SERIE 8 SOLUTIONS

28) (i) SINCE f IS A MEASURABLE FUNCTION, λf IS ALSO A MEASURABLE FUNCTION.

LET US NOTE THAT
$$(\lambda f)^+ = \begin{cases} \lambda f^+ , & \lambda \geq 0 \\ |\lambda| f^- , & \lambda < 0 \end{cases}$$
$$(\lambda f)^- = \begin{cases} \lambda f^- , & \lambda \geq 0 \\ |\lambda| f^+ , & \lambda < 0 \end{cases}$$

SINCE f IS "UNEIGENTLICH" μ- INTEGRABLE, IT ~~IT~~ FOLLOWS THA
EITHER $\int_\Omega f^+ \, d\mu < +\infty$ OR $\int_\Omega f^- \, d\mu < \infty$.
HENCE DEPENDING ON THE SIGN OF λ, EITHER $\int_\Omega (\lambda f)^+ \, d\mu < +$
OR $\int_\Omega (\lambda f)^- \, d\mu < +\infty$.

LET US CONSIDER THE CASE WHEN $|f| < \infty$ μ-a.e., THE OTHE
CASES FOLLOW SIMILARLY.

IF $\lambda > 0$,
$$\left.\begin{aligned} (\lambda f)^+ &= \lambda f^+ = \sum_{a_k > 0} \lambda a_k \chi_{A_k} \\ (\lambda f)^- &= \lambda f^- = \sum_{a_k < 0} \lambda (-a_k) \chi_{A_k} \end{aligned}\right\} \Rightarrow \begin{aligned} \lambda f &= (\lambda f)^+ - (\lambda f) \\ &= \sum_{a_k > 0} \lambda a_k \chi_{A_k} + \sum_{a_k < 0} \lambda a_k \chi \\ &= \lambda \sum_{a_k \neq 0} a_k \chi_{A_k} \end{aligned}$$

IF $\lambda < 0$,
$$\left.\begin{aligned} (\lambda f)^+ &= |\lambda| f^- = \sum_{a_k < 0} |\lambda| (-a_k) \chi_{A_k} \\ (\lambda f)^- &= |\lambda| f^+ = \sum_{a_k > 0} |\lambda| a_k \chi_{A_k} \end{aligned}\right\} \Rightarrow \begin{aligned} \lambda f &= (\lambda f)^+ - (\lambda f)^- \\ &= \sum_{a_k < 0} |\lambda| (-a_k) \chi_{A_k} - \sum_{a_k > 0} |\lambda| a_k \chi_{A_k} \\ &= \underbrace{-|\lambda|}_{\lambda} \left(\sum_{a_k > 0} a_k \chi_{A_k} + \sum_{a_k < 0} a_k \chi_{A_k} \right) \\ &= \lambda \sum_{a_k \neq 0} a_k \chi_{A_k} \end{aligned}$$

$$\therefore \int_\Omega \lambda f \, d\mu = \lambda \sum_{a_k \neq 0} a_k \, \mu(A_k) = \lambda \int_\Omega f \, d\mu .$$

IF $\lambda = 0$, THEN $\lambda f = 0$ ~~a.e.~~ a.e. AND

$$\int_\Omega \lambda f \, d\mu = \int_\Omega 0 \, d\mu = 0 = 0 \cdot \int_\Omega f \, d\mu .$$

ii) SINCE $A_k \cap A_j = \emptyset = B_k \cap B_j$, $\forall j \neq k$, AND $\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} B_k = \Omega$,

WE CAN WRITE $A_k = \bigcup_{j=1}^{\infty} \underbrace{(A_k \cap B_j)}_{\text{DISJOINT}}$, $B_l = \bigcup_{k=1}^{\infty} \underbrace{(A_k \cap B_l)}_{\text{DISJOINT}}$.

THEN $(f+g)(x) = f(x) + g(x) = \sum_{j,k} (a_k + b_j) \chi_{A_k \cap B_j}$ IS A SIMPLE FUNCT

AND HENCE $$\int_\Omega (f+g) \, d\mu = \sum_{j,k} (a_k + b_j) \, \mu(A_k \cap B_j) .$$

AGAIN $$\int_\Omega f \, d\mu + \int_\Omega g \, d\mu = \sum_k a_k \, \mu(A_k) + \sum_j b_j \, \mu(B_j)$$

$$= \sum_{j,k} (a_k + b_j) \, \mu(A_k \cap B_j)$$

$$\therefore \int_\Omega (f+g) \, d\mu = \int_\Omega f \, d\mu + \int_\Omega g \, d\mu .$$

iii) LET $f \le g$ μ a.e. AND $a, b \in \mathbb{R}$ BE SUCH THAT $a > b$.

