(A9) GIVEN $f(x) = \dfrac{(x-\pi)^2}{4}$, $x \in [0, 2\pi]$

WE HAVE ALREADY COMPUTED THAT $\forall k \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$,

$$\hat{f}(k) = \frac{1}{2k^2}$$

AND $\hat{f}(0) = \dfrac{\pi^2}{12}$.

SINCE $f \in L^2([0, 2\pi])$, FROM PARSEVAL'S IDENTITY, WE HAVE

$$\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2 \, dt = \|f\|_{L^2}^2 = \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2$$

$$= |\hat{f}(0)|^2 + 2\sum_{k=1}^{\infty} \frac{1}{(2k^2)^2}$$

$$= \frac{\pi^4}{144} + \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k^4} \quad \text{--- } (*)$$

ALSO, $\|f\|_{L^2}^2 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \frac{(x-\pi)^4}{16} \, dx = \frac{1}{32\pi} \cdot \frac{(x-\pi)^5}{5}\Bigg|_0^{2\pi} = \frac{\pi^4}{80}$.

THEREFORE, FROM $(*)$, WE HAVE

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = 2\left(\frac{\pi^4}{80} - \frac{\pi^4}{144}\right) = \frac{\pi^4}{90}.$$

(A10) $f_k(t) = \sin(kt)$, $g_k(t) = \cos(kt)$, $h_k(t) = e^{ikt}$.

(i) $\langle f_k, f_l \rangle = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \sin(kt)\sin(lt) \, dt$

$$= \frac{1}{4\pi}\left[\int_{-\pi}^{\pi} \cos((k-l)t) \, dt - \int_{-\pi}^{\pi} \cos((k+l)t) \, dt\right]$$

(USING THE IDENTITY $2\sin\theta\sin\varphi = \cos(\theta-\varphi) - \cos(\theta+\varphi)$)

IF $k \neq l$, WE THEN HAVE

$$\langle f_k, f_l \rangle = \frac{1}{4\pi}\left[\frac{\sin((k-l)t)}{k-l}\Bigg|_{-\pi}^{\pi} - \frac{\sin((k+l)t)}{k+l}\Bigg|_{-\pi}^{\pi}\right]$$

$$= 0$$

IF $k = \pm l$, THEN IT CAN BE EASILY CHECKED THAT

$$\langle f_k, f_l \rangle = \pm \frac{1}{2}.$$

$$\therefore \boxed{f_k \perp f_l \quad \forall k \neq \pm l}$$

(ii) $$\langle f_k, g_l \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(kt) \cos(lt)\, dt$$

$$= \frac{1}{4\pi} \left[\int_{-\pi}^{\pi} \sin((k+l)t)\, dt + \int_{-\pi}^{\pi} \sin((k-l)t)\, dt \right]$$

IF $k \neq \pm l$,

$$\langle f_k, g_l \rangle = \frac{1}{4\pi} \left[-\frac{1}{k+l} \cos((k+l)t) \Big|_{-\pi}^{\pi} - \frac{1}{k-l} \cos((k-l)t) \Big|_{-\pi}^{\pi} \right]$$

$$= 0.$$

SIMILARLY WE CAN SHOW THAT IF $k = \pm l$.

$$\langle f_k, g_l \rangle = 0.$$

$$\therefore \boxed{\langle f_k, g_l \rangle = 0 \quad \forall k, l}$$

A10) (iii) LET $k, l \in \mathbb{Z}$.

WE HAVE THAT $$\langle h_k, h_l \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikt} \overline{e^{ilt}} \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-l)t} \, dt$$

CASE 1: $k = l$

THEN $$\langle h_k, h_k \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{0} \, dt = 1.$$

CASE 2: $k \neq l$

WE USE THE FACT THAT $t \mapsto e^{i(k-l)t}$ IS 2π-PERIODIC WITH ZERO MEAN.

$$\langle h_k, h_l \rangle = \frac{1}{2\pi} \left\{ \underbrace{\int_{-\pi}^{\pi} \cos((k-l)t) \, dt}_{= 0} + i \underbrace{\int_{-\pi}^{\pi} \sin((k-l)t) \, dt}_{= 0} \right\}$$

$$= 0$$

$$\therefore \boxed{h_k \perp h_l \quad \forall k \neq l}$$

A11) (i) CLEARLY $\mu_1 : 2^X \to [0,\infty) \cup \{\infty\}$, AS THE NUMBER OF ELEMENTS OF $A \subset X$ IS EITHER FINITE OR INFINITE.

