1. Exercises 5.3.8, 6.1.1, 6.1.2, 6.1.5, 6.1.6 of the book.

   all further questions (below) are fun, but due to the current situation they optional

2. (optional) For $f : D \to \mathbb{R}$ denote the L^∞-norm as follows

$$\|\cdot\|_{L^\infty(D)} := \sup_D |f|.$$

   (a) Show that if $D = [a, b]$ is a closed interval, then any continuous function $f : [a, b] \to \mathbb{R}$ has finite L^∞-norm, i.e. show that

$$\|f\|_{L^\infty([a,b])} < \infty.$$

   (b) Given an example for $D = [0, 1]$ of a discontinuous function $f : [0, 1] \to \mathbb{R}$ with finite L^∞-norm.

   (c) For $D \subset \mathbb{R}$ a nonempty set denote the set

$$C^0(D) := \{f : D \to \mathbb{R}; f \text{ is continuous on } D\}$$

   Show that $C^0(D)$ is a linear vector space. Namely show: If $f, g \in C^0(D)$ and $\lambda, \mu \in \mathbb{R}$ then $\lambda f + \mu g \in C^0(D)$.

   (d) For $D \subset \mathbb{R}$ a nonempty set denote the set

$$L^\infty(D) := \left\{f : D \to \mathbb{R}; \quad \|f\|_{L^\infty(\mathbb{R})} < \infty\right\}$$

   Show that $L^\infty(D)$ is a linear vector space. Namely show: If $f, g \in L^\infty(D)$ and $\lambda, \mu \in \mathbb{R}$ then $\lambda f + \mu g \in L^\infty(D)$.

   (e) Show $\|\cdot\|_{L^\infty(D)}$ defines a *norm* on the space $L^\infty(D)$ (making $(L^\infty(D), \|\cdot\|_{L^\infty(D)})$ a normed vector space). Namely for any $f, g \in L^\infty(D)$ and $\lambda \in \mathbb{R}$ we have

   - $\|f + g\|_{L^\infty(D)} \leq \|f\|_{L^\infty(D)} + \|g\|_{L^\infty(D)}$ (triangle inequality)
   - $\|\lambda f\|_{L^\infty(D)} = |\lambda| \, \|f\|_{L^\infty(D)}$ (positive homogeneity)
   - $\|f\|_{L^\infty(D)} \geq 0$. Moreover, if $\|f\|_{L^\infty(D)} = 0$ then $f \equiv 0$.

3. (optional) Let $L^\infty(D)$ be the normed vector space from above. We say that a map $T : L^\infty(D) \to \mathbb{R}$, $T : f \mapsto Tf$, is continuous

$$\forall f \in L^\infty(D) \, \forall \varepsilon > 0 \quad \exists \delta = \delta(f, \varepsilon) > 0 \quad |Tf - Tg| < \varepsilon \quad \forall g \in L^\infty(D) : \, \|f - g\|_{L^\infty(D)} < \delta.$$

   Show that

   (a) T_1 is a continuous map, where

$$T_1 f := \|f\|_{L^\infty(D)}$$

   (b) Fix $x_0 \in D$. Show that T_2 is a continuous map, where

$$T_2 f := f(x_0)$$

4. (optional) A vector space V is called finite dimensional, if there exists a finite basis, i.e. finitely many vectors $v_1, \ldots, v_n \in V$ such that for any $v \in V$ we have some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $v = \lambda_1 v_1 + \ldots \lambda_n v_n$.

   Show that $L^\infty([0, 1])$ and $C^0([0, 1])$ defined above are not finite dimensional Vector spaces. You can argue by contradiction and use that

$$f = \lambda_1 f_1 + \ldots \lambda_n f_n$$

   is equivalent to saying

$$f(x)\lambda_1 f_1(x) + \ldots \lambda_n f_n(x).$$

   (how many points x do you have?)