SINCE $f \le g$ μ a.e., $\exists\, N \,(\in \Omega)$ WITH $\mu(N) = 0$ SUCH THAT

$$f(x) \le g(x) \quad \forall\, x \in \Omega \setminus N.$$

NOW LET $x_0 \in f^{-1}(a) \cap g^{-1}(b)$. THEN $f(x_0) = a$, $g(x_0) = b$.

SINCE $a > b$, WE HAVE $f(x_0) > g(x_0)$.

$$\Rightarrow x_0 \in N$$

$$\Rightarrow f^{-1}(a) \cap g^{-1}(b) \subseteq N$$

$$\Rightarrow \mu\left(f^{-1}(a) \cap g^{-1}(b)\right) \le \mu(N) = 0$$

$$\therefore \mu\left(f^{-1}(a) \cap g^{-1}(b)\right) = 0.$$

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iv) LET $f \le g$ μ a.e. AND $a \in \mathbb{R} \setminus \{0\}$.

THEN,
$$\sum_{\substack{l=1 \\ b_l \ne 0}}^{\infty} b_l\, \mu\left(g^{-1}(a) \cap f^{-1}(b_l)\right) = \sum_{\substack{b_l \ne 0 \\ b_l > a}} b_l\, \mu\left(g^{-1}(a) \cap f^{-1}(b_l)\right) + \sum_{\substack{b_l \ne 0 \\ b_l \le a}} b_l\, \mu\left(g^{-1}(a) \cap f^{-1}(b_l)\right)$$

USING (iii), WE GET THAT FOR $b_l > a$, $\mu\left(g^{-1}(a) \cap f^{-1}(b_l)\right) = 0$

$$\therefore \sum_{\substack{l=1 \\ b_l \ne 0}}^{\infty} b_l\, \mu\left(g^{-1}(a) \cap f^{-1}(b_l)\right) \le \sum_{\substack{b_l \ne 0 \\ b_l \le a}} b_l\, \mu\left(g^{-1}(a) \cap f^{-1}(b_l)\right)$$

$$\le \sum_{\{l:\, b_l \ne 0,\, b_l \le a\}} a\, \mu\left(g^{-1}(a) \cap f^{-1}(b_l)\right)$$

$$= \cancel{\Sigma}\; a\, \mu\left(g^{-1}(a) \cap \left(\bigcup_{\{l:\, b_l \ne 0,\, b_l \le a\}} f^{-1}(b_l)\right)\right)$$

$[\because f^{-1}(b_l) \cap f^{-1}(b_k) = \varphi,\ \forall k \ne l]$ — (*)

IF $a > 0$, WE CAN PROCEED AS FOLLOWS USING (*),

$$\sum_{\substack{l=1 \\ b_l \ne 0}}^{\infty} b_l\, \mu\left(g^{-1}(a) \cap f^{-1}(b_l)\right) \le a\, \mu\left(g^{-1}(a) \cap \Omega\right) = a\, \mu\left(g^{-1}(a)\right)$$

$$\Rightarrow a\, \mu\left(g^{-1}(a)\right) \ge \sum_{\substack{l=1 \\ b_l \ne 0}}^{\infty} b_l\, \mu\left(g^{-1}(a) \cap f^{-1}(b_l)\right), \quad a > 0 ~~\in \mathbb{R}\setminus\{0\}~~.$$

IF $a < 0$, $\quad g^{-1}(a) \cap \Omega = g^{-1}(a) \cap \left(\bigcup_i f^{-1}(b_i) \right)$

$$\Rightarrow \mu\left(g^{-1}(a) \cap \Omega\right) = \mu\left(g^{-1}(a) \cap \left(\bigcup_i f^{-1}(b_i)\right)\right) \overset{\text{DISJOINT}}{=} \sum_i \mu\left(g^{-1}(a) \cap f^{-1}(b_i)\right)$$

