

Exercise (3.1.9). Let c_1 be a cluster point of $A \subset \mathbb{R}$ and c_2 be a cluster point of $B \subseteq \mathbb{R}$. Suppose that $f : A \rightarrow B$ and $g : B \rightarrow \mathbb{R}$ are functions such that $f(x) \rightarrow c_2$ as $x \rightarrow c_1$ and $g(y) \rightarrow L$ as $y \rightarrow c_2$. Let $h(x) := g(f(x))$ and show $h(x) \rightarrow L$ as $x \rightarrow c_1$.

Proof. Given any $\epsilon > 0$, choose $\hat{\delta} > 0$ such that whenever $y \in B$ and $0 < |y - c_2| < \hat{\delta}$ we have that $|g(y) - L| < \epsilon$.

Next, since $\hat{\delta}$ is a positive number, we can choose $\delta > 0$ such that whenever $x \in A$ and $0 < |x - c_1| < \delta$ we have $|f(x) - c_2| < \hat{\delta}$.

Therefore we have, given any $\epsilon > 0$, found $\delta > 0$ such that whenever $x \in A$ and $0 < |x - c_1| < \delta$ we have $|g(f(x)) - L| < \epsilon$ (since $|f(x) - c_2| < \hat{\delta}$). \square

Exercise (3.2.1). Using the definition of continuity directly, prove that $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) := x^2$ is continuous.

Proof. Let $c \in \mathbb{R}$ be arbitrary. Given $\epsilon > 0$, choose $\delta = \min \left\{ 1, \frac{\epsilon}{1 + 2|c|} \right\}$.

Suppose that $x \in \mathbb{R}$ such that $|x - c| < \delta$. Then, since $\delta \leq 1$,

$$|x| \leq |x - c + c| \leq |x - c| + |c| < \delta + |c| \leq 1 + |c|.$$

Then, we have

$$\begin{aligned} |x^2 - c^2| &= |x - c||x + c| \leq |x - c|(|x| + |c|) \\ &< \delta(|x| + |c|) \\ &< \delta(1 + 2|c|) \\ &\leq \frac{\epsilon}{1 + 2|c|}(1 + 2|c|) \\ &= \epsilon \end{aligned}$$

Therefore, $f(x) = x^2$ is continuous at c . Since c was arbitrary, f is continuous. \square

Exercise (3.2.3). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ by defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational.} \end{cases}$$

Using the definition of continuity directly prove that f is continuous at 1 and discontinuous at 2.

Proof. First, we show that f is continuous at 1.

Let $\epsilon > 0$ be given. Choose $\delta = \min\{1, \epsilon/3\}$.

Let $x \in \mathbb{R}$ be such that $|x - 1| < \delta$. (Note that this implies that $0 < x < 2$.) Then,

$$\begin{aligned} |f(x) - f(1)| &= |f(x) - 1| = \begin{cases} |x - 1| & \text{if } x \in \mathbb{Q} \\ |x^2 - 1| & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \\ &\leq \begin{cases} |x - 1| & \text{if } x \in \mathbb{Q} \\ (|x| + 1)|x - 1| & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \\ &\leq \begin{cases} |x - 1| & \text{if } x \in \mathbb{Q} \\ 3|x - 1| & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \\ &\leq 3|x - 1| \\ &< 3 \cdot \delta \leq 3 \left(\frac{\epsilon}{3}\right) = \epsilon. \end{aligned}$$

Hence f is continuous at 1.

Next we show that f is not continuous at 2.

Let $\epsilon = 1$. Let δ be any positive number. Let $n \in \mathbb{N}$ be such that

$$\frac{\sqrt{2}}{\delta} < n \iff \frac{\sqrt{2}}{n} < \delta.$$

Let $x = 2 + \sqrt{2}/n$. Then $x \in \mathbb{R} \setminus \mathbb{Q}$ and $|x - 2| < \delta$. We also have that,

$$\begin{aligned} |f(x) - f(2)| &= |f(x) - 2| = |x^2 - 2| \\ &= |(2 + \sqrt{2}/n)^2 - 2| \\ &= 2 + \frac{4\sqrt{2}}{n} + \frac{2}{n^2} \\ &> 2 \\ &> 1 = \epsilon. \end{aligned}$$

Hence, f is not continuous at 2. □

Exercise (3.2.4). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Is f continuous? Prove your assertion.

Proof. We claim that f is not continuous at 0, and therefore not continuous.

Let $x_n := \frac{1}{(2n-1)\frac{\pi}{2}} = \frac{2}{(2n-1)\pi}$. Then $\{x_n\}$ converges to 0, but

$$\begin{aligned} f(x_n) &= \sin(1/x_n) = \sin\left(\frac{(2n-1)\pi}{2}\right) \\ &= (-1)^{n+1} \end{aligned}$$

which (famously) doesn't converge. Therefore f is not continuous at 0. \square

Exercise (3.2.5). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Is f continuous? Prove your assertion.

Proof. We will show that f is continuous at all real numbers.

Note that since $1/x$, $\sin(x)$, and x are all continuous on their respective domains, $f(x)$ is continuous at all nonzero numbers. (This is due to Propositions 3.2.5 and 3.2.7).

