Rules:

- Each question requires a proof or a counterexample.
- Each problem is worth 10 points.
- The 6 problems with the most points count towards your total
  - In particular the maximum points possible are 60.
  - Example: If you get 5 points on each question, your total is 6*5 = 30 (not good!).
  - Example: If you get 10 points on three questions, and 5 points on all the other question your total is (45).
  - So, it is important that you try to do 6 questions perfect; do not try to do 10 questions 'a little bit'.
- You can ask questions about this exam only to your class' TA or the instructor – these questions are like in an in-class exam: TA or instructor will not help you to solve these questions.
- You can use your textbook, the lecture notes, your notes from class and recitation. You can refer to theorems of the lecture, but only if it does not trivialize the question.
- Solutions are here: solutions – do not look at them before the end of the exam period.

Problems

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Let $Z(f) := \{x : f(x) = 0\}$. Show that $Z(f)$ is closed, which by definition means that it is the complement of an open set.

Solution: Solution 1:

First, note that $(-\infty, 0) \cup (0, \infty)$ is an open set. Then since f is continuous, $f^{-1}((-\infty, 0) \cup (0, \infty))$ is open (by previous homework problem). Then the complement of this set is closed, but the complement is $Z(f)$.

Solution 2:

Let $c \in \mathbb{R} \backslash Z(f)$. Without loss of generality, we can assume that $f(c) > 0$.

Let $\varepsilon = f(c)$. Since f is continuous at c, there is a $\delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < f(c)$. In other words, when $x \in (c - \varepsilon, c + \varepsilon)$, $f(x) > 0$, and so $(c - \varepsilon, c + \varepsilon) \subset \mathbb{R} \backslash Z(f)$. Since c was arbitrary, this shows that $\mathbb{R} \backslash Z(f)$ is open, and so $Z(f)$ is closed.

2. Suppose that f, g are continuous functions on an interval I. Show that

$$(f \vee g)(x) := \max\{f(x), g(x)\}$$
$$(f \wedge g)(x) := \min\{f(x), g(x)\}$$

are continuous on I.

Solution: We give two proofs here.

Solution 1:

Note that

$$(f \vee g)(x) = \frac{f(x) + g(x)}{2} + \frac{|f(x) - g(x)|}{2} \quad \text{and} \quad (f \wedge g)(x) = \frac{f(x) + g(x)}{2} - \frac{|f(x) - g(x)|}{2}$$

Given $c \in I$ and $\varepsilon > 0$, choose $\delta > 0$ such that when $|x-c| < \delta$, $|f(x) - f(c)| < \varepsilon/2$ and $|g(x) - g(c)| < \varepsilon/2$. Then, for any $|x-c| < \delta$,

$$\begin{aligned}
|(f \vee g)(x) - (f \vee g)(c)| &= \left| \left(\frac{f(x)+g(x)}{2} + \frac{|f(x)-g(x)|}{2} \right) - \left(\frac{f(c)+g(c)}{2} + \frac{|f(c)-g(c)|}{2} \right) \right| \\
&= \left| \left(\frac{f(x)+g(x)}{2} - \frac{f(c)+g(c)}{2} \right) + \left(\frac{|f(x)-g(x)|}{2} - \frac{|f(c)-g(c)|}{2} \right) \right| \\
&= \left| \left(\frac{f(x)-f(c)}{2} + \frac{g(x)-g(c)}{2} \right) + \left(\frac{|f(x)-g(x)|}{2} - \frac{|f(c)-g(c)|}{2} \right) \right| \\
&\le \frac{1}{2}|(f(x)-f(c)) + (g(x)-g(c))| + \frac{1}{2}||f(x)-g(x)| - |f(c)-g(c)|| \\
&\le \frac{1}{2}|(f(x)-f(c)) + (g(x)-g(c))| + \frac{1}{2}|(f(x)-g(x)) - (f(c)-g(c))| \\
&= \frac{1}{2}|(f(x)-f(c)) + (g(x)-g(c))| + \frac{1}{2}|(f(x)-f(c)) - (g(x)-g(c))| \\
&\le \frac{1}{2}|f(x)-f(c)| + \frac{1}{2}|g(x)-g(c)| + \frac{1}{2}|f(x)-f(c)| + \frac{1}{2}|g(x)-g(c)| \\
&= |f(x)-f(c)| + |g(x)-g(c)| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon
\end{aligned}$$