$$= \underbrace{\mu\left(g^{-1}(a) \cap f^{-1}(0)\right)}_{\substack{= 0 \;\; [\because\, a<0 \\ \text{AND USING (iii)}]}} + \underbrace{\sum_{\substack{b_i \neq 0 \\ b_i > a}} \mu\left(g^{-1}(a) \cap f^{-1}(b_i)\right)}_{\substack{= 0 \\ [\text{USING (iii)}]}} + \sum_{\substack{b_i \neq 0 \\ b_i \leq a}} \mu\left(g^{-1}(a) \cap f^{-1}(b_i)\right)$$

$$\Rightarrow \mu\left(g^{-1}(a)\right) = \mu\left(g^{-1}(a) \cap \Omega\right) = \sum_{\substack{b_i \neq 0 \\ b_i \leq a}} \mu\left(g^{-1}(a) \cap f^{-1}(b_i)\right)$$

$$\Rightarrow a\,\mu\left(g^{-1}(a)\right) = \sum_{\substack{b_i \neq 0 \\ b_i \leq a}} a\,\mu\left(g^{-1}(a) \cap f^{-1}(b_i)\right)$$

$$\geq \sum_{\substack{i=1 \\ b_i \neq 0}}^{\infty} b_i\,\mu\left(g^{-1}(a) \cap f^{-1}(b_i)\right) \quad \left(\text{ARGUING AS IN } (*)\right)$$

$$\therefore\; a\,\mu\left(g^{-1}(a)\right) \geq \sum_{\substack{i=1 \\ b_i \neq 0}}^{\infty} b_i\,\mu\left(g^{-1}(a) \cap f^{-1}(b_i)\right), \quad a \in \mathbb{R} \setminus \{0\}.$$

(v) FIRST OF ALL, LET US NOTE THAT $a_\ell \, \mu\left(g^{-1}(0) \cap f^{-1}(a_\ell)\right) = \begin{cases} 0 & \text{if } a_\ell > 0 \text{ (by (iii))} \\ \leq 0 & \text{if } a_\ell \leq 0 \end{cases}$

$$\therefore \int_\Omega g \, d\mu = \sum_{b_k \neq 0} b_k \, \mu\left(g^{-1}(b_k)\right) \overset{\text{by (ii)}}{\geq} \sum_{b_k \neq 0} \sum_{\substack{\ell=1 \\ a_\ell \neq 0}}^{\infty} a_\ell \, \mu\left(g^{-1}(b_k) \cap f^{-1}(a_\ell)\right)$$

$$\geq \sum_{b_k \neq 0} \sum_{\substack{\ell=1 \\ a_\ell \neq 0}}^{\infty} a_\ell \, \mu\left(g^{-1}(b_k) \cap f^{-1}(a_\ell)\right) + \sum_{\substack{\ell=1 \\ a_\ell \neq 0}}^{\infty} a_\ell \, \mu\left(g^{-1}(0) \cap f^{-1}(a_\ell)\right)$$

$$= \sum_{\substack{\ell=1 \\ a_\ell \neq 0}}^{\infty} a_\ell \, \mu\Big(\underbrace{\Big(\bigcup_k g^{-1}(b_k)\Big)}_{= \Omega} \cap f^{-1}(a_\ell)\Big) = \sum_{\substack{\ell=1 \\ a_\ell \neq 0}}^{\infty} a_\ell \, \mu\left(f^{-1}(a_\ell)\right) = \int_\Omega f \, d\mu$$

$$\Rightarrow \int_\Omega f \, d\mu \leq \int_\Omega g \, d\mu .$$

(vi) <u>CASE 1</u>: $\int_\Omega f^+ \, d\mu < +\infty$, $\int_\Omega f^- \, d\mu < +\infty$

IN THIS CASE, SINCE f IS ITSELF A SIMPLE FUNCTION AND OBVIOUSLY SATISFIES $f \leq f$ a.e., WE CAN CONSIDER IT AS AN "APPROXIMATING" SIMPLE FUNCTION FROM ABOVE AND BELOW. HENCE WE HAVE

$$\left.\begin{aligned} \int_\Omega f \, d\mu &\leq \underline{\int}_\Omega f \, d\mu , \\ \overline{\int}_\Omega f \, d\mu &\leq \int_\Omega f \, d\mu \end{aligned}\right\}$$

$$\Rightarrow \overline{\int}_\Omega f \, d\mu \leq \underline{\int}_\Omega f \, d\mu$$

$$\Rightarrow \int_\Omega f \, d\mu = \underline{\int}_\Omega f \, d\mu = \overline{\int}_\Omega f \, d\mu .$$

<u>CASE 2</u>: $\int_\Omega f^+ \, d\mu = +\infty$, $\int_\Omega f^- \, d\mu < +\infty$.

