Rules:

- Each question requires a proof or a counterexample.
- Each problem is worth 10 points.
- You can ask questions about this exam only to your class' TA or the instructor – these questions are like in an in-class exam: TA or instructor will not help you to solve these questions.
- You can use your textbook, the lecture notes, your notes from class and recitation. You can refer to theorems of the lecture, but only if it does not trivialize the question.
- You are in particular not allowed to look for the solutions online.
- If you think there is a typo, explain why and give a counterexample.

Grading

Your total points consist are computed as follows:

Let

$$\text{Part}_i := \{k : \quad \text{Problem } k \text{ belongs to Part } i\}.$$

and

$$\text{points}(k) := \text{points in Problem } k.$$

Let

$$I := \Big\{(k_1, \ldots, k_6) : \quad k_\ell \neq k_{\tilde{\ell}} \text{ if } \ell \neq \tilde{\ell}, \quad \{k_1, \ldots, k_6\} \cap \text{Part}_\text{i} \neq \emptyset \text{ for all } i = 1, 2, 3\Big\}$$

Then

$$\text{totalpoints} := \max_{(k_1,\ldots,k_6)\subset I} \sum_{\ell=1}^{6} \text{points}(k_\ell)$$

If that is not totally clear, we take the maximal sum of 6 problems which meet the following condition

- at least one problem is from Part 1, at least one problem from Part 2, at least one problem from Part 3
- the sum of the best three of the remaining problems.

Problems

Part 1

1. Let $S \subset \mathbb{R}$ and $f, g : S \to \mathbb{R}$. Suppose that c is a cluster point of S, $\lim\limits_{x \to c} f(x)$ exists and $\lim\limits_{x \to c} g(x)$ does not exist. Prove or give a counterexample for each of the following statements.
   (a) $\lim\limits_{x \to c}(f(x) - g(x))$ does not exist.
   (b) $\lim\limits_{x \to c}(f(x)g(x))$ does not exist.

2. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that that is continuous at every integer but discontinuous at every point in $\mathbb{R}\backslash\mathbb{Z}$. That is, a function such that
   - for any $x \in \mathbb{Z}$ f is continuous in x
   - for any $x \notin \mathbb{Z}$ f is discontinuous in x.

   Prove that your functions satisfies the given requirements.

3. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous and bounded on $\mathbb{R}$, but not uniformly continuous on $\mathbb{R}$. Include a proof to show that it is not uniformly continuous.

4. Let $f : [0, \infty) \to \mathbb{R}$ be a nonnegative and continuous function such that $\lim_{x \to \infty} f(x) = 0$. Prove that f has an absolute maximum value.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be homogeneous of order $\sigma \in \mathbb{R}$. That is, assume that
$$f(\lambda x) = \lambda^\sigma f(x) \quad \forall \lambda > 0 \; \forall x \in \mathbb{R}.$$
   (a) Show that $f(0) = 0$.
   (b) Show that f is continuous in $\mathbb{R}\backslash\{0\}$.
   (c) Assume $\sigma > 0$. Show that f is then necessarily continuous in all of $\mathbb{R}$.
   (d) Assume $\sigma \leq 0$. Show that f is either discontinuous in $x = 0$ or f is constantly zero.

Part 2

6. Let $f : [a, b] \to \mathbb{R}$ continuous on $[a, b]$
   (a) Prove the following: if $f(x) \geq 0$ for all $x \in [a, b]$ and $\int_{[a,b]} f = 0$ then $f(x) = 0$ for all $x \in [a, b]$ (*hint: argue by contradiction. What happens if $f(x) > 0$ for some x?)*
   (b) Disprove the following: if $\int_{[a,b]} f = 0$ then $f(x) = 0$ for all $x \in [a, b]$

7. Let $f : [-1, 1] \to \mathbb{R}$ be continuous. Assume that for any $a, b \in (-1, 1)$ we have
$$\int_{[a,b]} f \geq 0.$$
   Show that $f(x) \geq 0$ for all $x \in [-1, 1]$.

   Hint: Argue by contradiction, assuming that there exists $x \in [-1, 1]$ such that $f(x) < 0$.

8. Give an example of $f \in [0, 1]$ continuous and Riemann-integrable, but
$$U(P, f) \neq L(P, f)$$
   for any partition $P = \{x_0, \ldots, x_N\}$ of $[0, 1]$. Justify your answer with a proof.

9. Let f be a differentiable function on $(0, \infty)$ such that $\lim_{x\to\infty} f'(x) = 0$. Let $g(x) = f(x+\pi) - f(x)$. Prove that $\lim_{x\to\infty} g(x) = 0$. Give an example to show that f may be unbounded.

Part 3

10. Let I be an interval. Suppose that $\{f_n\}$ converges uniformly to f on I and $\{g_n\}$ converges uniformly to g on I.
    (a) Prove that $\{f_n + g_n\}$ converges uniformly to $f + g$ on I.
    (b) Show that $\{f_n g_n\}$ converges uniformly to fg if f and g are bounded.
    (c) Show that $\{f_n g_n\}$ may not converge uniformly to fg on I.

11. Show that $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2(x+1)}$ is continuous and differentiable.

    Hint: start by showing that the series is uniformly convergent on $[a, \infty)$ for $a > 0$.

12. For $x \in \mathbb{R}$ set

$$\zeta(x) := \sum_{n=1}^{\infty} \frac{1}{n^x}$$

    (a) for which $x \in \mathbb{R}$ is $\zeta(x)$ well-defined (i.e. finite)?
    (b) for which $x \in \mathbb{R}$ is $\zeta(x)$ is continuous?
    (c) for which $x \in \mathbb{R}$ is $\zeta(x)$ is differentiable and

$$\zeta'(x) := \sum_{n=1}^{\infty} \frac{1}{n^x \log n}$$

    Prove each of your claims!

Bonus question

13. (bonus 2 points): draw a funny uplifting picture relating the current situation and the topics of the Math420 course