Reelle Analysis - Serie 5

A17: (i) Def. $\mathcal{B}(\mathbb{R}^n)$ = smallest σ-algebra generated by open subsets of $\mathbb{R}^n$
= " " " closed " "

Recall def. $G \subset \mathbb{R}^n$ is open iff $\forall x \in G, \exists r>0$ s.t. $B(x,r) \subset G$ where $B(x,r) = \{y \in \mathbb{R}^n : \|y-x\| < r\}$
$F \subset \mathbb{R}^n$ is closed iff $\mathbb{R}^n \setminus F = F^c$ is open.

(i) $\{x\} \in \mathcal{B}(\mathbb{R}^n), \forall x \in \mathbb{R}^n$: because $\{x\}$ is closed since $\mathbb{R}^n \setminus \{x\}$ is open: $y \in \mathbb{R}^n \setminus \{x\}$
$\Rightarrow B\left(y, \frac{\|y-x\|}{4}\right) \subset \mathbb{R}^n \setminus \{x\}$

Alternatively, $\{x\} = \bigcap_{k=1}^{\infty} \underbrace{\left]x - \frac{1}{k}, x + \frac{1}{k}\right[}_{\in \mathcal{B}(\mathbb{R}^n) \text{ (open intervals)}} \in \mathcal{B}(\mathbb{R}^n)$ where: $]x-\varepsilon, x+\varepsilon[=]x_1 - \varepsilon, x_1+\varepsilon[\times \dots \times]x_n - \varepsilon, x_n + \varepsilon[$

$\mathcal{L}^n(\{x\}) = 0$: $\forall \varepsilon > 0$, $\{x\} \subset]x-\varepsilon, x+\varepsilon[$, so by monotonicity of the measure $\mathcal{L}^n$:
$\mathcal{L}^n(\{x\}) \leq \mathcal{L}^n(]x-\varepsilon, x+\varepsilon[) = (2\varepsilon)^n$ for arbitrary small $\varepsilon > 0$.

(ii) $\mathbb{Q}$ is countable and can be described as $\mathbb{Q} = \bigcup_{k \in \mathbb{N}} \underbrace{\{q_k\}}_{\in \mathcal{B}(\mathbb{R}) \text{ by (i)}} \in \mathcal{B}(\mathbb{R})$

By σ-subadditivity it implies: $\mathcal{L}^1(\mathbb{Q}) \leq \sum_{k \in \mathbb{N}} \underbrace{\mathcal{L}^1(\{q_k\})}_{=0} = 0$.

(iii) We have $\mathbb{R}^n = \bigcup_{k \in \mathbb{N}} [-k,k] \in \mathcal{B}(\mathbb{R}^n)$ where $[-k,k] = \prod_{i=1}^{n} [-k,k] = [-k,k] \times \dots \times [-k,k]$

and in particular: $\forall k \in \mathbb{N}$, $[-k,k] \subset \mathbb{R}^n$, so that by monotonicity:
$\underbrace{(2k)^n}_{\longrightarrow +\infty \text{ as } k \to +\infty} = \mathcal{L}^n([-k,k]) \leq \mathcal{L}^n(\mathbb{R}^n)$ which implies $\mathcal{L}^n(\mathbb{R}^n) = \infty$.

A18: (i) $\Rightarrow$ (ii) Let $\varepsilon > 0$ be fixed. We use Proposition 3.28:

for A: $\exists G \subset \mathbb{R}^n$ open s.t. $A \subset G$ and $\mathcal{L}^n(G \setminus A) < \varepsilon$ (a)

for A^c: $\exists \tilde{G} \subset \mathbb{R}^n$ open s.t. $A^c \subset \tilde{G}$ and $\mathcal{L}^n(\tilde{G} \setminus A^c) < \varepsilon$. (b)

Note that by definition $\left(\forall \underset{\text{borel}}{B} \subset \mathbb{R}^n, \mathcal{L}^n(B) = \mathcal{L}^n(B \cap A) + \mathcal{L}^n(B \cap A^c) \right)$ A $\mathcal{L}^n$-measurable $\Rightarrow A^c$ $\mathcal{L}^n$-measurable

