Reelle Analysis - Serie 6.

A21: Def.: $\cdot)$ μ is Borel-regular if μ is a Borel-measure s.t. $\forall A \subseteq \mathbb{R}^n$, $\exists B \in \mathcal{B}(\mathbb{R}^n)$: $A \subseteq B$ and $\mu(A) = \mu(B)$

$\cdot)$ μ is a Radon-measure if μ is Borel-regular and locally-finite i.e. $K \subseteq \mathbb{R}^n$ compact $\Rightarrow \mu(K) < \infty$.

(i) $\mathcal{L}^n$ Borel-regular: ($\mathcal{L}^n$ is defined on $\mathcal{B}(\mathbb{R}^n)$ = Borel sets). Let $A \subseteq \mathbb{R}^n$.

If $\mathcal{L}^n(A) = \infty$: $B = \mathbb{R}^n \in \mathcal{B}(\mathbb{R}^n)$ is such that $A \subseteq B$ and $\mathcal{L}^n(A) = \mathcal{L}^n(B)$

If $\mathcal{L}^n(A) < \infty$: then since $\mathcal{L}^n(A) = \inf\{\mathcal{L}^n(G) : A \subseteq G, G \text{ open}\}$ (from def. of the pre-measure Vol and a decomposition $A \subseteq \bigcup_{l=1}^{\infty} I_l$ into cuboids). (I_l Quader).

we know: $\forall k \in \mathbb{N}^*$, $\exists G_k \subseteq \mathbb{R}^n$ open s.t. $A \subseteq G_k$ and $\mathcal{L}^n(G_k) \leq \mathcal{L}^n(A) + \frac{1}{k}$.

Let $B := \bigcap_{k=1}^{\infty} G_k \in \mathcal{B}(\mathbb{R}^n)$, ~~from $G_{k+1} \subseteq G_k$, $\forall k \in \mathbb{N}$ (i.e. (G_k) is decreasing for "$\subseteq$" in 2^X).~~

we deduce $\mathcal{L}^n(B) = \lim_{k \to +\infty} \mathcal{L}^n(G_k) \leq \mathcal{L}^n(A)$.

Moreover, $A \subseteq G_k$, $\forall k \in \mathbb{N} \Rightarrow A \subseteq B$ so that $\mathcal{L}^n(A) \leq \mathcal{L}^n(B)$ by Monotonicity.

$\mathcal{L}^n$ locally-finite: Let $K \subseteq \mathbb{R}^n$ compact. (i.e. closed and bounded in $\mathbb{R}^n$)

We know $\exists r > 0$: $K \subset B(x,r) \subset [-r,r]^n$ and thus by monotonicity:

ball for the euclidean norm; ball for the sup norm

$\mathcal{L}^n(K) \leq \mathcal{L}^n([-r,r]^n) = (2r)^n < \infty$.

(ii) $\mathcal{H}^s$ is Borel-regular, but not locally finite when $s < n$.

Counter-example: $K = [0,1]^n$ is compact with $\mathcal{H}^n([0,1]^n) = \omega_n \mathcal{L}^n([0,1]^n) = \omega_n > 0$

but then Lemma 3.39 $\Rightarrow \mathcal{H}^s([0,1]^n) = \infty$, $\forall s < n$.

(iii) $\mu \llcorner A$ is a measure:

$\cdot)$ $\mu \llcorner A(\emptyset) = \mu(A \cap \emptyset) = \mu(\emptyset) = 0$ since μ is a measure

$\cdot)$ If $B \subseteq \bigcup_{k=1}^{\infty} B_k$ then: $\mu \llcorner A(B) = \mu(A \cap B) \underset{\text{Monotonicity}}{\leq} \mu\left(A \cap \bigcup_{k=1}^{\infty} B_k\right) = \mu\left(\bigcup_{k=1}^{\infty} (A \cap B_k)\right)$

Distributivity of $\cap$ on $\cup$

$$\leq \sum_{k=1}^{\infty} \mu(A \cap B_k) \underset{\text{def.}}{=} \sum_{k=1}^{\infty} \mu \llcorner A(B_k).$$

σ-subadditivity for the measure μ

$\mu \llcorner A$ is locally-finite: Let $K \subset \mathbb{R}^n$ compact. Then $\mu \llcorner A(K) \leq \mu(K) < \infty$.
by mononotonicity $A \cap K \subset K$.

(ii) A22: (i) $\Rightarrow$ (ii): Let $A \subset \mathbb{R}^n$ μ-measurable and $\varepsilon > 0$ be fixed.

By definition of the infimum, $\exists\, G \subset \mathbb{R}^n$ open s.t. $A \subseteq G$ and $\mu(G) \leq \mu(A) + \varepsilon$.