THEREFORE $\int_\Omega f \, d\mu = +\infty$ AND SINCE $\underline{\int}_\Omega f \, d\mu \leq \overline{\int}_\Omega f \, d\mu$, IT ALSO FOLLOWS THAT $\int_\Omega f \, d\mu = \overline{\int}_\Omega f \, d\mu = +\infty$.

HENCE WE NEED TO PROVE THAT $\underline{\int}_\Omega f \, d\mu = +\infty$.

LET $A_{+\infty} = \{x \in \Omega : f(x) = +\infty\}$, $A_{-\infty} = \{x \in \Omega : f(x) = -\infty\}$.

SINCE $\int_\Omega f^- \, d\mu < +\infty$, WE HAVE $\mu(A_{-\infty}) = 0$.

WITHOUT LOSS OF GENERALITY, WE MAY ASSUME THAT $\mu(A_{+\infty}) < +\infty$

$\mu(A_k) < +\infty, \forall k$ SUCH THAT ~~$\mu($~~ a_k

OTHERWISE WE HAVE TO REPEAT THE ARGUMENTS BY CUTTING TH DOMAINS WITH BALLS OF RADIUS n.

LET US WRITE $f = \sum_{a_k<0} a_k \chi_{A_k} + \sum_{a_k>0} a_k \chi_{A_k} + (+\infty)\chi_{A_{+\infty}} + (-\infty)\chi_{A_-}$

WE DEFINE $e_n = \sum_{a_k<0} a_k \chi_{A_k} + \sum_{\substack{k=1 \\ a_k>0}}^{n} a_k \chi_{A_k} + n \chi_{A_{+\infty}}$

THEN $\forall n \in \mathbb{N}$, (i) $e_n \leq f$ μ-a.e.,

(ii) $\int_\Omega e_n^+ \, d\mu = \sum_{\substack{k=1 \\ a_k>0}}^{n} a_k \mu(A_k) + n \mu(A_{+\infty}) < +\infty$.

~~So~~ BUT $\int_\Omega e_n \, d\mu \to +\infty$ as $n \to \infty$.

$\Rightarrow \int_\Omega f \, d\mu = +\infty$.

$\boxtimes$

<u>CASE 3</u>: $\int_\Omega f^+ \, d\mu < +\infty$, $\int_\Omega f^- \, d\mu = +\infty$.

PROOF IS ANALOGOUS TO CASE 2 -

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(29) LET $f, g : \Omega \to \overline{\mathbb{R}}$ BE μ-INTEGRABLE FUNCTIONS AND LET $a \in \mathbb{R}$

i) LET $\varepsilon > 0$. THEN THERE EXIST SIMPLE FUNCTIONS $f_\varepsilon, f^\varepsilon, g_\varepsilon, g^\varepsilon$ SUCH THAT $\int_\Omega (f^\varepsilon)^- \, d\mu < \infty$, $\int_\Omega (g^\varepsilon)^- \, d\mu < \infty$,

$$\int_\Omega (f_\varepsilon)^+ \, d\mu < \infty, \quad \int_\Omega (g_\varepsilon)^+ \, d\mu < \infty$$

AND $\int_\Omega f^\varepsilon \, d\mu - \int_\Omega f \, d\mu < \varepsilon$, $\int_\Omega f \, d\mu - \int_\Omega f_\varepsilon \, d\mu < \varepsilon$,

$$\int_\Omega g^\varepsilon \, d\mu - \int_\Omega g \, d\mu < \varepsilon, \quad \int_\Omega g \, d\mu - \int_\Omega g_\varepsilon \, d\mu < \varepsilon.$$

ALSO, $f_\varepsilon + g_\varepsilon$, $f^\varepsilon + g^\varepsilon$ ARE SIMPLE FUNCTIONS SUCH THAT

$$f_\varepsilon + g_\varepsilon \le f + g \le f^\varepsilon + g^\varepsilon$$

AND $\int_\Omega (f_\varepsilon + g_\varepsilon)^+ \, d\mu < \infty$, $\int_\Omega (f^\varepsilon + g^\varepsilon)^- \, d\mu < \infty$.

