Rules and Regulations:

- Each question requires a proof or a counterexample.
- Each problem is worth 10 points.
- The 6 problems with the most points count towards your total
  - In particular the maximum points possible are 60.
  - Example: If you get 5 points on each question, your total is 6*5 = 30 (not good!).
  - Example: If you get 10 points on three questions, and 5 points on all the other question your total is (45).
  - So, it is important that you try to do at least 6 questions perfect; do not try to do 10 questions 'a little bit' (but do as many problems as you can)
  - To avoid cheating (e.g. copying off answers of the internet), the instructor reserves the right to interview any student on any problem which they solved. If the student cannot properly explain what their solution was, points shall be deducted accordingly.
- You can ask questions about this exam only to your class' TA or the instructor – these questions are like in an in-class exam: TA or instructor will not help you to solve these questions.
- You can use your textbook, the lecture notes, your notes from class and recitation. You can refer to theorems of the lecture, but only if it does not trivialize the question.
- You are, in particular, not allowed to look for the solutions online.
- Just in case it is not clear ... **you may only use class notes and the textbook**. Dr. Schikorra's notes count as class notes – you can use them too.

Gradescope:

- Please upload your completed exam via **gradescope** (www.gradescope.com). You can login with your Pitt credentials, and should find Math420 as a course, and within that the "midterm 2".

  (Any issues, contact your instructor).
- Please write **one problem per page**
- Preferred format is pdf.
  - For Latex: please use the template we provide
  - for handwritten exams: you can use your phone to make pdfs, there are several free apps (usually called "camscanner") to make a pdf of your submission.

Problems

Here are some useful hints

1. Assume that $f : [0, \infty) \to \mathbb{R}$ is continuous and differentiable, and $f'(x)$ is bounded, $\sup |f'(x)| = M < \infty$. Show that there exists polynomials p_1, p_2 of degree one such that

$$p_1(x) \leq f(x) \leq p_2(x) \quad \forall x \in [0, \infty)$$

Solution: For every $x \neq 0$ there exists c_x such that

$$\frac{f(x) - f(0)}{x - 0} = f'(c_x)$$

or equivalently

$$f(x) = f(0) + f'(c_x)x$$

Setting

$$\Lambda := \sup f', \quad \lambda := \inf f'$$

we thus have for all $x \geq 0$

$$f(0) - \lambda x \leq f(x) \leq f(0) + \Lambda x$$

2. Assume that $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$. Show that the sequence

$$\{n(f(c + 1/n) - f(c))\}_{n=1}^{\infty}$$

converges to $f'(c)$.

Solution: Note that $n(f(c + 1/n) - f(c))$ can be written as

$$\frac{f(c + \frac{1}{n}) - f(c)}{\frac{1}{n}}$$

Since f is differentiable at c, $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$. By the sequential characterization of limits, we know that for any sequence $\{x_n\}$ that converges to c, $\frac{f(x_n) - f(c)}{x_n - c}$ converges to $f'(c)$. Since $\{x_n\} = \{c + 1/n\}$ is one such sequence that converges to c,

$$\frac{f(c + \frac{1}{n}) - f(c)}{(c + \frac{1}{n}) - c} = \frac{f(c + \frac{1}{n}) - f(c)}{\frac{1}{n}}$$

converges to $f'(c)$, as desired.

3. Suppose that f is differentiable on an interval I. Show that for all $n \in \mathbb{N}$, f^n is differentiable on I. Note that

$$f^n(x) := (f(x))^n$$

by definition.

Solution: When $n = 1$, $f^n = f$ is differentiable on I by assumption.

Next, assume that f^k is differentiable on I for $k \in \mathbb{N}$. By the Product Rule (Proposition 4.1.6), since f^k and f are both differentiable on I, $f^k \cdot f = f^{k+1}$ is differentiable on I.

Therefore, by induction, f^n is differentible on I for all $n \in \mathbb{N}$.

4. Show that $\sqrt{1+x} < 1 + x/2$ for all $x > 0$.

Solution: Fix $x > 0$ and define $f(t) = \sqrt{1+t}$ on $[0, x]$. Since f is continuous on $[0, x]$ and differentiable on $(0, x)$, by the Mean Value Theorem, $\exists c \in (0, x)$ such that $f(x) - f(0) = f'(c)(x - 0)$.