Then A μ-measurable $\Rightarrow$ $\mu(G) = \mu(\underbrace{G \cap A}_{= A}) + \mu(G \setminus A) = \mu(A) + \mu(G \setminus A)$.

Case 1: $\mu(A) < \infty$: we find $\mu(G \setminus A) \leq \varepsilon$.

Case 2: $\mu(A) = \infty$: we localize $A = \bigcup_{k=1}^{\infty} A_k$ where $A_k := A \cap [-k,k]^n$.

clearly A_k is μ-measurable and $\mu(A_k) \underset{\text{Monotonicity}}{\leq} \mu([-k,k]^n) < \infty$ since μ is Radon

so Case 1 $\Rightarrow$ given $2^{-k} \cdot \varepsilon > 0$, $\exists\, G_k \subset \mathbb{R}^n$ open s.t. $A_k \subseteq G_k$ and $\mathcal{L}^n(G_k \setminus A_k) \leq 2^{-k} \cdot \varepsilon$

Now let $G := \bigcup_{k=1}^{\infty} G_k$ open (union of open sets), we ~~have~~ : by monotonicity:

$$\mathcal{L}^n(G \setminus A) \leq \sum_{k=1}^{\infty}{}' \mu\Big(\underbrace{G_k \cap \bigcap_{k=1}^{\infty} A_k^c}_{\subset G_k \setminus A_k}\Big) \leq \sum_{k=1}^{\infty}{}' \mu(G_k \setminus A_k) \leq \sum_{k=1}^{\infty}{}' 2^{-k} \cdot \varepsilon = \varepsilon.$$

(ii) $\Rightarrow$ (i): Let $\varepsilon > 0$ be fixed. Let $B \subset \mathbb{R}^n$ and G open s.t. $\mu(G \setminus A) \leq \varepsilon$.

◦) By monotonicity: $\mu(B) \leq \mu(B \cap A) + \mu(B \setminus A)$

◦) μ Radon-measure $\Rightarrow$ μ Borel-measure; G open $\Rightarrow$ $G \in \mathcal{B}(\mathbb{R}^n)$ (Borel set) $\Big\} \Rightarrow$ G μ-measurable and:
$$\mu(B) = \mu(B \cap G) + \mu(B \setminus G). \quad (*)$$

Then: $A \subseteq G \Rightarrow \mu(B \cap A) \leq \mu(B \cap G)$

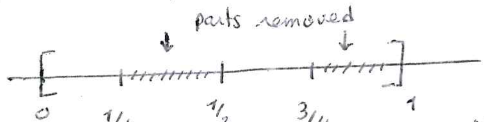
$B \setminus A \subseteq (B \setminus G) \cup (G \setminus A) \Rightarrow \mu(B \setminus A) \leq \mu(B \setminus G) + \mu(G \setminus A)$

$\Rightarrow \mu(B \setminus G) \geq \mu(B \setminus A) - \mu(G \setminus A)$ since $\mu(G \setminus A) \leq \varepsilon < \infty$

so: $\mu(B) \underset{(*)}{\geq} \mu(B \cap A) + \mu(B \setminus A) - \mu(G \setminus A) \geq \mu(B \cap A) + \mu(B \setminus A) - \varepsilon$.

and this holds for $\varepsilon > 0$ arbitrary small.

A23: We define the Cantor set $C = \bigcap_{k=0}^{\infty} A_k$ where $\begin{cases} A_0 = [0,1] \\ A_{k+1} = \frac{1}{4} A_k \cup \left(\frac{1}{2} + \frac{1}{4} A_k\right) \end{cases}$



We prove that $\dim_{\mathcal{H}}(C) = \frac{\ln(2)}{\ln(3)} =: \alpha$

By induction, we prove that $A_k = \bigcup_{l=1}^{2^k} I_{k,l}$: union of 2^k intervals of length $|I_{k,l}| = 4^{-k}$. In other words, $\forall k \in \mathbb{N}$, $(I_{k,l})_{l=1}^{2^k}$ is a covering (überdeckung) of C.

Recall: $\mathcal{H}^s(C) = \sup_{\delta > 0} \mathcal{H}^s_\delta(C) = \sup_{\delta > 0} \inf \left\{ \sum_{k=1}^{\infty}{}' r_k^s : A \subset \bigcup_{k=1}^{\infty} B(x_k, r_k),\ r_k < \delta \right\}$

$(B(x_k, r_k))_{k=1}^{\infty}$ being a particular type of cover with balls only

or $\mathcal{H}^s(C) = \inf \Big\{ \underbrace{\sum_{k=1}^{\infty}{}' |C_k|^s}_{} : (C_k)$ is a countable cover of $C \Big\}$.

