Exercise (2.1.3)**.** *Is the sequence* $\left\{\frac{(-1)^n}{2n}\right\}$ *convergent? If so, what is the limit?*

Answer: The sequence converges to 0.

Proof. Note that, for all $n \in \mathbb{N}$,

$$-\frac{1}{n} \leq \frac{(-1)^n}{2n} \leq \frac{1}{n}.$$

Since $\left\{\pm\frac{1}{n}\right\}$ converges to 0, $\left\{\frac{(-1)^n}{2n}\right\}$ converges to 0 by the Squeeze Theorem. □

Exercise (2.1.15)**.** *Let* $\{x_n\}$ *be a sequence defined by*

$$x_n := \begin{cases} n & \text{if } n \text{ is odd,} \\ 1/n & \text{if } n \text{ is even.} \end{cases}$$

a) Is the sequence bounded? (prove or disprove)

b) Is there a convergent subsequence? If so, find it.

Solution. a) $\{x_n\}$ is not bounded.

Proof. For any $M \in \mathbb{R}$, by the Archimedean property, $\exists n \in \mathbb{N}$ such that $n > M$. If n is odd, $x_n = n > M$. If n is even, $n+1$ is odd and $x_{n+1} = n+1 > M$. Hence, $\{x_n\}$ cannot be bounded. □

b) Let $n_i = 2i$ for all $i \in \mathbb{N}$. Then $\{x_{n_i}\}_i$=$\{1/2i\}_i$, which converges to zero. □

Exercise (2.2.7)**.** *True or false, prove or find a counterexample. If* (x_n) *is a sequence such that* (x_n^2) *converges, then* (x_n) *converges.*

Solution. This is false. One counterexample is given by $x_n := (-1)^n$. □

Exercise (2.3.7)**.** *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences.*

a) Show that $\{x_n + y_n\}$ *is bounded.*

b) Show that

$$(\liminf_{n\to\infty} x_n) + (\liminf_{n\to\infty} y_n) \le \liminf_{n\to\infty}(x_n + y_n).$$

c) Find explicit $\{x_n\}$ *and* $\{y_n\}$ *such that*

$$(\liminf_{n\to\infty} x_n) + (\liminf_{n\to\infty} y_n) < \liminf_{n\to\infty}(x_n + y_n).$$

Proof. a) Since $\{x_n\}$ and $\{y_n\}$ are bounded, there are numbers B_1 and B_2 such that for all $n \in \mathbb{N}$,

$$|x_n| \le B_1 \quad \text{and} \quad |y_n| \le B_2.$$

Then, for all $n \in \mathbb{N}$,

$$\begin{aligned} |x_n + y_n| &\le |x_n| + |y_n| \\ &\le B_1 + B_2. \end{aligned}$$

Hence, $\{x_n + y_n\}$ is a bounded sequence.

b) Note that, for any $n \in \mathbb{N}$ and $j \ge n$,

$$x_j + y_j \ge \inf\{x_k : k \ge n\} + \inf\{y_k : k \ge n\}.$$

Hence, $\inf\{x_k : k \ge n\} + \inf\{y_k : k \ge n\}$ is a lower bound of the set $\{x_k + y_k : k \ge n\}$. Therefore, for all $n \in \mathbb{N}$,

$$\inf\{x_k : k \ge n\} + \inf\{y_k : k \ge n\} \le \inf\{x_k + y_k : k \ge n\}.$$

Taking the limit as $n \to \infty$ of both sides of this inequality, we obtain

$$(\liminf_{n\to\infty} x_n) + (\liminf_{n\to\infty} y_n) \le \liminf_{n\to\infty}(x_n + y_n).$$

c) Let $\{x_n\} = \{(-1)^{n+1}\}$ and $\{y_n\} = \{(-1)^n\}$. Then $\{x_n + y_n\} = \{0\}$, for all $n \in \mathbb{N}$. Hence,

$$\begin{aligned} (\liminf_{n\to\infty} x_n) + (\liminf_{n\to\infty} y_n) &= (-1) + (-1) \\ &= -2 \\ &< 0 \\ &= \liminf_{n\to\infty}(x_n + y_n). \end{aligned}$$

□

Exercise. *Show that if $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence then every subsequence of $(x_n)_{n\in\mathbb{N}}$ is also convergent. Moreover if*

$$x := \lim_{n\to\infty} x_n$$

then for any subsequence $(x_{n_i})_i$,

$$x = \lim_{i\to\infty} x_{n_i}$$

Proof. Assume that (x_n) converges to x as $n \to \infty$. Let (x_{n_i}) be any subsequence.

Since (x_n) converges, given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow |x_n - x| < \epsilon$. Then, for any $i \in N$ such that $i \geq N$, since $n_i > i \geq N$, $|x_{n_i} - x| < \epsilon$. Hence, (x_{n_i}) also converges to x. □

Exercise. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence and assume one of the following properties:*

a) there is some x such that any subsequence $(x_{n_i})_i$ contains another subsequence $(x_{n_{i_j}})_j$ which is convergent to x.

b) any subsequence $(x_{n_i})_i$ contains another subsequence $(x_{n_{i_j}})_j$ which is convergent (a priori not necessarily to the same x)

Show in which cases $(x_n)_n$ is convergent. Give a counterexample for the other cases.

Solution. Assume that there is some x such that any subsequence $(x_{n_i})_i$ contains another subsequence $(x_{n_{i_j}})_j$ which converges to x.

If (x_n) is not convergent, there must be some $\epsilon > 0$ and a subsequence (x_{n_i}) such that

$$|x_{n_i} - x| > \epsilon \quad \forall i \in \mathbb{N}.$$

But then this $(x_{n_i})_i$ cannot have another subsequence $(x_{n_{i_j}})_j$ that converges to x, because still

$$|x_{n_{i_j}} - x| > \epsilon \quad \forall i \in \mathbb{N}.$$

This is a contradiction. Hence, $\lim_{n\to\infty} x_n = x$.

b) Assume that any subsequence $(x_{n_i})_i$ contains another subsequence $(x_{n_{i_j}})_j$ which is convergent (a priori not necessarily to the same x). In this case, (x_n) does not have to converge.

Counterexample: $x_n := (-1)^n$. Not convergent, but any subsequence has another subsequence which is convergent.

□

Exercise (bonus)**.** *Find the following limit. Show all work.*

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots+\frac{1}{\sqrt{n^2+2n}}\right)$$

Proof. Set

$$x_n := \left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots+\frac{1}{\sqrt{n^2+2n}}\right)$$

Then

$$\frac{2n}{\sqrt{n^2+2n}} \geq x_n \leq \frac{2n}{\sqrt{n^2+1}}$$

Now

$$\lim_{n\to\infty}\frac{2n}{\sqrt{n^2+2n}} = 2\lim_{n\to\infty}\frac{n}{n}\frac{1}{\sqrt{1+\frac{2}{n}}} = 2.$$

and

$$\lim_{n\to\infty}\frac{2n}{\sqrt{n^2+1}} = 2\lim_{n\to\infty}\frac{n}{n}\frac{1}{\sqrt{1+\frac{1}{n^2}}} = 2.$$

By the squeeze theorem

$$\lim_{n\to\infty} x_n = 2.$$

□