Reelle Analysis - Serie 2.

Aufgabe 4: (i) We proceed by induction. Let $P(m) := \{ \widehat{f^{(m)}}(k) = (ik)^m \hat{f}(k),\ \forall f \in C^m(\mathbb{T}), \forall k \in \mathbb{Z}$ for $m \in \mathbb{N},\ m \le l$.

$P(0)$ is true : $\widehat{f^{(0)}}(k) = (ik)^0 \hat{f}(k) = \hat{f}(k),\ \forall f \in C^0(\mathbb{T}),\ \forall k \in \mathbb{Z}$.

Suppose $P(m)$ is true for some $m \in \{0; 1; \dots; l-1\}$. We show that $P(m+1)$ is true.

Let $f \in C^{m+1}(\mathbb{T})$ and $k \in \mathbb{Z}$. We have:

$$\widehat{f^{(m+1)}}(k) = \widehat{(f')^{(m)}}(k) \underset{\substack{\uparrow \\ f' \in C^m(\mathbb{T});\ \text{using } P(m)}}{=} (ik)^m \hat{f'}(k) = (ik)^m \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(t) e^{-ikt}\, dt$$

$$\underset{\text{i.b.p.}}{=} (ik)^m \frac{1}{2\pi} \Big\{ \underbrace{\underbrace{f(t) e^{-ikt}}_{2\pi\text{-periodic}} \Big|_{-\pi}^{\pi}}_{=0} - \int_{-\pi}^{\pi} f(t) \underbrace{\frac{d}{dt}(e^{-ikt})}_{= -ik e^{-ikt}} dt \Big\} = (ik)^{m+1} \hat{f}(k).$$ This shows $P(m+1)$ holds

(ii) From (i): $\hat{f}(k) = (ik)^{-l} \widehat{f^{(l)}}(k)$ with $f^{(l)} \in C^0(\mathbb{T})$ hence $f^{(l)} \in R(\mathbb{T})$. (integrable).

So by Riemann-Lebesgue: $\widehat{f^{(l)}}(k) \longrightarrow 0$ as $|k| \to +\infty$ hence $|\hat{f}(k)| \le |k|^{-l} \underbrace{|\widehat{f^{(l)}}(k)|}_{\longrightarrow 0 \text{ as } |k| \to +\infty}$

Recall: Def.: $\Big| f(k) = o(g(k))$ as $|k| \to +\infty \iff \forall \varepsilon > 0, \exists A > 0$ s.t. $|k| > A \Rightarrow |f(k)| \le \varepsilon |g(k)|$. $(*)$

In conclusion we have: $\hat{f}(k) = o(|k|^{-l})$ as $|k| \to +\infty$.

(iii) When $f \in C^2(\mathbb{T})$, by (ii) we know that $|\hat{f}(k)| = o(|k|^{-2})$, and moreover $\sum'_{k \in \mathbb{Z}^*} |k|^{-2} < +\infty$, so that by comparison of series we have $\sum'_{k \in \mathbb{Z}^*} |\hat{f}(k)| < +\infty$.

$\Big[\varepsilon = 1$ in $(*) \Rightarrow \exists A > 0 : \sum'_{k \in \mathbb{Z}^*} |\hat{f}(k)| \le \sum'_{|k| < A} |\hat{f}(k)| + \sum'_{|k| \ge A} |k|^{-2} < +\infty \Big]$

Aufgabe 5: Let $f(x) = \dfrac{(x-\pi)^2}{4}$, $x \in [0, 2\pi]$: $f(0) = \dfrac{\pi^2}{4} = f(2\pi)$, so we may extend f by 2π-periodicity on $\mathbb{R}$ into a continuous function. We also have: $f \in C^\infty([0, 2\pi])$.

