Exercise (3.2.10)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Suppose that for all rational numbers r, $f(r) = g(r)$. Show that $f(x) = g(x)$ for all x.*

Proof. We want to show that $f(x) = g(x)$ for all $x \in \mathbb{R}$. We already have that $f(r) = g(r)$ if $r \in \mathbb{Q}$. Let x be an arbitrary irrational number. If we can show that $f(x) = g(x)$ we are done.

Since $x \in \mathbb{R}$, there exists a sequence $\{r_n\} \subset \mathbb{Q}$ such that $r_n \to x$ as $n \to \infty$. Both f and g are continuous, and so

$$f(x) = \lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} g(r_n) = g(x).$$

□

Exercise (3.3.4)**.** *Let*

$$f(x) := \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f has the intermediate value property. That is, for any $a < b$, if there exists a y such that $f(a) < y < f(b)$ or $f(b) < y < f(a)$, then there exists $c \in (a, b)$ such that $f(c) = y$.

Proof. Let $a < b$ and assume that there exists a y such that $f(a) < y < f(b)$ or $f(b) < y < f(a)$. Note that $f(a), f(b) \in [-1, 1]$ and so $-1 < y < 1$.

Case 1 $0 \in (a, b)$. There exists $t \in (a, b)$ such that $t > 0$ and $f(t) = f(a)$. Since $f\big|_{(t,b)}$ is continuous and $f(t) < y < f(b)$ or $f(b) < y < f(t)$, $\exists\, c \in (t, b) \subset (a, b)$ such that $f(c) = y$ by the Intermediate Value Theorem.

Case 1 $0 \notin (a, b)$. Then $f\big|_{(a,b)}$ is continuous and $\exists\, c \in (a, b)$ such that $f(c) = y$ by the Intermediate Value Theorem. □

Exercise (3.3.7)**.** *Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function. Prove that the direct image $f([a, b])$ is a closed and bounded interval or a single number.*

Proof. If $f([a, b])$ is a single number, we are done. So, suppose otherwise. By the Min-Max Theorem, f attains both an absolute maximum and an absolute minimum. Suppose that the minimum occurs at $c_1 \in [a, b]$ and the maximum at $c_2 \in [a,b]$. Then, $f(c_1) \le f(x) \le f(c_2)$, for all $x \in [a,b]$. I.e. $f([a,b]) \subseteq [f(c_1), f(c_2)]$.

Without loss of generality, suppose that $c_1 < c_2$. Note that the restriction of f to $[c_1, c_2]$ is continuous. Therefore, by the Intermediate Value Theorem, for any y such that $f(c_1) < y < f(c_2)$, there is a $c \in (c_1, c_2)$ such that $f(c) = y$. Therefore, $f([a,b]) = [f(c_1), f(c_2)]$, which is a closed and bounded interval. □

Exercise (3.4.3)**.** *Show that* $f : (c, \infty) \to \mathbb{R}$ *for some* $c > 0$ *and defined by* $f(x) := 1/x$ *is Lipschitz continuous.*

Proof. Let $K := 1/c^2$. Then, for any $x, y \in (c, \infty)$,

$$|f(x) - f(y)| = |1/x - 1/y| = \frac{|y-x|}{xy} \le \frac{|x-y|}{c^2} = K\,|x-y|.$$

Hence, f is Lipschitz continuous on (c, ∞). □

Exercise (3.4.4)**.** *Show that* $f : (0, \infty) \to \mathbb{R}$ *defined by* $f(x) := 1/x$ *is not Lipschitz continuous.*

Proof. Assume for contradiction that $\exists\, K \in \mathbb{R}$ such that $|f(x)-f(y)| \le K\,|x-y|$ for all $x, y \in (0, \infty)$. Then, for all $x, y \in (0, \infty)$,

$$\begin{aligned}
&|1/x - 1/y| \le K\,|x-y| \\
\Rightarrow &|1/x - 1| \le K\,|x-1| \qquad \text{for all } x \in (0,\infty) \\
\Rightarrow &\frac{|1/x - 1|}{|x-1|} \le K \qquad \text{for all } x \in (0,\infty) \\
\Rightarrow &\frac{|1-x|}{|x(x-1)|} \le K \qquad \text{for all } x \in (0,\infty) \\
\Rightarrow &\frac{1}{x} \le K \qquad \text{for all } x \in (0,\infty)
\end{aligned}$$

This is clearly a contradiction, and hence f is not Lipschitz continuous on $(0, \infty)$. □

Alternatively, you can show that $f(x) = 1/x$ is not uniformly continuous on $(0, \infty)$ and hence cannot be Lipschitz continuous.

Exercise. *A function* $f : D \subset \mathbb{R} \to \mathbb{R}$ *is called* (sequentially) lower semicontinuous *at a point* $x \in D$ *if we have*

$$f(x) \leq \liminf_{D \ni y \to x} f(y),$$

in the sense that for any sequence $(y_n)_{n\in\mathbb{N}} \subset D$ *with* $\lim_{n\to\infty} y_n = x$ *we have*

$$f(x) \leq \liminf_{n\to\infty} f(y_n).$$

In a similar spirit, a function is called (sequentially) upper semicontinuous *if*

$$f(x) \geq \limsup_{D \ni y \to x} f(y).$$

(a) Give an example of a lower semicontinuous function which is not continuous.

(b) Give an example of an upper semicontinuous function which is not continuous.

(c) Show that f *is continuous at* $x \in D$ *if and only if* f *is lower and upper semicontinuous at* x.

(d) Show that if $f : [a, b] \to \mathbb{R}$ *is lower semicontinuous in every* $x \in [a, b]$*, then* f *attains its minimum value in* $[a, b]$.

(e) Show that if $f : [a, b] \to \mathbb{R}$ *is upper semicontinuous in every* $x \in [a, b]$*, then* f *attains its maximum value in* $[a, b]$.

Note: proofs of parts (d) and (e) will be given after the take-home exam.

Proof. (a)

$$f(x) := \begin{cases} -1 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

is lower semicontinuous but not continuous (not continuous at 0).

(b)

$$f(x) := \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

is upper semicontinuous but not continuous (not continuous at 0).

(c) Assume that f is continuous at $x \in D$. For any sequence $(y_n) \subset D$ such that $y_n \to x$ as $n \to \infty$, $f(y_n) \to f(x)$ and hence

$$\liminf_{n\to\infty} f(y_n) = f(x) = \limsup_{n\to\infty} f(y_n).$$

Therefore, f is lower and upper semicontinuous at x.

Next assume that f is lower and upper semicontinuous at x. Then for any sequence $(y_n) \subset D$ such that $y_n \to x$,

$$\limsup_{n\to\infty} f(y_n) \le f(x) \le \liminf_{n\to\infty} f(y_n).$$

However, since $\liminf_{n\to\infty} f(y_n) \le \limsup_{n\to\infty} f(y_n)$, we must have

$$\limsup_{n\to\infty} f(y_n) = f(x) = \liminf_{n\to\infty} f(y_n)$$

and so $\lim_{n\to\infty} f(y_n) = f(x)$. Since (y_n) was arbitrary, this shows that f is continuous at x.

$\square$