Note that

$$\begin{aligned} f(x) - f(0) = f'(c)(x-0) &\Leftrightarrow \sqrt{1+x} - \sqrt{1+0} = \frac{1}{2} \cdot \frac{1}{\sqrt{1+c}} \cdot (x-0) \\ &\Leftrightarrow \sqrt{1+x} - 1 = \frac{x}{2} \cdot \frac{1}{\sqrt{1+c}} \end{aligned}$$

Since $c > 0$, $\frac{1}{\sqrt{1+c}} < 1$, and so $\sqrt{1+x} - 1 < \frac{x}{2}$ or equivalently $\sqrt{1+x} < 1 + \frac{x}{2}$.

Therefore, since $x > 0$ was arbitrary, $\sqrt{1+x} < 1 + x/2$ for all $x > 0$.

5. Show that it is not possible for a function f to exist such that f is differentiable on $(0, \infty)$, and $f'(x) = \lfloor x \rfloor$.

(Hint: You can use the Mean Value Theorem to show this)

Solution: Assume for contradiction that there is a function f such that $f'(x) = \lfloor x \rfloor$ on $(0, \infty)$. Then, in particular f is differentiable at 1. I.e.

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = f'(1) = \lfloor 1 \rfloor = 1$$

By the Mean Value Theorem, for any $x \in (0, 1)$, $\exists c \in (x, 1)$ such that

$$f(x) - f(1) = f'(c)(x-1) = \lfloor c \rfloor (x-1) = 0 \cdot (x-1) = 0$$

I.e. for any x in $(0, 1)$, $\frac{f(x) - f(1)}{x-1} = 0$. Therefore,

$$1 = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} 0 = 0$$

a contradiction. Hence, there can be no such f.

6. Let $f : [-1, 1] \to \mathbb{R}$ be a Riemann integrable function. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence defined by

$$x_n := \int_{[-\frac{1}{n}, \frac{1}{n}]} f(z)\, dz$$

Compute

$$\lim_{n \to \infty} x_n.$$

Prove your answer.

Solution: Since f is Riemann-integrable, f is bounded, that is

$$\Lambda := \sup_{[-1,1]} |f(x)| < \infty.$$

Then,

$$|x_n| \leq \frac{2}{n}\Lambda \xrightarrow{n\to\infty} 0.$$

7. Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function. Set for $x \in (a,b)$

$$g(x) := \int_a^x f(z)\, dz$$

Give a direct proof that g is uniformly continuous in (a,b) (without using the fundamental theorem).

Solution: Since f is Riemann integrable, f is bounded. Then we have

$$g(x) - g(y) = \int_y^x f(z)dz.$$

Since f is bounded and integrable, we have learned that

$$|g(x) - g(y)| = \left| \int_y^x f(z)dz \right| \leq \sup_{(a,b)} |f|\, |x-y|.$$

So for any $\varepsilon > 0$ set $\delta := \frac{\varepsilon}{\sup_{(a,b)} |f| + 1}$. Then, whenever $|x-y| < \delta$ we have

$$|g(x) - g(y)| < \varepsilon.$$

8. Let $f, g, h : [a,b] \to \mathbb{R}$ be Riemann integrable, and $f(z) \leq g(z) + h(z)$ for all $z \in [a,b]$. Using the definition of the integral via Darboux sums, show that

$$\int_{[a,b]} f \leq \int_{[a,b]} g + \int_{[a,b]} h$$

Solution: Let $P = \{x_0, \ldots, x_N\}$ be any partition of $[a,b]$. By assumption for any $i = 1, \ldots, N$

