace{(\partial_\alpha \eta)\, f}_{L^q(\mathbb{R}^n)} + \partial_\alpha(\underbrace{\eta f}_{L^q(\mathbb{R}^n)})$$

□

Moreover we use the following global result:

Proposition XI.2.4. *Assume that $v \in W^{1,p}(\mathbb{R}^n)$,*

$$\Delta v = g + \partial_\alpha f \quad \text{in } \mathbb{R}^n$$

with $f \in L^q(\mathbb{R}^n)$, $g \in L^r(\mathbb{R}^n)$ all with compact support.

Then for $1 < \sigma \leq q$ and if $r < n$ additionally $\sigma \leq \frac{nr}{n-r}$

$$\|\nabla v\|_{W^{1,\sigma}(\mathbb{R}^n)} \lesssim C(f, g).$$

PROOF. By the compact support of v we only need to estimate ∇v (the rest follows from Poincaré). By the boundedness of the Riesz transform,

$$\|\nabla v\|_{L^\sigma(\mathbb{R}^n)} \lesssim \|(-\Delta)^{\frac{1}{2}} v\|_{L^\sigma(\mathbb{R}^n)}.$$

Now we estimate for any $\Phi \in C_c^\infty(\operatorname{supp}(v), \mathbb{R}^n)$ with $\|\Phi\|_{L^{\sigma'}(\mathbb{R}^n)} \leq 1$ the expression

$$\int_{\mathbb{R}^n} \nabla v \cdot \Phi$$

Set

$$\varphi := I^1 \mathcal{R}_\beta \Phi^\beta.$$

By the compact support of Φ one can show that $\varphi \in L^{\sigma'}(\mathbb{R}^n)$, by the boundedness of the Riesz transforms $\nabla\varphi \in L^{\sigma'}(\mathbb{R}^n)$. Moreover $\nabla\varphi \in L^{\tilde{\sigma}}(\mathbb{R}^n)$ for any $\tilde{\sigma} \in (1, \sigma')$ by boundedness of Riesz transform and the compact support of Φ. In particular $\varphi \in W^{1,\sigma'}(\mathbb{R}^n) \equiv W_0^{1,\sigma'}(\mathbb{R}^n)$, and

$$\int_{\mathbb{R}^n} \nabla v \cdot \Phi = \int_{\mathbb{R}^n} (-\Delta)^{\frac{1}{2}} v \, (-\Delta)^{\frac{1}{2}} \varphi = c \int_{\mathbb{R}^n} \nabla v \, \nabla\varphi$$

the last equality can be seen by Fourier transform. From the equation of v we get

$$\begin{aligned}\int_{\mathbb{R}^n} \nabla v \, \nabla\varphi = &\int_{\mathbb{R}^n} f \partial_\alpha \varphi + g\varphi \\ &\lesssim \|f\|_{L^\sigma(\mathbb{R}^n)} \|\partial_\alpha \varphi\|_{L^{\sigma'}(\mathbb{R}^n)} + \|g\|_{L^r(\mathbb{R}^n)} \|\varphi\|_{L^{r'}(\mathbb{R}^n)}.\end{aligned}$$

Since $\sigma \leq q$, using compact support of f,

$$\|f\|_{L^\sigma(\mathbb{R}^n)} \|\partial_\alpha \varphi\|_{L^{\sigma'}(\mathbb{R}^n)} \lesssim \|f\|_{L^q(\mathbb{R}^n)} \underbrace{\|\varphi\|_{W^{1,\sigma'}(\mathbb{R}^n)}}_{\leq 1}.$$

As for the second term, if $r < n$ then since $\sigma < \frac{nr}{n-r}$ we have that $\frac{n\sigma'}{n-\sigma'} > r'$ and by Sobolev embedding (since also $\nabla\varphi \in W^{1,\tilde{\sigma}}(\mathbb{R}^n)$ for any $\tilde{\sigma} \in (1, \sigma')$)

$$\|g\|_{L^r(\mathbb{R}^n)} \|\varphi\|_{L^{r'}(\mathbb{R}^n)} \lesssim \|g\|_{L^r(\mathbb{R}^n)} \underbrace{\|\varphi\|_{W^{1,\sigma'}(\mathbb{R}^n)}}_{\lesssim 1}.$$

We have shown that for any $\Phi \in C_c^\infty(\operatorname{supp} v, \mathbb{R}^n)$, $\|\Phi\|_{L^{\sigma'}(\mathbb{R}^n)} \leq 1$ we have

$$\int_{\mathbb{R}^n} \nabla v \cdot \Phi \lesssim \left(\|g\|_{L^r(\mathbb{R}^n)} + \|f\|_{L^q(\mathbb{R}^n)} \right).$$

By duality/Riesz representation theorem this means that $\nabla v \in L^\sigma(\mathbb{R}^n)$ and

$$\|\nabla v\|_{L^\sigma(\mathbb{R}^n)} \lesssim \left(\|g\|_{L^r(\mathbb{R}^n)} + \|f\|_{L^q(\mathbb{R}^n)} \right).$$

Since v has compact support, by Poincaré inequality, we finally obtain

$$\|v\|_{W^{1,\sigma}(\mathbb{R}^n)} \lesssim \left(\|g\|_{L^r(\mathbb{R}^n)} + \|f\|_{L^q(\mathbb{R}^n)} \right).$$

□

Proof of Theorem XI.2.1. Let $\Omega \supset\supset \Omega_1 \supset\supset \Omega_2 \supset\supset \dots$ and take $\eta_i \in C_c^\infty(\Omega_i)$ with $\eta \equiv 1$ in Ω_{i+1}.