NOW, $\overline{\int_\Omega} (f+g) \, d\mu \le \int_\Omega (f^\varepsilon + g^\varepsilon) \, d\mu = \int_\Omega f^\varepsilon \, d\mu + \int_\Omega g^\varepsilon \, d\mu$

$$\le \int_\Omega f \, d\mu + \int_\Omega g \, d\mu + 2\varepsilon. \quad \text{— (1)}$$

SIMILARLY, $\underline{\int_\Omega} (f+g) \, d\mu \ge \int_\Omega f \, d\mu + \int_\Omega g \, d\mu - 2\varepsilon$. — (2)

SINCE ε IS ARBITRARY, (1) AND (2) GIVES

$$\overline{\int_\Omega} (f+g) \, d\mu \le \int_\Omega f \, d\mu + \int_\Omega g \, d\mu \le \underline{\int_\Omega} (f+g) \, d\mu$$

ALSO BY DEFINITION, $\underline{\int_\Omega} (f+g) \, d\mu \le \overline{\int_\Omega} (f+g) \, d\mu$

$$\therefore \int_\Omega (f+g) \, d\mu = \underline{\int_\Omega} (f+g) \, d\mu = \overline{\int_\Omega} (f+g) \, d\mu = \int_\Omega f \, d\mu + \int_\Omega g \, d\mu$$

ALSO, APPLYING THIS TO THE FUNCTIONS $|f|$ AND $|g|$, WE HAVE $\int_\Omega |f+g| \, d\mu \le \int_\Omega |f| \, d\mu + \int_\Omega |g| \, d\mu < \infty \Rightarrow f+g$ IS μ-INTEGRABLE

(ii) LET $\varepsilon > 0$. AS IN (i) WE CAN CHOOSE A SIMPLE FUNCTION SUCH THAT $f_\varepsilon \le f \le f^\varepsilon$.

$\boxed{\lambda \ge 0}$ IN THIS CASE WE SEE THAT $\lambda f_\varepsilon, \lambda f^\varepsilon$ ARE SIMPLE FUNCTIONS SUCH THAT $\lambda f_\varepsilon \le \lambda f \le \lambda f^\varepsilon$, AND

$$\int_\Omega (\lambda f^\varepsilon)^- \, d\mu = \lambda \int_\Omega (f^\varepsilon)^- \, d\mu < +\infty, \quad \int_\Omega (\lambda f_\varepsilon)^+ \, d\mu = \lambda \int_\Omega (f_\varepsilon)^+ \, d\mu < +\infty$$

THEN $\overline{\int_\Omega} (\lambda f) \, d\mu \le \int_\Omega (\lambda f^\varepsilon) \, d\mu = \lambda \int_\Omega f^\varepsilon \, d\mu \le \lambda \int_\Omega f \, d\mu + \lambda \varepsilon$

AND SIMILARLY $\underline{\int_\Omega} (\lambda f) \, d\mu \ge \lambda \int_\Omega f \, d\mu - \lambda \varepsilon$.

SINCE ε IS ARBITRARY, $\overline{\int_\Omega} (\lambda f) \, d\mu \le \lambda \int_\Omega f \, d\mu \le \underline{\int_\Omega} (\lambda f) \, d\mu$

$$\Rightarrow \int_\Omega (\lambda f) \, d\mu = \underline{\int_\Omega} (\lambda f) \, d\mu = \overline{\int_\Omega} (\lambda f) \, d\mu = \lambda \int_\Omega f \, d\mu$$

$\boxed{\lambda < 0}$ IN THIS CASE WE NOTE THAT $\lambda f^\varepsilon \le \lambda f \le \lambda f_\varepsilon$, AND

$$\int_\Omega (\lambda f^\varepsilon)^+ \, d\mu = |\lambda| \int_\Omega (f^\varepsilon)^- \, d\mu < +\infty,$$

$$\int_\Omega (\lambda f_\varepsilon)^- \, d\mu = |\lambda| \int_\Omega (f_\varepsilon)^+ \, d\mu < +\infty.$$

THEN WE CAN PROCEED AS IN THE PREVIOUS CASE.

$\therefore \int_\Omega (\lambda f) \, d\mu = \lambda \int_\Omega f \, d\mu$, AND

$\int_\Omega |\lambda f| \, d\mu = |\lambda| \int_\Omega |f| \, d\mu < +\infty$; THAT IS, λf IS μ-INTEGRABLE.

