Rules:

- Each question requires a proof or a counterexample.
- Each problem is worth 10 points.
- The 6 problems with the most points count towards your total
  - In particular the maximum points possible are 60.
  - Example: If you get 5 points on each question, your total is 6*5 = 30 (not good!).
  - Example: If you get 10 points on three questions, and 5 points on all the other question your total is (45).
  - So, it is important that you try to do 6 questions perfect; do not try to do 10 questions 'a little bit'.
- You can ask questions about this exam only to your class' TA or the instructor – these questions are like in an in-class exam: TA or instructor will not help you to solve these questions.
- You can use your textbook, the lecture notes, your notes from class and recitation. You can refer to theorems of the lecture, but only if it does not trivialize the question.
- Solutions are here: solutions – do not look at them before the end of the exam period.

Problems

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Let $Z(f) := \{x : f(x) = 0\}$. Show that $Z(f)$ is closed, which by definition means that it is the complement of an open set.
2. Suppose that f, g are continuous functions on an interval I. Show that

$$(f \vee g)(x) := \max\{f(x), g(x)\}$$
$$(f \wedge g)(x) := \min\{f(x), g(x)\}$$

   are continuous on I.
3. Let f and $g : [0, 1] \to \mathbb{R}$ be continuous, and assume $f(x) = g(x)$ for all $x < 1$. Does this imply that $f(1) = g(1)$? Provide a proof or a counterexample.
4. Use the ε-δ-definition of continuity to show that if $f, g : D \to \mathbb{R}$ are continuous then $h(x) := f(x)g(x)$ is continuous in D.
5. Let $g(x) = \dfrac{\sin x}{\sqrt{x}}$. Prove that g has a maximum and a minimum value on the set $(0, \pi]$.

   Hint: You can assume that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$.
6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{h\to 0} [f(x + h) - f(x - h)] = 0$$

   for all $x \in \mathbb{R}$. Does this imply that f is continuous?

   If we change the property to say that

$$\lim_{h\to 0} [f(x) - f(x - h)] = 0$$

   for all $x \in \mathbb{R}$, then is f continuous in this case?

7. A function $f : D \subset \mathbb{R} \to \mathbb{R}$ is called *(sequentially) lower semicontinuous* at a point $x \in D$ if we have

$$f(x) \leq \liminf_{D \ni y \to x} f(y),$$

in the sense that for any sequence $(y_n)_{n\in\mathbb{N}} \subset D$ with $\lim_{n\to\infty} y_n = x$ we have

$$f(x) \leq \liminf_{n\to\infty} f(y_n).$$

Let $-\infty < a < b < \infty$. Show that if $f : [a,b] \to \mathbb{R}$ is lower semicontinuous in every $x \in [a,b]$, then f attains its minimum value in $[a,b]$.

Give the full proof, you cannot use the min-max property for upper semicontinuous or continuous functions

8. Prove the "Banach Fixed Point Theorem":

Assume that $f : \mathbb{R} \to \mathbb{R}$ is a function that for some $L \in [0,1)$ satisfies

$$|f(x) - f(y)| \leq L|x - y| \quad \forall x, y \in \mathbb{R}.$$

Show that there exists a unique $x \in \mathbb{R}$ such that $f(x) = x$. More precisely

(a) Show that f is uniformly continuous in $\mathbb{R}$.

(b) To show that there exists (at least) one $z \in \mathbb{R}$ with $f(z) = z$ consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined as $x_{n+1} := f(x_n)$ with $x_1 = 0$. Show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, and conclude that its limit $z := \lim_{n\to\infty} x_n$ satisfies $f(z) = z$.

(c) To show that there exist exactly one solution $x = f(x)$ assume there are two solutions x and y and compute $|f(x) - f(y)|$.

9. Let $f : A \subset \mathbb{R} \to \mathbb{R}$ be uniformly continuous.

(a) Show that if A is a nonempty bounded set, then f is a bounded function on A, i.e. $\sup_A f < \infty$ and $\inf_A f > -\infty$.

(b) Show that the conclusion is false if A is not bounded.

10. Let A and B be subsets of $\mathbb{R}$. Suppose that $f : B \to \mathbb{R}$ and $g : A \to \mathbb{R}$ such that $g(A) \subset B$. If f and g are both uniformly continuous, will $f \circ g$ be uniformly continuous?