Following Lemma XI.2.3 and Proposition XI.2.4 we obtain that

$$\eta_1 u \in W^{1,\sigma_1}(\Omega_1)$$

where we take $\sigma_1 \leq p$ and $\sigma_1 \leq \frac{2n}{n-2}$ (if $n \geq 3$). If we can take $\sigma_1 = p$ we are done, otherwise we observe that $\sigma_1 = \frac{2n}{n-2} > 2$. We then repeat the argument for $\eta_2\eta_1 u$: from Lemma XI.2.3 and Proposition XI.2.4 we then obtain

$$\eta_2\eta_1 u W^{1,\sigma_2}(\mathbb{R}^n)$$

for $\sigma_2 \leq p$ and $\sigma_2 \leq \frac{\sigma_1 n}{n-\sigma_1}$ (if $\sigma_1 < n$). Again, either we can choose $\sigma_2 = p$ or $\sigma_2 = \frac{\sigma_1 n}{n-\sigma_1}$.

In this way we obtain a sequence

$$v_k := \eta_k\eta_{k-1}\dots\eta_1 u \in W^{1,\sigma_k}(\mathbb{R}^n)$$

where

$$\sigma_k = \begin{cases} p & \text{if } \sigma_{k-1} < n \text{ or } p \leq \frac{\sigma_k n}{n-\sigma_k} \\ \frac{\sigma_{k-1} n}{n-\sigma_{k-1}} & \text{else.} \end{cases}$$

This sequence terminates after finitely many steps. Indeed, let

$$\tilde{\sigma}_k := \begin{cases} \frac{\tilde{\sigma}_{k-1} n}{n-\tilde{\sigma}_{k-1}} & \text{if } \sigma_{k-1} < n \\ \infty & \text{otherwise.} \end{cases}$$

The sequence $\tilde{\sigma}_k$ is increasing, $\tilde{\sigma}_k \geq \tilde{\sigma}_{k-1}$ and strictly increasing unless $\sigma_k = n$. The only possibility that $\tilde{\sigma}_k$ is not ∞ after finitely many steps k, is that $\sigma_k < n$ for all k – then we have a monotone, bounded sequence which has a limit $\tilde{\sigma} \leq n$, which has to satisfy

$$\tilde{\sigma} = \frac{\tilde{\sigma} n}{n - \tilde{\sigma}}$$

There is no positive, finite solution to this equation. Contradiction. So $\tilde{\sigma}_k$ is infinite for $k \geq K$ for some K, which means that $\sigma_k = p$ for $k \geq K$.

That is, we have shown that $v_K \in W^{1,p}(\mathbb{R}^n)$, and since $v_K \equiv u$ in Ω_K we have $u \in W^{1,p}(\mathbb{R}^n)$ in Ω_K.

The number K is independent of the equation, it just depends on the dimension and p, so if we choose Ω_K well, then we get the claim. $\square$

CHAPTER 12

Short introduction to Viscosity solutions

Cf. Koike

- There is no C^1-solution to the equation
$$\begin{cases} |u'(t)|^2 = 1 & t \in (-1,1) \\ u(-1) = u(1) = 0. \end{cases}$$
- If we consider a.e. solutions $u \in C^{0,1}$ there are many.
$$u_1(t) = \begin{cases} 1+t & t \in (-1,0) \\ 1-t & t \in [0,1). \end{cases}$$
or
$$u_2(t) = -u_1(t)$$
but also really any saw-tooth-shaped function is a solution.
- In particular there are sequences of solutions approximating the function $u \equiv 0$ (which in no way is a solution)

u_1 and u_2 are special solutions, they are in some sense maximal. Why do we choose u_1 over u_2? well this is a choice of convexity we make (and for our purposes we will want to make things concave).

In order to define notion a solution one can use the vanishing viscosity method: one adds a viscosity term $-\varepsilon u''$ to the equation, i.e.
$$\begin{cases} -\varepsilon u_\varepsilon''(t) + |u_\varepsilon'(t)|^2 = 0 & t \in (-1,1) \\ u_\varepsilon(-1) = u_\varepsilon(1) = 0. \end{cases}$$
There is one unique solution to this equation, and it is
$$u_\varepsilon(x) = -\varepsilon \log\left(\frac{\cosh(x/\varepsilon)}{\cosh(1/\varepsilon)}\right).$$
One can show that $u_\varepsilon \xrightarrow{\varepsilon \to 0} 1 - |x|$ uniformly in $[-1,1]$.

We repeat this argument for general second-order PDEs

(XII.0.1) $$F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega.$$