Exercise (6.1.7)**.** *Suppose that there exists a sequence of functions* $\{g_n\}$ *uniformly converging to* 0 *on* A*. Now suppose that we have a sequence of functions* $\{f_n\}$ *and a function* f *on* A *such that*

$$|f_n(x) - f(x)| \leq g_n(x)$$

for all $x \in A$*. Show that* $\{f_n\}$ *converges uniformly to* f *on* A*.*

Proof. Let $\varepsilon > 0$ be given. Since $g_n \to 0$ uniformly, there exists an $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |g_n(x)| < \varepsilon$ for all $x \in A$.

For this same N, when $n \geq N$,

$$|f_n(x) - f(x)| \leq |g_n(x)| < \varepsilon$$

for all $x \in A$. Therefore, $\{f_n\}$ converges uniformly to f on A. □

Exercise (6.2.2)**.** *Let* $f_n(x) := \dfrac{x^n}{n}$*. Show that* f_n *converges uniformly to a differentiable function* f *on* $[0,1]$*. However, show that* $f'(1) \neq \lim_{n\to\infty} f_n'(1)$*.*

Proof. Note that

$$\begin{aligned} ||f_n - 0||_u &= \sup_{x\in[0,1]} \left|\frac{x^n}{n}\right| \\ &\leq \frac{1}{n} \\ &\longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore, $\{f_n\}$ converges uniformly to the zero function.

However,

$$\lim_{n\to\infty} f_n'(1) = \lim_{n\to\infty} 1^{n-1} = 1 \neq 0 = f'(1).$$

□

Exercise (6.2.3)**.** *Let* $f : [0,1] \to \mathbb{R}$ *be a Riemann integrable (hence bounded) function. Find* $\lim_{n\to\infty} \int_0^1 \frac{f(x)}{n}\, dx$*.*

Proof. Define $f_n(x) = \dfrac{f(x)}{n}$. Let M be a bound for f on $[0,1]$.

Since $|f_n(x) - 0| \le \dfrac{M}{n} \to 0$ as $n \to \infty$ independent of x, $f_n \to 0$ uniformly on $[0,1]$. Therefore,

$$\lim_{n\to\infty} \int_0^1 \frac{f(x)}{n}\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 0\,dx = 0$$

□

Exercise (6.2.4). *Show* $\lim_{n\to\infty} \int_1^2 e^{-nx^2}\,dx = 0$.

Solution. Let $f_n(x) := e^{-nx^2}$ and note that $f_n'(x) = -2nxe^{-nx^2} < 0$ on $[1,2]$. Then we have that

$$\begin{aligned} ||f_n - 0||_u &= \sup_{x\in[1,2]} |f_n(x)| \\ &\le e^{-n} \longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore, $\{f_n\}$ converges uniformly to the zero function, and

$$\lim_{n\to\infty} \int_1^2 e^{-nx^2}\,dx = \int_1^2 0\,dx = 0.$$

□

Exercise (6.5.5). *(a) Show that*

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{1+n^2x}$$

defines a continuous function on $(0,\infty)$.

(b) Prove that $\lim_{x\to 0^+} f(x) = \infty$ *and* $\lim_{x\to\infty} f(x) = 1$.

Proof. (a) Fix $a > 0$. Then, on $[a,\infty)$, $\left|\dfrac{1}{1+n^2x}\right| \le \dfrac{1}{a}\cdot\dfrac{1}{n^2}$. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, the series $\displaystyle\sum_{n=0}^{\infty}\frac{1}{1+n^2x}$ converges uniformly by the M-Test.

Since each $f_n(x) := \dfrac{1}{1+n^2x}$ is continuous and the series converges uniformly, $f(x)$ is continuous on $[a,\infty)$. Since $a > 0$ was arbitrary, $f(x)$ is continuous on $(0,\infty)$.

(b) Note that $f(1/n) \geq n$. Therefore, $\lim_{x\to 0^+} f(x) = \infty$.

Also, note that

$$\begin{aligned} |f(x) - 1| = \left| \sum_{n=1}^{\infty} \frac{1}{1+n^2x} \right| &= \lim_{N\to\infty} \left| \sum_{n=1}^{N} \frac{1}{1+n^2x} \right| \\ &\leq \lim_{N\to\infty} \frac{1}{x} \sum_{n=1}^{N} \frac{1}{n^2} \\ &= \frac{1}{x} \sum_{n=1}^{\infty} \frac{1}{n^2} \end{aligned}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-test), this gives that $|f(x) - 1| \to 0$ as $x \to \infty$. I.e. $\displaystyle\lim_{x\to\infty} f(x) = 1$.

□