Rules:

- Each question requires a proof or a counterexample.
- Each problem is worth 10 points.
- You can ask questions about this exam only to your class' TA or the instructor – these questions are like in an in-class exam: TA or instructor will not help you to solve these questions.
- You can use your textbook, the lecture notes, your notes from class and recitation. You can refer to theorems of the lecture, but only if it does not trivialize the question.
- You are in particular not allowed to look for the solutions online.
- If you think there is a typo, explain why and give a counterexample.

Grading

Your total points consist are computed as follows:

Let

$$\text{Part}_i := \{k : \quad \text{Problem } k \text{ belongs to Part } i\}.$$

and

$$\text{points}(k) := \text{points in Problem } k.$$

Let

$$I := \Big\{(k_1, \ldots, k_6) : \quad k_\ell \neq k_{\tilde{\ell}} \text{ if } \ell \neq \tilde{\ell}, \quad \{k_1, \ldots, k_6\} \cap \text{Part}_\text{i} \neq \emptyset \text{ for all } i = 1, 2, 3\Big\}$$

Then

$$\text{totalpoints} := \max_{(k_1,\ldots,k_6) \subset I} \sum_{\ell=1}^{6} \text{points}(k_\ell)$$

If that is not totally clear, we take the maximal sum of 6 problems which meet the following condition

- at least one problem is from Part 1, at least one problem from Part 2, at least one problem from Part 3
- the sum of the best three of the remaining problems.

Problems

Part 1

1. Let $S \subset \mathbb{R}$ and $f, g : S \to \mathbb{R}$. Suppose that c is a cluster point of S, $\lim\limits_{x\to c} f(x)$ exists and $\lim\limits_{x\to c} g(x)$ does not exist. Prove or give a counterexample for each of the following statements.
   (a) $\lim\limits_{x\to c}(f(x) - g(x))$ does not exist.
   (b) $\lim\limits_{x\to c}(f(x)g(x))$ does not exist.

Solution:
(a) This is true. If $\lim\limits_{x\to c}(f(x) - g(x))$ did exist, then by properties of limits, $\lim\limits_{x\to c} g(x) = \lim\limits_{x\to c} f(x) - \lim\limits_{x\to c}(f(x) - g(x))$ would exist.
(b) False. Let $S = (0, \infty)$, $f(x) = x$ and $g(x) = 1/x$.

2. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that that is continuous at every integer but discontinuous at every point in $\mathbb{R}\backslash\mathbb{Z}$. That is, a function such that
   - for any $x \in \mathbb{Z}$ f is continuous in x
   - for any $x \notin \mathbb{Z}$ f is discontinuous in x.

   Prove that your functions satisfies the given requirements.

Solution: E.g. $f(x) := \sin(\pi x)D(x)$ where D is the Dirichlet function.

By the sequeze theorem, for any $x \in \mathbb{Z}$ we have $\lim_{y\to x} \sin(\pi y)D(y) = 0 = \underbrace{\sin(\pi x)}_{=0} D(x)$. For any $x \notin \mathbb{Z}$ $\frac{1}{\sin(\pi x)}$ is continuous and if f was continous so was $\frac{1}{\sin(\pi x)} f(x) = D(x)$, which is however nowhere continuous.

3. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous and bounded on $\mathbb{R}$, but not uniformly continuous on $\mathbb{R}$. Include a proof to show that it is not uniformly continuous.

Solution: $f(x) = \sin(x^2)$

Let $\varepsilon = 1$ and $\delta > 0$ arbitrary.

Let $x = \sqrt{n\pi}$ and $y = \sqrt{n\pi + \delta/2}$. Then $|x - y| = \dfrac{\pi/2}{\sqrt{n\pi + \delta/2} + \sqrt{n\pi}} < \dfrac{1}{\sqrt{n}}$ and so $|x - y| < \delta$ for large n. However, $|f(x) - f(y)| = 1$.