ALSO, (a) $\mu_1(\varphi) = 0$ AS φ HAS NO ELEMENT

(b) IF $A \subset \bigcup_{k=1}^{\infty} A_k$, A HAS ATMOST AS MANY ELEMENTS AS THE COLLECTION $\{A_k\}_{k \in \mathbb{N}}$, WHICH IF THE A_k's ARE DISJOINT HAS $\sum_{k=1}^{\infty} \mu_1(A_k)$ ELEMENTS.

$$\therefore \mu_1(A) \leq \sum_{k=1}^{\infty} \mu_1(A_k).$$

(ii) $\mu_2 : 2^X \to \{0,1\} \subset [0,\infty) \cup \{\infty\}$.

(a) BY DEFINITION, $\mu_2(\varphi) = 0$.

(b) IF $A \subset \bigcup_{k=1}^{\infty} A_k$, EITHER $A = \varphi$ AND $\mu_2(A) = 0 \leq \sum_{k=1}^{\infty} \mu_2(A_k)$

OR $A \neq \varphi$ AND $\exists k_0 \in \mathbb{N}^*$ SUCH THAT $A_{k_0} \neq \varphi$

(OTHERWISE $A \subset \bigcup_{k=1}^{\infty} A_k = \varphi$, CONTRADICTION)

$$\therefore \mu_2(A) = 1 \leq \underbrace{\mu_2(A_{k_0})}_{=1} + \sum_{k \neq k_0} \underbrace{\mu_2(A_k)}_{\geq 0}.$$

(iii)

LET $A := \mathbb{Q} \cap [0,1]$. SINCE $A \subset \bigcup_{k=1}^{N} (a_k, b_k)$, WITHOUT LOSS OF GENERALITY WE CAN ASSUME THAT

$$a_1 < 0 < b_1 \leq a_2 < b_2 \leq a_3 < \cdots < b_{N-1} \leq a_N < 1 < b_N$$

CLAIM 1: $\mu_3(A) \leq 1$

PROOF: $\forall n \in \mathbb{N}$, $A \subset \left(-\frac{1}{n}, 1+\frac{1}{n}\right)$

$$\Rightarrow \mu_3(A) \leq \left| -\frac{1}{n} - \left(1+\frac{1}{n}\right) \right| = 1 + \frac{2}{n}$$

SINCE THIS IS TRUE FOR ALL $n \in \mathbb{N}$, WE HAVE

$$\boxed{\mu_3(A) \leq 1} \longrightarrow (*)$$

CLAIM 2: IF $A \subset \bigcup_{k=1}^{N} (a_k, b_k)$, THEN $b_{i-1} = a_i$, $i = 2, \ldots, N$

PROOF: SUPPOSE ON THE CONTRARY, $\exists i \in \{2, \ldots, N\}$ SUCH THAT $b_{i-1} < a_i$.

SINCE $\mathbb{Q}$ IS DENSE IN $\mathbb{R}$, $\exists q \in \mathbb{Q}$ SUCH THAT

$$b_{i-1} < q < a_i.$$

BUT THIS IMPLIES $q \notin \bigcup_{k=1}^{N} (a_k, b_k)$ AND HENCE

$A \not\subset \bigcup_{k=1}^{N} (a_k, b_k)$ WHICH IS A CONTRADICTION.

$$\therefore \boxed{b_{i-1} = a_i \quad \forall i \in \{2, \ldots, N\}}.$$

USING CLAIM 2, WE CONCLUDE THAT

$$a_1 < 0 < b_1 = a_2 < b_2 = a_3 < \cdots < b_{N-1} = a_N < 1 < b_N$$

THEREFORE, $\sum_{k=1}^{N} |a_k - b_k| = b_N - a_1 > 1$

$$\Rightarrow \boxed{\mu_3(A) \geq 1} \longrightarrow (**)$$

FROM $(*)$ AND $(**)$, WE THEN HAVE $\boxed{\mu_3(A) = 1}$.

NOW GIVEN ANY POINT x, WE CAN WRITE

$$\{x\} \subset \left(x - \tfrac{1}{n}, x + \tfrac{1}{n}\right) \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \mu_3(\{x\}) \leq \frac{2}{n} \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \boxed{\mu_3(\{x\}) = 0}.$$

SINCE $\mathbb{Q}$ IS COUNTABLE, WE CAN WRITE $A = \bigcup_{k \in \mathbb{N}} \{q_k\}$

BUT $\sum_{k=1}^{\infty} \mu_3(\{q_k\}) = 0 \neq 1 = \mu_3(A)$

AND HENCE μ_3 IS NOT A MEASURE.

4.12) (i) IT FOLLOWS FROM AXIOM (ii) OF MEASURES WITH

$$A = \bigcup_{k \in \mathbb{N}} A_k \subset \bigcup_{k=1}^{\infty} A_k.$$

(ii) IF $A \subset B$, IT IS A PARTICULAR CASE OF AXIOM (ii) THAT $\mu(A) \leq \mu(B)$ HOLDS.