(iii) $f = g$ μ-a.e. $\Rightarrow f^+ = g^+$ μ-a.e., $f^- = g^-$ μ-a.e.

NOW, f, g INTEGRABLE $\Rightarrow f^+, g^+, f^-, g^-$ ARE INTEGRABLE

$$f^+ = g^+ \Rightarrow \begin{cases} f^+ \le g^+ \Rightarrow \int_\Omega f^+ \le \int_\Omega g^+ \\ f^+ \ge g^+ \Rightarrow \int_\Omega f^+ \ge \int_\Omega g^+ \end{cases} \Rightarrow \int_\Omega f^+ = \int_\Omega g^+$$

SIMILARLY, $\int_\Omega f^- = \int_\Omega g^-$ AND HENCE $\int_\Omega f \, d\mu = \int_\Omega g \, d\mu$

30 $f : \mathbb{R} \to [0, \infty)$ SUCH THAT $f(x) := \begin{cases} 0, & x \le 1, \\ \frac{1}{x}, & x > 1. \end{cases}$

(i) WE HAVE ALREADY SEEN THAT f IS A MEASURABLE FUNCTION. ALSO $f \geq 0$.

THEREFORE, f IS "UNEIGENTLICH" - $\mathcal{L}^1$-INTEGRABLE.

(ii) NEXT WE SHOW THAT f IS $\underline{\underline{\text{NOT}}}$ $\mathcal{L}^1$-INTEGRABLE.

TO SEE THIS, LET US DEFINE $a_1 = 0$, $a_k = \frac{1}{k}$, $k \geq 2$

$A_1 = (-\infty, 1]$, $A_k = (k-1, k]$, $k \geq 2$

AND $g(x) = \sum_{k=1}^{\infty} a_k \chi_{A_k}$.

THEN g IS A SIMPLE FUNCTION, AND $g(x) \leq f(x)$ a.e.

FURTHER, $\int_{\mathbb{R}} g(x)\, dx = \sum_{k=1}^{\infty} a_k \mathcal{L}^1(A_k) = \sum_{k=2}^{\infty} \frac{1}{k} \underbrace{\mathcal{L}^1(A_k)}_{=1}$

$= \sum_{k=2}^{\infty} \frac{1}{k}$ WHICH DIVERGES TO $+\infty$

$\therefore \int_{\mathbb{R}} f(x)\, dx \geq \int_{\mathbb{R}} g(x)\, dx = +\infty$

$\Rightarrow$ f IS NOT $\mathcal{L}^1$-INTEGRABLE.

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31 LET $a > 0$ AND $f : \Omega \to \overline{\mathbb{R}}$ BE A μ-INTEGRABLE FUNCTION

LET US DEFINE $\begin{cases} f_1 := a^{-1} |f| \\ f_2 := \chi_{\{x \in \Omega : |f(x)| \geq a\}} \end{cases}$

CLAIM: ~~SUPPOSE~~ $f_1 \geq f_2$ μ a.e.

SUPPOSE THAT THE CLAIM IS TRUE. THEN BY MONOTONICITY,

$\int_\Omega f_2 \, d\mu \leq \int_\Omega f_1 \, d\mu$

$\Rightarrow \mu(\{x \in \Omega : |f(x)| \geq a\}) \leq a^{-1} \int_\Omega |f| \, d\mu$.

PROOF OF CLAIM: LET $\tilde{\Omega} := \{x \in \Omega : |f(x)| \geq a\}$.

THEN $f_2 = \chi_{\tilde{\Omega}}$.

LET $x \in \tilde{\Omega}^c$. THEN $f_2(x) = 0$ AND BY DEFINITION, $f_1(x) \geq 0$.

$$\Rightarrow \quad f_1(x) \geq f_2(x) \quad \forall x \in \tilde{\Omega}^c.$$

LET $x \in \tilde{\Omega}$. THEN $f_2(x) = 1$.

NOW, $x \in \tilde{\Omega} \Rightarrow |f(x)| \geq a$

$$\Rightarrow a^{-1}|f(x)| \geq 1$$

$$\Rightarrow f_1(x) \geq 1$$

$$\therefore \quad f_1(x) \geq f_2(x) \quad \forall x \in \tilde{\Omega}$$

$$\Rightarrow \quad f_1 \geq f_2 \quad \mu \text{ a.e. } \Omega.$$

THIS PROVES THE CLAIM.
