University of Pittsburgh
Armin Schikorra (armin@pitt.edu)

Partial Differential Equations 1 – Spring 2019
Exercise Sheet 4 — Due Date: February 25

Work in groups, write in LaTeX!

Problem 11 Let $u \in C^1(\overline{\Omega})$. Denote by ν the outwards facing unit normal of Ω.

Assume that $u = 0$ on $\partial\Omega$. Show that then

$$Du = 0 \quad \text{on } \partial\Omega \quad \Leftrightarrow \quad \partial_\nu u = 0 \quad \text{on } \partial\Omega.$$

Problem 12 Extend Theorem 2.2. and Theorem 2.24 to the following equation

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

for $f \in C^0(\overline{\Omega})$, $\Omega \subset\subset \mathbb{R}^n$ with smooth boundary.

Here Δ^2 is the bi-Laplace operator,

$$\Delta^2 u = \Delta(\Delta u) = \sum_{i=1}^n \sum_{j=1}^n \partial_{x_i x_i} \partial_{x_j x_j} u.$$

More precisely,

(i) Find an energy $\mathcal{E}(u)$ and a set of permissible functions $X \subset C^4(\Omega) \cap C^1(\overline{\Omega})$ such that a minimizer of $\mathcal{E}$ is a solution to the PDE (1) and so that a solution to the PDE is a minimizer

(ii) Show uniqueness, i.e. if there are two solutions $u, v \in C^4(\Omega) \cap C^1(\overline{\Omega})$ of the PDE (1) then $u \equiv v$.

Problem 13 Show that for the equation

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

no uniqueness result holds and explain why the proof of Theorem 2.24 fails.

More precisely show

(i) if u solves (2) then for v a solution to

$$\begin{cases} \Delta v = 1 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

the for any $\lambda \in \mathbb{R}$ function

$$w := u + \lambda v$$

still solves (2).

(ii) Where does the proof of Theorem 2.24 fail?