Also,

$$\begin{aligned}
|(f \wedge g)(x) - (f \wedge g)(c)| &= \left| \left(\frac{f(x)+g(x)}{2} - \frac{|f(x)-g(x)|}{2} \right) - \left(\frac{f(c)+g(c)}{2} - \frac{|f(c)-g(c)|}{2} \right) \right| \\
&= \left| \left(\frac{f(x)+g(x)}{2} - \frac{f(c)+g(c)}{2} \right) - \left(\frac{|f(x)-g(x)|}{2} - \frac{|f(c)-g(c)|}{2} \right) \right| \\
&= \left| \left(\frac{f(x)-f(c)}{2} + \frac{g(x)-g(c)}{2} \right) - \left(\frac{|f(x)-g(x)|}{2} - \frac{|f(c)-g(c)|}{2} \right) \right| \\
&\le \frac{1}{2}|(f(x)-f(c)) + (g(x)-g(c))| + \frac{1}{2}(|f(x)-g(x)| - |f(c)-g(c)|) \\
&\le \frac{1}{2}|(f(x)-f(c)) + (g(x)-g(c))| + \frac{1}{2}|(f(x)-g(x)) - (f(c)-g(c))| \\
&= \frac{1}{2}|(f(x)-f(c)) + (g(x)-g(c))| + \frac{1}{2}|(f(x)-f(c)) - (g(x)-g(c))| \\
&\le \frac{1}{2}(|f(x)-f(c)| + |g(x)-g(c)|) + \frac{1}{2}(|f(x)-f(c)| + |g(x)-g(c)|) \\
&= |f(x)-f(c)| + |g(x)-g(c)| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon
\end{aligned}$$

Hence, $f \vee g$ and $f \wedge g$ are continuous at c. Since c was arbitrary, $f \vee g$ and $f \wedge g$ are continuous.

Solution 2: Let $c \in \mathbb{R}$ be arbitrary. Let $\varepsilon > 0$ be given.

Case 1: $f(c) \neq g(c)$

WLOG assume $f(c) > g(c)$. Let $\varepsilon' = \frac{f(c)-g(c)}{2}$.

$$\exists \delta_1 > 0 \text{ such that } |x-c| < \delta_1 \Rightarrow |f(x) - f(c)| < \varepsilon'$$
$$\exists \delta_2 > 0 \text{ such that } |x-c| < \delta_2 \Rightarrow |g(x) - g(c)| < \varepsilon'$$

Let $\delta' = \min\{\delta_1, \delta_2\}$. Then when $|x - c| < \delta'$,

$$f(x) > \frac{f(c) + g(c)}{2} > g(x) \qquad \text{(why? You should convice yourself of this)}$$

In other words, $f(x) > g(x)$ on $(c - \delta', c + \delta')$.

Next, since f is continuous at c we can choose $\delta^* > 0$ such that $|x - c| < \delta^* \Rightarrow |f(x) - f(c)| < \varepsilon$.

Let $\delta = \min\{\delta', \delta^*\}$. Then

$$|x - c| < \delta \Rightarrow |(f \vee g)(x) - (f \vee g)(c)| = |f(x) - f(c)| < \varepsilon$$

and hence $f \vee g$ is continuous at c.

Case 2: $f(c) \neq g(c)$

$$\exists \delta_1 > 0 \text{ such that } |x - c| < \delta_1 \Rightarrow |f(x) - f(c)| < \varepsilon$$
$$\exists \delta_2 > 0 \text{ such that } |x - c| < \delta_2 \Rightarrow |g(x) - g(c)| < \varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then when $|x - c| < \delta$,

$$|(f \vee g)(x) - (f \vee g)(c)| = \begin{cases} |f(x) - f(c)| & \text{if } f(x) \geq g(x) \\ |g(x) - g(c)| & \text{if } f(x) < g(x) \end{cases}$$
$$< \varepsilon$$

Hence, $f \vee g$ is continuous at c.