Now let $F := \tilde{G}^c \subset A$ with F closed. We find:
$\mathcal{L}^n(A \setminus F) = \mathcal{L}^n(A \cap F^c) = \mathcal{L}^n(A \cap \tilde{G}) = \mathcal{L}^n(\tilde{G} \setminus A^c) < \varepsilon$ from (b), so we conclude with (a):
that $\mathcal{L}^n(A \setminus F) + \mathcal{L}^n(G \setminus A) < 2\varepsilon$. (*)

(ii) $\Rightarrow$ (iii) Given that (*) holds $\forall \varepsilon > 0$, from $G \setminus F = (G \setminus A) \cup (A \setminus F)$, we see
by monotonicity that: $\mathcal{L}^n(G \setminus F) \leq \mathcal{L}^n(G \setminus A) + \mathcal{L}^n(A \setminus F) \underset{(*)}{<} 2\varepsilon$. (**)

(iii) $\Rightarrow$ (i): Given that (**) holds $\forall \varepsilon > 0$, from $G \setminus A \subset G \setminus F$ since $F \subset A$, we see
by monotonicity again that: $\mathcal{L}^n(G \setminus A) \leq \mathcal{L}^n(G \setminus F) < 2\varepsilon$, which is all there is to prove to show that A is $\mathcal{L}^n$-measurable, according to Proposition 3.28.

A19: Vitali set: $\exists V \subset [0,1]$ s.t. (1) $\forall x,y \in V,\ x \neq y \Rightarrow x-y \notin \mathbb{Q}$.
(2) $\forall x \in \mathbb{R},\ \exists z \in \mathbb{Q}$ s.t. $x - z \in V$

Existence of V may be ensured by the axiom of choice as $V = \{[x] : x \in [0,1]\} = \mathbb{R}/\sim$
where $y \in [x] \Leftrightarrow y \sim x \Leftrightarrow x - y \in \mathbb{Q}$.

We only use the following properties for the measure $\mathcal{L}^1$:

(Π1) $\forall (A_k)_{k\in\mathbb{N}}$ pairwise disjoint $\mathcal{L}^n$-measurable subsets: $\mathcal{L}^1\left(\bigcup_{k\in\mathbb{N}} A_k\right) = \sum_{k\in\mathbb{N}} \mathcal{L}^1(A_k)$
(σ-additivity)

(Π2) A congruent to $B \Rightarrow \mathcal{L}^1(A) = \mathcal{L}^1(B)$ — Def: A and B congruent iff $B = f(A)$ with f affine s.t. $f(x) = Ax + b$, $b \in \mathbb{R}^n$, $A \in \mathcal{O}(n)$

(Π3) $\mathcal{L}^1([0,1]) = 1$

By contradiction, we suppose V is $\mathcal{L}^1$-measurable. Let $\mathbb{Q} \cap (-1,1) = \bigcup_{k\in\mathbb{N}} \{q_k\}$
and $V_k = q_k + V$. V measurable (w.r.t. $\mathcal{L}^n$) $\overset{(\Pi 2)}{\Longrightarrow}$ V_k $\mathcal{L}^n$-measurable $\forall k \in \mathbb{N}$.

- $(V_k)_{k\in\mathbb{N}}$ pairwise disjoint: If $q_k + v_1 = q_\ell + v_2 \in V_k \cap V_\ell$, then $v_2 - v_1 = q_k - q_\ell \in \mathbb{Q}$
and $v_2 - v_1 \in \mathbb{Q} \overset{(1)}{\Longrightarrow} v_1 = v_2 \Rightarrow q_k = q_\ell$ i.e. $k = \ell$. CCL: $\boxed{V_k \cap V_\ell \neq \emptyset \Rightarrow k = \ell}$

- $\forall k \in \mathbb{N}$, V_k is $\mathcal{L}^1$-measurable, so $\bigcup_{k\in\mathbb{N}} V_k$ is $\mathcal{L}^1$-measurable and:
$[0,1) \underset{(a)}{\subset} \bigcup_{k\in\mathbb{N}} V_k \underset{(b)}{\subset} [-1,2)$: (b): $V_k = q_k + V \subset (-1,1) + [0,1] \subset (-1,2)$, $\forall k \in \mathbb{N}$

(a): Let $x \in [0,1]$. From (2), $\exists z \in \mathbb{Q}$ s.t. $x - z =: v \in V$, but $z = \underset{[0,1]}{x} - \underset{[0,1]}{v} \in (-1,1)$,
so $\exists k \in \mathbb{N}$ s.t. $z = q_k$ and $x = q_k + v \in V_k$.