Therefore f is not uniformly continuous.

4. Let $f : [0, \infty) \to \mathbb{R}$ be a nonnegative and continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that f has an absolute maximum value.

Solution: WLOG f is not the zero function.

Let c be a element of $[0, \infty)$ for which $f(c) > 0$.

Let $\varepsilon = f(c)$. By assumption, $\exists\, M \in \mathbb{R}$ such that $x > M \Rightarrow |f(x) - 0| < \varepsilon$, i.e. $f(x) < f(c)$.

Consider f restricted to $[0, M]$. Since f is continuous, it must have an absolute maximum on $[0, M]$ and that maximum must be greater than or equal to $f(c)$. From the way we chose M above, this value must also be an absolute maximum on $[0, \infty)$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be homogeneous of order $\sigma \in \mathbb{R}$. That is, assume that

$$f(\lambda x) = \lambda^\sigma f(x) \quad \forall \lambda > 0\ \forall x \in \mathbb{R}.$$

(a) Show that $f(0) = 0$ if $\sigma \in \mathbb{R}/\{0\}$

(b) Show that f is continuous in $\mathbb{R}\backslash\{0\}$.

(c) Assume $\sigma > 0$. Show that f is then necessarily continuous in all of $\mathbb{R}$.

(d) Assume $\sigma \leq 0$. Show that f is either discontinuous in $x = 0$ or f is constantly zero.

Solution:

(a) We have $f(0) = f(\frac{1}{2}0) = (\frac{1}{2})^\sigma f(0)$. If $f(0) \neq 0$ then we can divide by $f(0)$ and have $\frac{1}{2})^\sigma = \lambda = 1$, a contradiction whenever $\sigma = 0$.

(b) Let $x \in \mathbb{R}\backslash\{0\}$ and $\varepsilon > 0$. Let $\delta < |x|$ such that

$$\delta < \varepsilon |x| \frac{1}{|f(x)|}.$$

Then for any $|y - x| < \delta$ we have that $y = \frac{|y|}{|x|}x$. Thus

$$f(y) = \frac{|y|}{|x|} f(x),$$

that is

$$|f(y) - f(x)| = |1 - \frac{|y|}{|x|}||f(x)| \leq \frac{\delta}{|x|}|f(x)| < \varepsilon.$$

(c) Assume $\sigma > 0$. It only remains to show that (from the previous argument) that f is continuous in zero, i.e. $\lim_{x\to 0} f(x) = f(0) = 0$. But

$$|f(x)| = |x|^\sigma |f(x/|x|)| \leq |x|^\sigma \max\{f(1), f(-1)\} \xrightarrow{x\to 0} 0.$$

(d) Let $\sigma < 0$. Assume f is continuous at $x = 0$. We need to show that f is constantly zero. Let $x \in \mathbb{R}$. Then

$$f(x) = |x|^\sigma f(x/|x|).$$

So we only need to show that $f(1) = f(-1) = 0$ to conclude.

Now

$$f(1) = \lambda^{-\sigma} \underbrace{f(\lambda)}_{\xrightarrow{\lambda\to 0} 0} \xrightarrow{\lambda\to 0} 0.$$

Similarly,

$$f(-1) = \lambda^{-\sigma} \underbrace{f(-\lambda)}_{\xrightarrow{\lambda\to 0} 0} \xrightarrow{\lambda\to 0} 0.$$

Part 2

6. Let $f : [a, b] \to \mathbb{R}$ continuous on $[a, b]$

   (a) Prove the following: if $f(x) \geq 0$ for all $x \in [a, b]$ and $\int_{[a,b]} f = 0$ then $f(x) = 0$ for all $x \in [a, b]$ (*hint: argue by contradiction. What happens if $f(x) > 0$ for some x?*)

   (b) Disprove the following: if $\int_{[a,b]} f = 0$ then $f(x) = 0$ for all $x \in [a, b]$

