Exercise (2.4.1)**.** *Prove that* $\left(\frac{n^2-1}{n^2}\right)$ *is Cauchy using directly the definition of Cauchy sequences.*

Proof. Given $\epsilon > 0$, let $M \in \mathbb{N}$ be such that $\sqrt{\frac{2}{\epsilon}} < M$.

Then, for any $m, n \geq M$,

$$\begin{aligned}
|x_m - x_n| &= \left|\frac{m^2-1}{m^2} - \frac{n^2-1}{n^2}\right| \\
&= \left|\frac{1}{n^2} - \frac{1}{m^2}\right| \\
&\leq \frac{1}{n^2} + \frac{1}{m^2} \\
&\leq \frac{1}{M^2} + \frac{1}{M^2} \\
&= \frac{2}{M^2} \\
&< \epsilon.
\end{aligned}$$

Therefore, $\left(\frac{n^2-1}{n^2}\right)$ is a Cauchy sequence. □

Exercise (2.4.2)**.** *Let* $\{x_n\}$ *be a sequence such that there exists a* $0 < C < 1$ *such that*

$$|x_{n+1} - x_n| \leq C|x_n - x_{n-1}|.$$

Prove that $\{x_n\}$ *is Cauchy. Hint: You can freely use the formula (for* $C \neq 1$*)*

$$1 + C + C^2 + \cdots + C^n = \frac{1 - C^{n+1}}{1 - C}.$$

Proof. Let $\epsilon > 0$ be given. Note that

$$\begin{aligned}
&|x_3 - x_2| \leq C|x_2 - x_1| \\
&|x_4 - x_3| \leq C|x_3 - x_2| \leq C \cdot C|x_2 - x_1| = C^2|x_2 - x_1|
\end{aligned}$$

and in general, one could prove that

$$|x_{n+1} - x_n| \leq C|x_n - x_{n-1}| \leq C^2|x_{n-1} - x_{n-2}| \leq \cdots \leq C^{n-1}|x_2 - x_1|.$$

Now, for $m > n$, we can evaluate the quantity

$$\begin{aligned}
|x_m - x_n| &\le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\
&\le C^{m-2}|x_2 - x_1| + C^{m-3}|x_2 - x_1| + \cdots + C^{n-1}|x_2 - x_1| \\
&= (C^{m-2} + \cdots + C^{n-1})|x_2 - x_1| \\
&= C^{n-1}(1 + C + \cdots + C^{m-n-1})|x_2 - x_1| \\
&= C^{n-1}\left(\frac{1 - C^{m-n}}{1 - C}\right)|x_2 - x_1| \\
&\le C^{n-1}\left(\frac{1}{1 - C}\right)|x_2 - x_1|
\end{aligned}$$

Now, since $0 < C < 1$, $C^{n-1} \to 0$ as $n \to \infty$. Therefore, there exists $N \in \mathbb{N}$ such that whenever $n \ge N$,

$$|C^{n-1} - 0| < \frac{\epsilon}{\left(\frac{1}{1-C}\right)|x_2 - x_1|}.$$

For this same N, whenever $m > n \ge N$

$$\begin{aligned}
|x_m - x_n| &\le C^{n-1}\left(\frac{1}{1 - C}\right)|x_2 - x_1| \\
&< \frac{\epsilon}{\left(\frac{1}{1-C}\right)|x_2 - x_1|} \cdot \left(\frac{1}{1 - C}\right)|x_2 - x_1| \\
&= \epsilon.
\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence. □

Exercise. *Prove the following statement using Bolzano-Weierstrass theorem.*

Assume that $(x_n)_{n\in\mathbb{N}}$ *is a bounded sequence in* $\mathbb{R}$ *and that there exists* $x \in \mathbb{R}$ *such that any convergent subsequence* $(x_{n_i})_{i\in\mathbb{N}}$ *converges to* x*. Then* $\lim_{n\to\infty} x_n = x$.

Proof. Assume for contradiction that $x_n \not\to x$. Then $\exists \epsilon > 0$ such that $|x_n - x| \ge \epsilon$ for infinitely many n. From this, we can create a subsequence $\{x_{n_j}\}$ such that $|x_{n_j} - x| \ge \epsilon$ for all $j \in \mathbb{N}$.

Since our original sequence is bounded, this subsequence is bounded, and so, by Bolzano-Weierstrass, there is a convergent subsequence of this subsequence, $\{x_{n_{j_k}}\}$. By assumption, $\{x_{n_{j_k}}\}$ converges to x. However, this is a contradiction since $|x_{n_{j_k}} - x| \ge \epsilon$ for all $k \in \mathbb{N}$.

Hence, we must have $x_n \to x$ as $n \to \infty$. □

Exercise. *Show that*

a) the set $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ *has no cluster points.*

b) every point in $\mathbb{R}$ *is a cluster point of* $\mathbb{Q}$.

Proof. a) If $x \in \mathbb{Z}$, then $(x - 1/2, x + 1/2) \cap \mathbb{Z} \backslash \{x\} = \emptyset$ and so x is not a cluster point of $\mathbb{Z}$.

If $x \notin \mathbb{Z}$, then $\exists k \in \mathbb{Z}$ such that $x \in (k, k+1)$. Choose $\epsilon = \min\{|x-k|, |x-(k+1)|\}$. Then $(x-\epsilon, x+\epsilon) \cap \mathbb{Z} \backslash \{x\} = \emptyset$, and so x is not a cluster point of $\mathbb{Z}$.

Therefore $\mathbb{Z}$ has no cluster points.

b) Let $x \in \mathbb{R}$. Let $\epsilon > 0$. By the density of $\mathbb{Q}$, $\exists r \in \mathbb{Q}$ such that $x < r < x + \epsilon$. Then $(x-\epsilon, x+\epsilon) \cap \mathbb{Q} \backslash \{x\} \neq \emptyset$. Since ϵ was arbitrary, this shows that x is a cluster point of $\mathbb{Q}$. Since x was arbitrary, every point in $\mathbb{R}$ is a cluster point of $\mathbb{Q}$.

□

Exercise. *In the lecture we have shown*

$$\text{any Cauchy sequence } (x_n)_{n\in\mathbb{N}} \subset \mathbb{R} \text{ has a limit in } \mathbb{R}, \\ \text{i.e. there exists } x \in \mathbb{R} \text{ with } \lim_{n\to\infty} x_n = x. \tag{1}$$

The same statement is false in $\mathbb{Q}$, *the following is false:*

$$\text{any Cauchy sequence } (x_n)_{n\in\mathbb{N}} \subset \mathbb{Q} \text{ has a limit in } \mathbb{Q}, \\ \text{i.e. there exists } x \in \mathbb{Q} \text{ with } \lim_{n\to\infty} x_n = x. \tag{2}$$

a) Give a counterexample to (2).

b) Which part of the proof of (1) (from the lecture) fails when we attempt to prove (2)?

Proof. a) Define a sequence $\{x_n\}$ as follows. For all $n \in \mathbb{N}$, choose $x_n \in \mathbb{Q}$ such that $\sqrt{2} - 1/n < x_n < \sqrt{2}$. Then $\{x_n\}$ is a Cauchy sequence in $\mathbb{Q}$ that does not have a limit in $\mathbb{Q}$ (since the limit "should be" $\sqrt{2}$).

b) Depending on which proof you are looking at, you can either point out that the Bolzano-Weierstrass Theorem does not guarantee a subsequence that converges to some $x \in Q$

... or ...

$\limsup x_n$ and $\liminf x_n$ may not be members of $\mathbb{Q}$.

□