$=$ "s-length" of the cover $(C_k)_{k \geq 1}$.

and $\dim_{\mathcal{H}}(C) = \inf\{s>0 : \mathcal{H}^s(C)=0\}$.

$\underline{\dim_{\mathcal{H}}(C) \leq \alpha :}$ $\quad$ $\underline{\text{i.e.}}$ $s > \alpha \Rightarrow \mathcal{H}^s(C) = 0$.

Let $k \in \mathbb{N}$. The s-length of the cover $(I_{k,\ell})_{\ell=1}^{2^k}$ is:

$$\sum_{\ell=1}^{2^k}{}' |I_{k,\ell}|^s = 2^k \cdot (4^{-k})^s = \exp\left(k\left(\ln(2) - s\ln(4)\right)\right)$$

In particular, $s>\alpha \iff \ln(2) - s\ln(4) < 0$, so that:

$$\mathcal{H}^s(C) \leq \lim_{k\to+\infty} \sum_{\ell=1}^{2^k}{}' |I_{k,\ell}|^s = \lim_{k\to+\infty} \exp\left(k\left(\ln(2)-s\ln(4)\right)\right) = 0.$$

$\underline{\dim_{\mathcal{H}}(C) \geq \alpha :}$ $\quad$ Let $(C_n)_{n\geq 1}$ be a $\underline{\text{countable}}$ cover of C with $C_n \subset [0,1]$.

$\underline{\text{claim}}$: Given $\varepsilon > 0$, $\exists$ $\underline{\text{open}}$ intervals $(D_j)_{j=1}^m$ s.t. $\bigcup_{n=1}^{\infty} C_n \subseteq \bigcup_{j=1}^{m} D_j$

with $\sum_{j=1}^{m}{}' |D_j|^\alpha \leq \sum_{n=1}^{\infty}{}' |C_n|^\alpha + \varepsilon$ $\quad (*)$

Up to enlarging the C_n, this follows from $\underline{\text{Borel-Lebesgue}}$ for the cover $(C_n)_{n\geq1}$ of the compact $[0,1]$.

Now select $k \in \mathbb{N}$ s.t. $4^{-k} \leq \min_{1\leq j\leq m} |D_j|$ and for $\ell \in [\![1,k]\!]$, let:

$N_\ell := \#\{D_j : 4^{-\ell} \leq |D_j| \leq 4^{-\ell+1}\}$ $\quad$ (clearly $\sum_{\ell=1}^{k}{}' N_\ell = m$).

We have $\displaystyle\sum_{j=1}^{m}{}' |D_j|^\alpha \geq \sum_{\ell=1}^{k}{}' N_\ell \cdot (4^{-\ell})^\alpha \underset{\alpha = \frac{\ln(2)}{\ln(4)}}{=} \sum_{\ell=1}^{k}{}' N_\ell \cdot 2^{-\ell}$. $\quad (**)$

Now $\underline{\text{fix } \ell \in [\![1,k]\!]}$, so we estimate N_ℓ:

Suppose $D_{j'}$ is such that $4^{-\ell} \leq |D_{j'}| \leq 4^{-\ell+1}$. Then $D_{j'}$ can intersect at most 2 of the intervals $I_{\ell,j}$, $1 \leq j \leq 2^\ell$, in the decomposition $A_\ell = \bigcup_{j=1}^{2^\ell} I_{\ell,j}$

(or at most 4 if we don't bother counting empty spaces).

So, $\underline{\text{at step } k}$: $\underline{\text{each}}$ of those $I_{\ell,j}$ that $D_{j'}$ intersects produces $2^{k-\ell}$ more intervals, and $D_{j'}$ thus intersects at most $2 \cdot 2^{k-\ell}$ of the $I_{k,j}$, $1 \leq j \leq 2^k$.

In other words: $\quad 2^k \leq \displaystyle\sum_{\ell=1}^{k}{}' N_\ell \cdot 2 \cdot 2^{k-\ell}$

Number of $I_{k,j}$ such $D_{j'}$ can intersect.

Total number of $I_{k,j}$

Number of $D_{j'}$ s.t. $4^{-\ell} \leq |D_{j'}| \leq 4^{-\ell+1}$

$$\Rightarrow \sum_{\ell=1}^{k}{}' N_\ell \cdot 2^{-\ell} \geq \frac{1}{2} \quad \underset{(**)}{\Longrightarrow} \quad \sum_{j=1}^{m}{}' |D_j|^\alpha \geq \frac{1}{2}.$$

From $(*)$ with $\varepsilon = 1/4$ we deduce $\sum_{n=1}^{\infty}{}' |C_n|^{\alpha} \geq \frac{1}{4}$

It follows that $\mathcal{H}^{\alpha}(C) \geq \frac{1}{4}$. and thus $\dim_{\mathcal{H}}(C) \geq \alpha$.