The proof for $(f \wedge g)$ is similar.

3. Let f and $g : [0, 1] \to \mathbb{R}$ be continuous, and assume $f(x) = g(x)$ for all $x < 1$. Does this imply that $f(1) = g(1)$? Provide a proof or a counterexample.

Solution: Let (x_n) be a sequence in $[0, 1)$ such that $x_n \to 1$ as $n \to \infty$. Since f and g are both continuous at 1, and $f(x) = g(x)$ for $x < 1$,

$$f(1) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(1).$$

4. Use the ε-δ-definition of continuity to show that if $f, g : D \to \mathbb{R}$ are continuous then $h(x) := f(x)g(x)$ is continuous in D.

Solution: Let $x \in D$ and $\varepsilon > 0$. Since f and g are continuous for any $\tilde{\varepsilon} > 0$ (to be chosen later) there exist δ_f and δ_g such that

$$|f(x) - f(y)| < \tilde{\varepsilon} \quad \forall y \in D : |x - y| < \delta_f$$

and

$$|g(x) - g(y)| < \tilde{\varepsilon} \quad \forall y \in D : |x - y| < \delta_g$$

Now

$$\begin{aligned}h(x)-h(y)=&f(x)g(x)-f(y)g(y)\\=&(f(x)-f(y))\,g(x)+f(y)\,(g(x)-g(y))\\=&(f(x)-f(y))\,g(x)+f(x)(g(x)-g(y))+(f(y)-f(x))\,(g(x)-g(y))\end{aligned}$$

That if $\delta := \min\{\delta_f, \delta_g\}$ and $|x-y|<\delta$ then

$$\begin{aligned}|h(x)-h(y)|\le&|f(x)-f(y)|\,|g(x)|+|f(x)|\,|g(x)-g(y)|+|f(y)-f(x)|\,|g(x)-g(y)|\\\le&\tilde{\varepsilon}|g(x)|+\tilde{\varepsilon}|f(x)|+(\tilde{\varepsilon})^2.\end{aligned}$$

So if we choose $\tilde{\varepsilon} := \frac{1}{4}\min\{1, \varepsilon, \frac{\varepsilon}{|f(x)|}, \frac{\varepsilon}{|g(x)|}\}$, then

$$|h(x)-h(y)|<\varepsilon \quad \forall y\in D:\ |x-y|<\delta.$$

5. Let $g(x) = \dfrac{\sin x}{\sqrt{x}}$. Prove that g has a maximum and a minimum value on the set $(0, \pi]$.

   Hint: You can assume that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$.

Solution: Define

$$h(x) := \begin{cases} \frac{\sin x}{\sqrt{x}} & \text{if } x\in(0,\pi] \\ 0 & \text{if } x = 0\end{cases}$$

Note that $h(x) = g(x)$ on $(0,\pi]$, $h(x)\ge 0$ on $[0,\pi]$, and $0 = h(0) = h(\pi) = g(\pi)$.

Since

$$\lim_{x\to 0}\frac{\sin x}{\sqrt{x}} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\sqrt{x} = 1\cdot 0 = 0$$

h is continuous on $[0,\pi]$. So, by the Min-Max Theorem, h attains both an absolute maximum and an absolute minimum in $[0,\pi]$. We immediately see that the absolute minimum is 0 and the absolute maximum is a positive number, attained for some $c\in(0,\pi)$. Hence, g has an absolute minimum at π and an absolute maximum at c.

6. Suppose that $f:\mathbb{R}\to\mathbb{R}$ such that

$$\lim_{h\to 0}\,[f(x+h)-f(x-h)] = 0$$

   for all $x\in\mathbb{R}$. Does this imply that f is continuous?

   If we change the property to say that

$$\lim_{h\to 0}\,[f(x)-f(x-h)] = 0$$

   for all $x\in\mathbb{R}$, then is f continuous in this case?