- On the one hand: $\mathcal{L}^1\left(\bigcup_{k\in\mathbb{N}} V_k\right) \overset{(\Pi 1)}{=} \sum_{k\in\mathbb{N}} \mathcal{L}^1(V_k) \overset{(\Pi 2)}{=} \sum_{k\in\mathbb{N}} \mathcal{L}^1(V) \in \{0;\infty\}$. (α)

and on the other, by monotonicity (implied by (Π1)):

$$1 \overset{(\Pi 3)}{=} \mathcal{L}^1([0,1)) \underset{(a)}{\leq} \mathcal{L}^1\left(\bigcup_{k\in\mathbb{N}} V_k\right) \underset{(b)}{\leq} \mathcal{L}^1([-1,2)) \overset{(\Pi 1)}{\leq} \mathcal{L}^1([-1,0)) + \mathcal{L}^1([0,1)) + \mathcal{L}^1([1,2)) \overset{(\Pi 2)}{=} 3\,\mathcal{L}^1([0,1)) \overset{(\Pi 3)}{=} 3.$$

i.e. $\mathcal{L}^1\left(\bigcup_{k\in\mathbb{N}} V_k\right) \in [1,3]$ (β). But (α) and (β) are contradictory

CCL: $\boxed{V \text{ is not } \mathcal{L}^1\text{-measurable}}$

A20: The Cantor set is defined as $C = \bigcap_{k \in \mathbb{N}} A_k$ where A_k can recursively be defined as: $A_0 = [0,1]$, $A_{k+1} = \frac{1}{3} A_k \underset{\substack{\uparrow \\ \text{disjoint}}}{\cup} \left(\frac{2}{3} + \frac{1}{3} A_k \right)$, $k \in \mathbb{N}$

In particular by σ-additivity of $\mathcal{L}^1$ (Vol):

$$\mathcal{L}^1(A_{k+1}) = \mathcal{L}^1\left(\tfrac{1}{3} A_k\right) + \mathcal{L}^1\left(\tfrac{2}{3} + \tfrac{1}{3} A_k\right) \underset{\substack{\uparrow \\ \text{invariance by translation / dilation}}}{=} \frac{2}{3} \mathcal{L}^1(A_k), \text{ whence } \mathcal{L}^1(A_k) = \left(\frac{2}{3}\right)^k.$$

Alternatively, one may prove by induction that A_k is the union of 2^k intervals of length 3^{-k}:

$P(1)$: $A_1 = [0, 1/3] \cup [2/3, 1]$: 2 intervals of length $\frac{1}{3}$

$P(k) \Rightarrow P(k+1)$: for every of the 2^k interval of A_k, we split it in 3 and remove the middle part, leaving 2 more intervals of $1/3$ of the initial length; that is $2 \cdot 2^k = 2^{k+1}$ intervals of length $\frac{1}{3} \cdot 3^{-k} = 3^{-(k+1)}$.

In any case, by monotonicity: $C \subset A_k$, $\forall k \in \mathbb{N} \Rightarrow \mathcal{L}^1(C) \leq \mathcal{L}^1(A_k) = \left(\frac{2}{3}\right)^k \xrightarrow[\text{as } k \to +\infty]{} 0$

ccl $\boxed{\mathcal{L}^1(C) = 0}$

Rk: (One may prove that $C \underset{\substack{\uparrow \\ \text{bijection}}}{\simeq} \left\{ \sum_{k=1}^{\infty} a_k \cdot 3^{-k} : a_k \in \{0;2\} \right\} \underset{\uparrow}{\simeq} 2^{\mathbb{N}}$ is uncountable