Exercise (5.1.3)**.** *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Suppose that there exists a sequence of partitions $\{P_k\}$ of $[a, b]$ such that*

$$\lim_{k\to\infty} (U(P_k, f) - L(P_k, f)) = 0.$$

Show that f is Riemann integrable and that

$$\int_a^b f = \lim_{k\to\infty} U(P_k, f) = \lim_{k\to\infty} L(P_k, f).$$

Proof. Given $\varepsilon > 0$, by assumption there exists a $N \in \mathbb{N}$ such that when $k \geq N$, $U(P_k, f) - L(P_k, f) < \varepsilon$. Specifically, $U(P_N, f) - L(P_N, f) < \varepsilon$. Then,

$$0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f \leq U(P_N, f) - L(P_N, f) < \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, we must have

$$\overline{\int_a^b} f - \underline{\int_a^b} f = 0 \qquad \text{or} \qquad \overline{\int_a^b} f = \underline{\int_a^b} f.$$

Therefore, f is Riemann integrable.

Next,we show that $\int_a^b f = \lim_{k\to\infty} U(P_k, f)$. Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that when $k \geq N$, $U(P_k, f) - L(P_k, f) < \varepsilon$. Then, when $k \geq N$

$$\begin{aligned} \left| U(P_k, f) - \int_a^b f \right| &= U(P_k, f) - \int_a^b f \\ &= U(P_k, f) - \underline{\int_a^b} f \\ &\leq U(P_k, f) - L(P_k, f) < \epsilon. \end{aligned}$$

Hence, we have that $\int_a^b f = \lim_{k\to\infty} U(P_k, f)$.

Finally, since $\lim_{k\to\infty} (U(P_k, f) - L(P_k, f)) = 0$ and $\lim_{k\to\infty} U(P_k, f) = \int_a^b f$,

$$\begin{aligned} \lim_{k\to\infty} L(P_k, f) &= \lim_{k\to\infty} [U(P, k) - (U(P_k, f) - L(P_k, f))] \\ &= \lim_{k\to\infty} U(P, k) - \lim_{k\to\infty} (U(P_k, f) - L(P_k, f)) \\ &= \int_a^b f - 0 \\ &= \int_a^b f. \end{aligned}$$

□

Exercise (5.1.6)**.** *Let* $c \in (a, b)$ *and let* $d \in \mathbb{R}$*. Define* $f : [a, b] \to \mathbb{R}$ *as*

$$f(x) := \begin{cases} d & \text{if } x = c, \\ 0 & \text{if } x \neq c. \end{cases}$$

Prove that $f \in \mathscr{R}[a, b]$ *and compute* $\int_a^b f$ *using the definition of the integral (and propositions of the section).*

Proof. Without loss of generality, assume that $d > 0$.

Let $\varepsilon > 0$ be given. Let $0 < \beta < \min\{c - a, b - c, \varepsilon/2d\}$.

Let $P := \{a, c - \beta, c + \beta, b\}$. Then,

$$\begin{aligned} U(P, f) - L(P, f) &= \sum_{j=1}^{3} (M_j - m_j)\Delta x_j \\ &= (0 - 0)\Delta x_1 + (d - 0)\Delta x_2 + (0 - 0)\Delta x_3 \\ &= d(2\beta) \\ &= 2d\beta \\ &< \varepsilon. \end{aligned}$$

Therefore, by the Cauchy criterion, f is integrable.

Next, for any $\varepsilon > 0$, if we define P as above, we have

$$0 = L(P, f) \leq \int_a^b f \leq U(P, f) < \varepsilon.$$

Since ε was arbitrary, this gives that $\int_a^b f = 0$. □

Exercise (5.2.1)**.** *Let* f *be in* $\mathscr{R}[a, b]$*. Prove that* $-f$ *is in* $\mathscr{R}[a, b]$ *and*

$$\int_a^b -f(x)\, dx = -\int_a^b f(x)\, dx.$$

Proof. Note that for any set $A \subseteq \mathbb{R}$, $\sup(-A) = -\inf A$ and $\inf(-A) = -\sup A$, where $-A = \{-a : a \in A\}$.

Let $P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$ be any partition of $[a, b]$, and let $I_k = [x_{k-1}, x_k]$ denote the kth subinterval created by the partition. Then, using the above fact, we have that for $k = 1, \ldots, n$,

$$\begin{aligned} U(P, -f) &= \sum_{k=1}^{n} \sup_{I_k}(-f(x))\Delta x_k \\ &= \sum_{k=1}^{n} -\inf_{I_k} f(x)\Delta x_k \\ &= -\sum_{k=1}^{n} \inf_{I_k} f(x)\Delta x_k \\ &= -L(P, f) \end{aligned}$$

Similarly, $L(P, -f) = -U(P, f)$.

Hence,

$$\begin{aligned} U(P, -f) - L(P, -f) &= -L(P, f) - (-U(P, f)) \\ &= U(P, f) - L(P, f) \end{aligned}$$

Since f is integrable, from problem 5.1.3, there is a sequence of partitions $(P_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}(U(P_k, f) - L(P_k, f)) = 0$. Then, for the same sequence,

$$\lim_{k\to\infty}(U(P_k, -f) - L(P_k, -f)) = \lim_{k\to\infty}(U(P_k, f) - L(P_k, f)) = 0$$

and so $-f$ is integrable if and only if f is integrable.

Finally, since f and $-f$ are both integrable we have that

$$\begin{aligned} \int_a^b -f(x)\,dx &= \underline{\int_a^b} - f(x)\,dx \\ &= \sup\{L(P, -f) : P \text{ is a partition of } [a, b]\} \\ &= \sup\{-U(P, f) : P \text{ is a partition of } [a, b]\} \\ &= -\inf\{U(P, f) : P \text{ is a partition of } [a, b]\} \\ &= -\overline{\int_a^b} f(x)\,dx \\ &= -\int_a^b f(x)\,dx. \end{aligned}$$

□

Exercise (5.2.4)**.** *Prove the* mean value theorem for integrals*. That is, prove that if $f : [a, b] \to \mathbb{R}$ is continuous, then there exists a $c \in [a, b]$ such that $\int_a^b f = f(c)(b - a)$.*

Proof. By the Min-Max Theorem, f achieves both an absolute maximum M and an absolute minimum m in $[a, b]$, say at points c_1 and c_2. Without loss of generality, $c_1 < c_2$. Then we have that

$$m(b - a) \leq \int_a^b f \leq M(b - a).$$

or

$$m \leq \frac{1}{(b - a)} \int_a^b f \leq M.$$

By the Intermediate Value Theorem, there must be a point $c \in (c_1, c_2) \subset [a, b]$ such that

$$f(c) = \frac{1}{(b - a)} \int_a^b f.$$

Therefore, for this c,

$$\int_a^b f = f(c)(b - a).$$

□

Exercise (5.2.6)**.** *Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function and $\int_a^b f = 0$. Prove that there exists a $c \in [a, b]$ such that $f(c) = 0$.*

Proof. Apply Exercise 5.2.4 with $\int_a^b f = 0$ to obtain the conclusion. □