Solution: The first property does not imply that f is continuous. Take for example $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x) := \begin{cases} x^2 & \text{if } x\neq 0 \\ 1 & \text{if } x = 0\end{cases}$$

Then, f is not continuous at zero, but when $x = 0$

$$\begin{aligned}
\lim_{h\to 0} [f(x+h) - f(x-h)] &= \lim_{h\to 0} [f(0+h) - f(0-h)] \\
&= \lim_{h\to 0} [h^2 - (-h)^2] \\
&= \lim_{h\to 0} 0 \\
&= 0
\end{aligned}$$

The second property does give continuity. Fix any $c \in \mathbb{R}$. Note that

$$\lim_{h\to 0} f(c) = f(c) \tag{1}$$

$$\lim_{h\to 0} -[f(c) - f(c-h)] = 0 \qquad \text{by assumption} \tag{2}$$

Then,

$$\begin{aligned}
\lim_{x\to c} f(x) &= \lim_{h\to 0} f(c-h) \qquad \text{(let } h = c - x\text{)} \\
&= \lim_{h\to 0} -[f(c) - f(c-h) - f(c)] \\
&= \lim_{h\to 0} -[f(c) - f(c-h] + \lim_{h\to 0} f(c) \qquad \text{since both limits exist} \\
&= 0 + f(c) \\
&= f(c)
\end{aligned}$$

Since c was arbitrary we have that f is continuous.

7. A function $f : D \subset \mathbb{R} \to \mathbb{R}$ is called *(sequentially) lower semicontinuous* at a point $x \in D$ if we have

$$f(x) \leq \liminf_{D\ni y\to x} f(y),$$

in the sense that for any sequence $(y_n)_{n\in\mathbb{N}} \subset D$ with $\lim_{n\to\infty} y_n = x$ we have

$$f(x) \leq \liminf_{n\to\infty} f(y_n).$$

Let $-\infty < a < b < \infty$. Show that if $f : [a, b] \to \mathbb{R}$ is lower semicontinuous in every $x \in [a, b]$, then f attains its minimum value in $[a, b]$.

Give the full proof, you cannot use the min-max property for upper semicontinuous or continuous functions

Solution: Let

$$I := \inf_{x\in[a,b]} f(x) \in [-\infty, \infty).$$

If $I = -\infty$, then there must be a sequence $(x_n)_{n\in\mathbb{N}} \subset [a, b]$ such that $f(x_n) \leq -n$ for all $n \in \mathbb{N}$. If $I \in (-\infty, \infty)$ there must be a sequence $(x_n)_{n\in\mathbb{N}} \subset [a, b]$ such that $I \leq f(x_n) \leq I + \frac{1}{n}$.

In either case $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence, so by Bolzano-Weierstrass there exists a converging subsequence $(x_{n_i})_{i\in\mathbb{N}}$. Let us call its limit $x := \lim_{i\to\infty} x_{n_i}$. Since $[a, b]$ is a closed set and $(x_{n_i})_i \subset [a, b]$, we have that $x \in [a, b]$.

By lowercontinuity we have

$$\lim_{i\to\infty} f(x_{n_i}) \geq \liminf_{n\to\infty} f(x_n) \geq f(x).$$

But by the choice of the sequence $(x_n)_{n\in\mathbb{N}}$ this means either that $f(x) \leq -\infty$ (if $I = -\infty$) (and thus $f(x) = -\infty$), which is impossible since f is a function, or $f(x) \leq I$. So the latter has to be the case. Observe that $x \in [a,b]$ so we also have $f(x) \geq I$, that is we have $f(x) = I$, which implies

$$f(x) = \inf_{[a,b]} f \leq f(y) \quad \forall y \in [a,b].$$

8. Prove the "Banach Fixed Point Theorem":

Assume that $f : \mathbb{R} \to \mathbb{R}$ is a function that for some $L \in [0,1)$ satisfies

$$|f(x) - f(y)| \leq L|x-y| \quad \forall x, y \in \mathbb{R}.$$

Show that there exists a unique $x \in \mathbb{R}$ such that $f(x) = x$. More precisely

(a) Show that f is uniformly continuous in $\mathbb{R}$.

