1. Exercises 5.3.8: Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Suppose that $\int_a^x f = \int_x^b$ for all $x \in [a, b]$. Show that $f(x) = 0$ for all $x \in [a, b]$.

Solution: Let $F(x) := \int_a^x f(t)\,dt$. By the Fundamental Theorem of Calculus, since f is continuous, F is differentiable and $F'(x) = f(x)$.

Since $\int_a^x f = \int_x^b$ for all $x \in [a, b]$, we have

$$F(x) = \int_a^b f - F(x),$$

or

$$F(x) = \frac{1}{2}\int_a^b f,$$

which means that F is a constant function.

Then $F'(x) = 0$ for all $x \in [a, b]$, i.e. $f(x) = 0$ for all $x \in [a, b]$.

2. 6.1.2:

   1. Find the pointwise limit $\dfrac{e^{x/n}}{n}$ for $x \in \mathbb{R}$.
   2. Is the limit uniform on $\mathbb{R}$?
   3. Is the limit uniform on $[0, 1]$?

Solution:

1. Let $f_n(x) := \dfrac{e^{x/n}}{n}$. For a fixed $x_0 \in \mathbb{R}$, $f_n(x_0) \longrightarrow 0$ as $n \to \infty$. Hence the pointwise limit is the zero function.
2. The limit is not uniform on $\mathbb{R}$. Let $\epsilon = 1$ and $n \in \mathbb{N}$ be arbitrary. Note that for $x = n \cdot \ln 2n$, $|f_n(x) - 0| = f_n(x) = 2 \geq \epsilon$.
3. The limit is uniform on $[0, 1]$. Note that in this case, each f_n and f are bounded functions. Furthermore,

$$\begin{aligned}||f_n - f||_u &= \sup_{x\in[0,1]} |f_n(x)| \\ &= \frac{e^{1/n}}{n} \quad \text{since } f_n \text{ is increasing on } [0,1] \\ &\longrightarrow 0 \text{ as } n \to \infty.\end{aligned}$$

3. 6.1.5: Suppose that $\{f_n\}$ and $\{g_n\}$ defined on some set A converge to f and g respectively uniformly on A. Show that $\{f_n + g_n\}$ converges uniformly to $f + g$ on A.

Solution: Let $\epsilon > 0$ be given.

Let $N \in \mathbb{N}$ be such that for all $n \geq N$,

$$|f_n(x) - f(x)| < \epsilon/2 \quad \text{and} \quad |g_n(x) - g(x)| < \epsilon/2$$

for all $x \in A$. (Why can we find N that works for both f and g?)

Then for all $n \geq N$, and for all $x \in A$,

$$\begin{aligned} |(f_n + g_n)(x) - (f+g)(x)| &= |(f_n(x) + g_n(x)) - (f(x) + g(x))| \\ &\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Hence, $\{f_n + g_n\}$ converges uniformly to $f + g$ on A.

4. 6.1.6: Find an example of sequences of functions $\{f_n\}$ and $\{g_n\}$ that converge uniformly to some f and g on some set A, but such that $\{f_n g_n\}$ does not converge uniformly to fg on A.

Solution: Let $A = \mathbb{R}$, and $f_n(x) = g_n(x) = x + \dfrac{1}{n}$. We will show that $f_n \longrightarrow f(x) = x$ uniformly, but $\{f_n^2\}$ does not converge uniformly to $f^2(x) = x^2$.

Let $\epsilon > 0$ be given. Let $N \in \mathbb{N}$ be such that $1/N < \epsilon$. Then for $n \geq N$, and for any $x \in \mathbb{R}$,

$$|f_n(x) - f(x)| = \left| x + \frac{1}{n} - x \right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

Hence, $f_n \longrightarrow f$ uniformly.

However, for $\epsilon = 1$ and any $n \in \mathbb{N}$, if we take $x = \dfrac{n^2 - 1}{2n}$, then

$$\begin{aligned} |f_n^2(x) - f^2(x)| = \left| \left(x^2 + \frac{2x}{n} + \frac{1}{n^2} \right) - x^2 \right| &= \left| \frac{2x}{n} + \frac{1}{n^2} \right| \\ &= \frac{2}{n} \left(\frac{n^2 - 1}{2n} \right) + \frac{1}{n^2} \\ &= 1 \geq \epsilon. \end{aligned}$$

Therefore, $\{f_n^2\}$ does not converge uniformly to $f^2(x) = x^2$.