1. (a) Show that there exists at least one solution to $x - \cos(x) = 0$.

   (b) $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim_{x\to\infty} f(x) = M_+$ and $\lim_{x\to-\infty} = M_-$ for some $M-, M_+ \in \mathbb{R} \cup \{\infty, -\infty\}$. Then for any $L \in (M_-, M_+)$ there exists $x \in \mathbb{R}$ with $f(x) = L$.

2. Prove the following statements

   (a) If $f : D \to \mathbb{R}$ is uniformly continuous, then f is continuous.

   (b) The converse is false (give a counterexample)

3. We showed in the lecture that continuous functions on closed finite intervals $[a, b]$ are continuous.

   Repeat the the proof from the lecture to show that any continuous map $f : D \to \mathbb{R}$ is actually uniformly continuous, if D is a compact set.

   A set $A \subset \mathbb{R}$ is (sequentially) compact if any sequence $(x_n)_{n\in\mathbb{N}} \subset A$ has a converging subsequence $(x_{n_i})_{i\in\mathbb{N}}$ with $\lim_{i\to\infty} x_{n_i} = x \in A$.

4. Let $f : [a, b] \to \mathbb{R}$ be continuous and assume that f has a local maximum at $c \in [a, b]$. Assume that f is differentiable at c. Show that

   (a) If $c \in (a, b)$ then $f'(c) = 0$

   (b) If $c = a$ then $f'(c) \leq 0$

   (c) If $c = b$ then $f'(c) \geq 0$

   (d) What can we say about the derivatives if f has a local minimum at c?

5. Let

$$f(x) = \begin{cases} 0 & |x| \geq 1 \\ e^{\frac{1}{x^2-1}} & |x| < 1. \end{cases}$$

   (a) Show that f is a differentiable function in $\mathbb{R}$

   (b) Show that $f^{(k)}$, the k-th derivative of f, is still differentiable for any $k \in \mathbb{N}$.

   Hint: You can use without proof that any function $g : [a, b] \to \mathbb{R}$ which is continuous on $[a, b]$ and differentiable on $[a, b]\backslash\{c\}$ but whose derivative has a limit $\lim_{x\to c} g'(x) = L$ is actually differentiable.