(b) To show that there exists (at least) one $z \in \mathbb{R}$ with $f(z) = z$ consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined as $x_{n+1} := f(x_n)$ with $x_1 = 0$. Show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, and conclude that its limit $z := \lim_{n\to\infty} x_n$ satisfies $f(z) = z$.

(c) To show that there exist exactly one solution $x = f(x)$ assume there are two solutions x and y and compute $|f(x) - f(y)|$.

Solution:

(a) Let $\varepsilon > 0$. For $\delta := \frac{\varepsilon}{L}$ we have

$$|f(x) - f(y)| \leq L|x-y| < L\,\delta = \varepsilon \quad \forall x, y \in \mathbb{R} : |x-y| < \delta.$$

This is uniform continuity.

(b) Define the sequence $(x_n)_{n\in\mathbb{N}}$ as $x_{n+1} := f(x_n)$ with $x_1 = 0$.

Then

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \leq L|x_n - x_{n-1}|.$$

Since $L < 1$ this implies that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, as we have shown in an exercise already.

In $\mathbb{R}$ any Cauchy sequence converges, so there is $x \in \mathbb{R}$ with $x = \lim_{n\to\infty} x_n$. Since f is continuous, we have $f(x) = \lim_{n\to\infty} f(x_n)$. But by construction we also have $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f(x)$. That is $x = f(x)$ as claimed.

(c) Assume there are two solutions x and y, i.e. $x = f(x)$ and $y = f(y)$. Then $|x - y| = |f(x) - f(y)| \leq L|x-y|$. If $x \neq y$ then $|x-y| \neq 0$ so we can divide this equation by $|x-y|$ and obtain $L \geq 1$, a contradicition. So $x = y$ must be true.

9. Let $f : A \subset \mathbb{R} \to \mathbb{R}$ be uniformly continuous.

(a) Show that if A is a nonempty bounded set, then f is a bounded function on A, i.e. $\sup_A f < \infty$ and $\inf_A f > -\infty$.

(b) Show that the conclusion is false if A is not bounded.

Solution:

(a) Assume that A is a nonempty bounded set. Assume that $\sup_A f = \infty$, then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset A$ such that $f(x_n) > n$. Since A is bounded, the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded, and by Bolzano-Weierstrass there exists a convergent subsequence $(x_{n_i})_{i\in\mathbb{N}}$. In particular, $(x_{n_i})_{i\in\mathbb{N}}$ is a Cauchy sequence. From a theorem in class we know that since f is uniformly continuous, $(f(x_n))_{n\in\mathbb{N}}$ is also a Cauchy sequence. But Cauchy sequences are bounded, so $\sup_{n\in\mathbb{N}} f(x_n) < \infty$, which contradicts the construction of x_n. In the same way we can argue that $\inf_A f > -\infty$.

(b) Let $A = \mathbb{R}$ and $f(x) := x$. Then f is uniformly continuous (for any ε we choose $\delta := \varepsilon$ in the definition of uniform continuity). However $f(A) = \mathbb{R}$ is unbounded.

10. Let A and B be subsets of $\mathbb{R}$. Suppose that $f : B \to \mathbb{R}$ and $g : A \to \mathbb{R}$ such that $g(A) \subset B$. If f and g are both uniformly continuous, will $f \circ g$ be uniformly continuous?

Solution: Fix $\varepsilon > 0$. Since f is uniformly continuous on B, $\exists\ \delta > 0$ such that when $x, y \in B$ and $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Since $\delta > 0$ and g is uniformly continuous on A, $\exists\ \delta' > 0$ such that when $x, y \in A$ and $|x - y| < \delta' \Rightarrow |g(x) - g(y)| < \delta$.

Then, for this $\delta' > 0$, when $x, y \in A$ and $|x - y| < \delta'$, $|g(x) - g(y)| < \delta$. Then, since $g(x), g(y) \in B$ and $|g(x) - g(y)| < \delta$, $|f(g(x)) - f(g(y))| < \varepsilon$.

In summary, given $\varepsilon > 0$, $\exists\ \delta' > 0$ such that $x, y \in A$ and $|x-y| < \delta'$ gives $|(f\circ g)(x) - (f\circ g)(y)| < \varepsilon$. I.e. the composition is uniformly continuous.