Exercise (1.2.10). *Let A and B be two nonempty bounded sets of nonnegative real numbers. Define the set $C := \{ab : a \in A, b \in B\}$. Show that B is a bounded set and that*

$$\sup C = (\sup A)(\sup B) \qquad \textit{and} \qquad \inf C = (\inf A)(\inf B)$$

Proof. First, let M be a bound for A and N a bound for B. Then for any $c = ab \in C$, $|c| = |ab| = |a||b| \leq MN$. Hence, C is bounded.

Next, let $a \in A$ and $b \in B$ be arbitrary, so that $c = ab$ is an arbitrary member of C. Since $c = ab \leq (\sup A)(\sup B)$, $(\sup A)(\sup B)$ is an upper bound of C, and therefore $\sup C \leq (\sup A)(\sup B)$.

Also, since $c = ab \in C$, $ab \leq \sup C$. Since all numbers are nonnegative, this gives that $a \leq \frac{\sup C}{b}$. Since a was arbitrary, $\frac{\sup C}{b}$ is an upper bound of A, and hence $\sup A \leq \frac{\sup C}{b}$.

Rearranging this equation gives $b \leq \frac{\sup C}{\sup A}$, and again, since b was an arbitrary member of B, $\frac{\sup C}{\sup A}$ is an upper bound of B. Hence, $\sup B \leq \frac{\sup C}{\sup A}$ or $(\sup A)(\sup B) \leq \sup C$.

Since we have $\sup C \leq (\sup A)(\sup B)$ and $(\sup A)(\sup B) \leq \sup C$, it must be true that $\sup C = (\sup A)(\sup B)$.

The proof of the statement involving infimums is similar.

□

Exercise (1.3.1). *Let $\epsilon > 0$. Show that $|x-y| < \epsilon$ if and only if $x-\epsilon < y < x+\epsilon$.*

Proof. Let $\epsilon > 0$. Using Proposition 1.3.1 (ii) and (v),

$$\begin{aligned} |x - y| < \epsilon &\Leftrightarrow |y - x| < \epsilon \\ &\Leftrightarrow -\epsilon < y - x < \epsilon \\ &\Leftrightarrow x - \epsilon < y < x + \epsilon \end{aligned}$$

□

Exercise (2.1.12)**.** *Let $S \subset \mathbb{R}$ be a nonempty bounded set. Then there exist monotone sequences (x_n) and (y_n) such that $x_n, y_n \in S$ and*

$$\sup S = \lim_{n\to\infty} x_n \qquad \textit{and} \qquad \inf S = \lim_{n\to\infty} y_n.$$

Proof. Let $a := \sup S$. By the definition of supremum, $a - 1$ is not an upper bound of S, and so there exists $y_1 \in S$ such that $a - 1 < y_1 \leq a$. Let $x_1 := y_1$.

Next, since $a - \frac{1}{2}$ is not an upper bound of S, there exists $y_2 \in S$ such that $a - \frac{1}{2} < y_2 \leq a$. Since we want to construct a monotone sequence, we let $x_2 := \max\{x_1, y_2\}$.

Continuing, since $a - \frac{1}{3}$ is not an upper bound of S, there exists $y_3 \in S$ such that $a - \frac{1}{3} < y_3 \leq a$. Let $x_3 := \max\{x_2, y_3\}$.

Continuing in this manner, we construct a monotone increasing sequence (x_n) such that for all $n \in \mathbb{N}$, $a - \frac{1}{n} < x_n \leq a$. By the Squeeze Lemma, (x_n) converges to a, as desired.

Note that we can construct a monotone decreasing sequence that converges to $\inf S$ in a similar way.

□

Exercise (2.2.9)**.** *Suppose that $\{x_n\}$ is a sequence and suppose that for some $x \in \mathbb{R}$, the limit*

$$L := \lim_{n\to\infty} \frac{|x_{n+1} - x|}{|x_n - x|}$$

exists and $L < 1$. Show that $\{x_n\}$ converges to x.

Proof. Let $\{x_n\}$ be a sequence and $x \in \mathbb{R}$. Define a sequence $\{y_n\}$ by $y_n = (x_n - x)$, for all $n \in \mathbb{N}$. Then, by our assumption

$$L := \lim_{n\to\infty} \frac{|y_{n+1}|}{|y_n|} = \lim_{n\to\infty} \frac{|x_{n+1} - x|}{|x_n - x|}$$

exists and $L < 1$. Hence, by the Ratio Test for sequences, $\{y_n\}$ converges to 0.

Note that this implies that

$$x_n = (x_n - x) + x$$

converges to $0 + x = x$, as desired. □