Next, note that $f(0) = 0$, and $\lim_{x \rightarrow 0} f(x) = 0$ by Example 3.1.8. Hence f is continuous at 0. \square

Exercise (3.2.9). Give an example of functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ such that the function h defined by $h(x) := f(x) + g(x)$ is continuous, but f and g are not continuous. Can you find f and g that are nowhere continuous, but h is a continuous function?

Solution. For a first example, consider

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases} \quad g(x) = \begin{cases} 1 & \text{if } x < 0 \\ -1 & \text{if } x \geq 0. \end{cases}$$

For an example for the second part of this problem, consider

$$f(x) = \begin{cases} -1 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases} \quad g(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational.} \end{cases}$$

\square

Exercise (3.2.11). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Suppose that $f(c) > 0$. Show that there exists an $\alpha > 0$ such that for all $x \in (c - \alpha, c + \alpha)$ we have $f(x) > 0$.

Proof. Let $\epsilon = f(c) > 0$. Since f is continuous, there exists $\delta > 0$ such that when $|x - c| < \delta$, $|f(x) - f(c)| < \epsilon$. In other words, when $x \in (c - \delta, c + \delta)$,

$$-\epsilon < f(x) - f(c) < \epsilon.$$

Let $\alpha = \delta$. Then, when $x \in (c - \alpha, c + \alpha)$, we have

$$-f(c) < f(x) - f(c) < f(c).$$

Adding $f(c)$ to both sides of the left-hand inequality, we have that for all $x \in (c - \alpha, c + \alpha)$, $f(x) > 0$. \square

Exercise. Show that any continuous map $f : \mathbb{R} \rightarrow \mathbb{Z}$ is constant.

Proof. An easy way to prove this is to use the Intermediate Value Theorem and proof by contradiction. However, we can also prove it by the definition (still using proof by contradiction).

Suppose for contradiction that f is not constant.

Let $c \in \mathbb{R}$ and $\epsilon = 1$. Since f is continuous at c , $\exists \delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < 1$. In other words, if $x \in (c - \delta, c + \delta)$, then $f(x) = f(c)$ since the codomain of f is \mathbb{Z} .

Let A be the set of all δ with the above property. I.e.

$$A = \{\delta > 0 : x \in (c - \delta, c + \delta) \Rightarrow f(x) = f(c)\}.$$

Since f is not constant, A is bounded above. Let $s = \sup A$. Since f is continuous at $c - s$, $\exists \gamma_1 > 0$ such that $|x - (c - s)| < \gamma_1 \Rightarrow |f(x) - f(c - s)| < 1$. In other words, $f(x) = f(c - s) = f(c)$ for all $x \in ((c - s) - \gamma_1, (c - s) + \gamma_1)$.

Also, since f is continuous at $c + s$, $\exists \gamma_2 > 0$ such that $|x - (c + s)| < \gamma_2 \Rightarrow |f(x) - f(c + s)| < 1$. Again, we can phrase this as $f(x) = f(c + s) = f(c)$ for all $x \in ((c + s) - \gamma_1, (c + s) + \gamma_1)$.

If we let $\gamma := \min\{\gamma_1, \gamma_2\}$, this gives that $s + \gamma \in A$. This is a contradiction since $s = \sup A$. Hence, we must have that f is constant. \square

Exercise. Recall the notion of open sets $A \subset \mathbb{R}$.

$$A \subset \mathbb{R} \text{ is open} \Leftrightarrow \forall x_0 \in A : \exists \epsilon > 0 : (x_0 - \epsilon, x_0 + \epsilon) \subset A.$$

Show the following. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Then the following are equivalent

1. $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous.
2. the inverse f^{-1} maps open sets into open sets. That is: whenever $A \subset \mathbb{R}$ is an open set, then the $f^{-1}(A)$ defined as

$$f^{-1}(A) \equiv \{x \in \mathbb{R} : f(x) \in A\}$$

is an open set.

Proof. We start by showing $1 \Rightarrow 2$. Assume f is continuous. Let $A \subset \mathbb{R}$ be an open set and let $c \in f^{-1}(A)$. Since A is open and $f(c) \in A$, $\exists \epsilon > 0$ such that $(f(c) - \epsilon, f(c) + \epsilon) \subset A$. Since f is continuous at c , $\exists \delta > 0$ such that $x \in (c - \delta, c + \delta) \Rightarrow |f(x) - f(c)| < \epsilon$. In other words, $(c - \delta, c + \delta) \subset f^{-1}(A)$, and so $f^{-1}(A)$ is an open set.

Next we show $2 \Rightarrow 1$. Let $c \in \mathbb{R}$ and let $\epsilon > 0$ be given. Since $A = (f(c) - \epsilon, f(c) + \epsilon)$ is an open set, $f^{-1}(A)$ is open. Next, since $c \in f^{-1}(A)$, $\exists \delta > 0$ be such that $(c - \delta, c + \delta) \subset f^{-1}(A)$. I.e. $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$. Therefore f is continuous at c . Since c was arbitrary, f is continuous. \square