$$f(z) \leq g(z) + h(z) \quad \forall z \in [x_{i-1}, x_i].$$

This implies that

$$\max_{[x_{i-1},x_i]} f \leq \max_{[x_{i-1},x_i]} g + \max_{[x_{i-1},x_i]} h.$$

and thus

$$U(P,f) \leq U(P,g) + U(P,h).$$

Since $\int_{[a,b]} f = \underline{\int_{[a,b]}} f = \inf_P U(P,f)$ we have

$$\int_{[a,b]} f \leq U(P,g) + U(P,h). \tag{1}$$

Let $\varepsilon > 0$. Take a partition P_1 such that

$$U(P_1, g) \leq \int_{[a,b]} g + \varepsilon$$

and a partition P_2 such that

$$U(P_2, g) \leq \int_{[a,b]} h + \varepsilon$$

Set $P := P_1 \cup P_2$ in (1), by the refinement property we then have

$$\int_{[a,b]} f \leq U(P_1 \cup P_2, g) + U(P_1 \cup P_2, h) \leq U(P_1, g) + U(P_2, h) \leq \int_{[a,b]} g + \int_{[a,b]} h + 2\varepsilon.$$

This holds for any $\varepsilon > 0$, letting $\varepsilon \to 0$ we conclude the claim.

9. Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable
   (a) Let $p > 0$. Show that $g(x) := |f(x)|^p$ is Riemann-integrable (Hint: Riemann-Lebesgue Theorem)
   (b) Let $q < 0$. Is it true that $h(x) := |f(x)|^q$ is necessarily integrable (proof or counterexample)?

Solution:

(a) By the Riemann-Lebesgue theorem, f is bounded and continuous in a set $[a,b]\backslash\Sigma$, where Σ is a set of measure zero. But then also $|f|^p$ is continuous in $[a,b]\backslash\Sigma$. Again by the Riemann-Lebesgue theorem, $|f|^p$ is integrable.

(b) No, let $h(x) := x$ and $[a,b] = [-1,1]$. Then $\frac{1}{x}$ is not integrable on $[-1,1]$ since it is not bounded.

10. Prove the *Cauchy Criterion*:

    If f is a bounded function on $[a,b]$, then f is Riemann integrable on $[a,b]$ if and only if for all $\varepsilon > 0$ there is a partition $\mathcal{P}$ of $[a,b]$ such that $U(\mathcal{P}, f) - L(\mathcal{P}, f) < \varepsilon$.

Solution: First assume that f is Riemann integrable on $[a,b]$. Let $\varepsilon > 0$. By the definition of the upper integral, there is a partition P_1 of $[a,b]$ such that

$$U(\mathcal{P}_1, f) < \overline{\int_a^b} f + \varepsilon/2$$

Similarly, by the definition of the lower integral, there is a partition P_2 of $[a,b]$ such that

$$\underline{\int_a^b} f - \varepsilon/2 < L(\mathcal{P}_2, f)$$

Define $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then $\mathcal{P}$ is a refinement of both $\mathcal{P}_1$ and $\mathcal{P}_2$, and so

$$\begin{aligned} U(\mathcal{P}, f) - L(\mathcal{P}, f) &\leq U(\mathcal{P}_1, f) - L(\mathcal{P}_2, f) \\ &< \left(\overline{\int_a^b} f + \varepsilon/2 \right) - \left(\underline{\int_a^b} f - \varepsilon/2 \right) \\ &= \overline{\int_a^b} f - \underline{\int_a^b} + \varepsilon \\ &= \int_a^b f - \int_a^b f + \varepsilon \\ &= \varepsilon \end{aligned}$$

Therefore, for all $\varepsilon > 0$ there is a partition $\mathcal{P}$ of $[a, b]$ such that $U(\mathcal{P}, f) - L(\mathcal{P}, f) < \varepsilon$.

Next, assume that for all $\varepsilon > 0$ there is a partition $\mathcal{P}$ of $[a, b]$ such that $U(\mathcal{P}, f) - L(\mathcal{P}, f) < \varepsilon$. Fix an arbitrary $\varepsilon > 0$ and consider such a partition $\mathcal{P}$. Then,

$$0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f \leq U(\mathcal{P}, f) - L(\mathcal{P}, f) < \varepsilon$$

Since $eps > 0$ was arbitrary, we must have $\overline{\int_a^b} f - \underline{\int_a^b} f = 0$, or $\overline{\int_a^b} f = \underline{\int_a^b} f$. Therefore, f is Riemann integrable on $[a, b]$.