> **Solution:**
>
> (a) Assume $f(x) > 0$ then there exist an $\varepsilon > 0$ and $\delta > 0$ such that $f(y) > \delta$ in $[x - \varepsilon, x + \varepsilon]$. Since $f \geq 0$ otherwise we have
>
> $$f(x) \geq g(x) := \begin{cases} \delta & [x - \varepsilon, x + \varepsilon] \\ 0 & \end{cases}$$
>
> g is clearly integrable. So
>
> $$\int_{[a,b]} f \geq \int_{[a,b]} g \geq 2\varepsilon\delta > 0,$$
>
> a contradiction.
>
> (b) Let $a = [-1, 1]$ and $f(x) = x$. Then $\int_{[-1,1]} f = 0$ but $f(x) \not\equiv 0$.

7. Let $f : [-1, 1] \to \mathbb{R}$ be continuous. Assume that for any $a, b \in (-1, 1)$ we have

$$\int_{[a,b]} f \geq 0.$$

Show that $f(x) \geq 0$ for all $x \in [-1, 1]$.

Hint: Argue by contradiction, assuming that there exists $x \in [-1, 1]$ such that $f(x) < 0$.

> **Solution:** Assume there exists $x \in [-1, 1]$ such that $f(x) < 0$. Set $c := f(x) < 0$. By continuity there must then exist $\varepsilon > 0$ such that $f < \frac{c}{2}$ in $[-1, 1] \cap (x - \varepsilon, x + \varepsilon)$. Take a and b such that $[a, b] \subset [-1, 1] \cap (x - \varepsilon, x + \varepsilon)$. Then
>
> $$\int_{[a,b]} f \leq (b - a)\,\frac{c}{2}.$$

8. Give an example of $f \in [0, 1]$ continuous and Riemann-integrable, but

$$U(P, f) \neq L(P, f)$$

for any partition $P = \{x_0, \ldots, x_N\}$ of $[0, 1]$. Justify your answer with a proof.

> **Solution:** Let $f(x) = x$. Indeed then
>
> $$\inf_{[x_{i-1},x_i]} f < \sup_{[x_{i-1},x_i]} f,$$
>
> and thus $L(P, f) < U(P, f)$ for any partition P.

9. Let f be a differentiable function on $(0, \infty)$ such that $\lim_{x\to\infty} f'(x) = 0$. Let $g(x) = f(x + \pi) - f(x)$. Prove that $\lim_{x\to\infty} g(x) = 0$. Give an example to show that f may be unbounded.

Solution: For $\varepsilon > 0$, pick M such that $|f'(x)| < \frac{\varepsilon}{\pi}$ when $x > M$.

For $x > M$, by the Mean Value Theorem, $\exists\, c > 0$ such that $g(x) = f'(c)((x+\pi) - x) < \pi \cdot \dfrac{\varepsilon}{\pi} = \varepsilon$.

Hence, $\lim_{x\to\infty} g(x) = 0$.

Part 3

10. Let I be an interval. Suppose that $\{f_n\}$ converges uniformly to f on I and $\{g_n\}$ converges uniformly to g on I.

(a) Prove that $\{f_n + g_n\}$ converges uniformly to $f + g$ on I.

(b) Show that $\{f_n g_n\}$ converges uniformly to fg if f and g are bounded.

(c) Show that $\{f_n g_n\}$ may not converge uniformly to fg on I.

Solution:

(a) $|(f_n+g_n)(x) - (f+g)(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$ and so $\{f_n+g_n\}$ converges uniformly to $f+g$.

(b) Let M be a bound for f and N be a bound for g.

$$\begin{aligned} |(f_n g_n)(x) - (fg)(x)| &\leq |g_n(x)||f_n(x) - f(x)| + |f(x)||g_n(x) - g(x)| \\ &\leq (1+N)|f_n(x) - f(x)| + M|g_n(x) - g(x)| \end{aligned}$$

and therefore $\{f_n g_n\}$ converges uniformly to fg.

(c) $f_n(x) = g_n(x) = x + \dfrac{1}{n}$ on $I = \mathbb{R}$.

$|(f_n g_n)(x) - (fg)(x)| = \dfrac{|2xn+1|}{n^2} \to \infty$ as $x \to \infty$.

11. Show that $f : (0,\infty) \to \mathbb{R}$ defined by $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2(x+1)}$ is continuous and differentiable.

Hint: start by showing that the series is uniformly convergent on $[a,\infty)$ for $a > 0$.

Solution: For $a > 0$, we will show that f is uniformly convergent on $[a,\infty)$.

Let $f_k(x) := \dfrac{1}{k^2(x+1)}$.

$|f_k(x)| \leq \dfrac{1}{k^2(a+1)} =: M_k$ on $[a,\infty)$.

Since $\sum M_k$ is convergent, f converges uniformly by the M-Test. Then, since each f_k is continuous on $[a,\infty)$, f is continuous on $[a,\infty)$. Since $a > 0$ was arbitrary, f is continuous on $(0,\infty)$.

Define $g_k(x) = f_k'(x) = -\dfrac{1}{k^2(x+1)^2}$

$$|g_k(x)| \leq \frac{1}{k^2(a+1)^2} =: N_k \text{ on } [a,\infty)$$

Since $\sum N_k$ is convergent, $g(x) := \sum_{k=1}^{\infty} g_k(x)$ converges uniformly.

Therefore, $f' = g$ on $[a,\infty)$. Again, since $a > 0$ was arbitrary, f is differentiable on $(0,\infty)$.

12. For $x \in \mathbb{R}$ set

$$\zeta(x) := \sum_{n=1}^{\infty} \frac{1}{n^x}$$

(a) for which $x \in \mathbb{R}$ is $\zeta(x)$ well-defined (i.e. finite)?

(b) for which $x \in \mathbb{R}$ is $\zeta(x)$ is continuous?

(c) for which $x \in \mathbb{R}$ is $\zeta(x)$ is differentiable and

$$\zeta'(x) := \sum_{n=2}^{\infty} \frac{1}{n^x \log n}$$

Prove each of your claims!

Solution: $\zeta(x)$ is well-defined for each $x > 1$ (Calculus); so is

$$h(x) := \sum_{n=1}^{\infty} \frac{1}{n^x \log n}.$$

We claim that for each $x > 1$ the function ζ is continuous and differentiable.

Clearly for each N

$$g_N(x) := \sum_{n=1}^{N} \frac{1}{n^x}$$

is a continuous and differentiable function. Its derivative is

$$h_N(x) := \sum_{n=1}^{N} \frac{1}{n^x \log n}$$

We claim that for any $a > 1$ the sequence g_N converges uniformly to ζ as $N \to \infty$ and the sequence h_N converges uniformly to h in $[a,\infty)$. Then, as we learned in the class ζ is continuous and differentiable and the limits commute so that

$$h(x) = \lim_{N\to\infty} h_N(x) = \lim_{N\to\infty} (g_N)'(x) = (\lim_{N\to\infty} g_N)'(x) = \zeta'(x).$$

For the uniform convergence we use the Weierstrass M-test. Observe that for any $a > 1$ and $x \geq a$ we have for any $n \geq 2$

$$\frac{1}{n^x} \leq \frac{1}{n^a}, \quad \frac{1}{n^x \log n} \leq \frac{1}{n^a \log n}.$$

Since $\sum_n \frac{1}{n^a}$ and $\sum_n \frac{1}{n^a \log n} < \infty$ (Calculus) we have from the Weierstrass M-test uniform convergence in $[a,\infty)$.

Bonus question

13. (bonus 2 points): draw a funny uplifting picture relating the current situation and the topics of the Math420 course