For $k \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$, we compute:

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{(t-\pi)^2}{4} e^{-ikt} dt \underset{\text{i.b.p.}}{=} \frac{1}{2\pi} \Big\{ \underbrace{\frac{(t-\pi)^2}{4} \frac{e^{-ikt}}{-ik} \Big|_0^{2\pi}}_{=0} - \int_0^{2\pi} \frac{t-\pi}{2} \cdot \frac{e^{-ikt}}{-ik} dt \Big\}$$

$$= \frac{1}{4\pi i k} \int_0^{2\pi} (t-\pi) e^{-ikt} dt \underset{\text{i.b.p.}}{=} \frac{1}{4\pi i k} \Big\{ \underbrace{(t-\pi) \frac{e^{-ikt}}{-ik} \Big|_0^{2\pi}}_{= \frac{2\pi}{-ik}} - \underbrace{\int_0^{2\pi} \frac{e^{-ikt}}{-ik} dt}_{=0} \Big\} = \frac{1}{2k^2}.$$

By Dirichlet's theorem, since $f \in R(\mathbb{T})$ and $C^1(\mathbb{T})$ we find:

$$\forall x \in [0, 2\pi],\ f(x) = \hat{f}(0) + \sum'_{k \in \mathbb{Z}^*} \frac{e^{+ikt}}{2k^2} \; ; \text{ where } \hat{f}(0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(t-\pi)^2}{4} dt = \frac{1}{2\pi} \frac{(t-\pi)^3}{12} \Big|_0^{2\pi} = \frac{\pi^2}{12}.$$

In particular: $\dfrac{\pi^2}{4} = f(0) = \dfrac{\pi^2}{12} + 2 \sum'^{\infty}_{k=1} \dfrac{1}{2k^2}$. hence:

$$\sum^{\infty}_{k=1} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Aufgabe 6: For $k \in \mathbb{Z}^*$, we calculate:

$$\widehat{(f - f_{\pi/k})}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(f(t) - f(t+\tfrac{\pi}{k})\right)e^{-ikt}\,dt = \hat{f}(k) - \frac{1}{2\pi}\int_{-\pi+\frac{\pi}{k}}^{\pi+\frac{\pi}{k}} f(s)\,e^{-iks+i\pi}\,ds.$$

Substitution in the 2nd integral:

$$\left|\begin{aligned} s &= t + \tfrac{\pi}{k} \\ ds &= dt \end{aligned}\right.$$

$= 2\hat{f}(k)$, since $e^{i\pi} = -1$, and by 2π-periodicity of $s \mapsto f(s)e^{-iks}$.

It follows that:

$$|\hat{f}(k)| = \frac{1}{2}\left|\widehat{(f - f_{\pi/k})}(k)\right| \le \frac{1}{4\pi}\int_{-\pi}^{\pi}\underbrace{\left|f(t) - f(t+\tfrac{\pi}{k})\right|}_{\le C\left|\frac{\pi}{k}\right|^{\alpha}}dt \le \underbrace{\frac{C\pi^{\alpha}}{2}}_{=:\,K\,=\,K(C,\alpha)}\cdot|k|^{-\alpha}$$

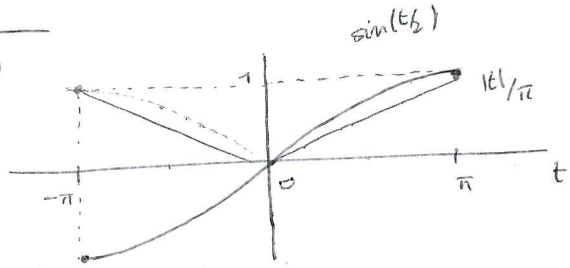
dependence on f.

Aufgabe 7: Recall that $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(t)\,dt = 1$ where $D_n(t) = \frac{\sin((n+\frac{1}{2})t)}{\sin(\frac{t}{2})}$ (induction for instance).

We deduce that for $x \in \mathbb{R}$:

$$S_n[f](x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\,D_n(t)\,dt - \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(t)\,dt\right)\cdot f(x)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(f(x-t) - f(t)\right)D_n(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{\frac{f(x-t) - f(x)}{\sin(\frac{t}{2})}}_{=:\,F_x(t)}\cdot\sin((n+\tfrac{1}{2})t)\,dt$$



On $[-\pi;\pi]$, we have: $|\sin(\frac{t}{2})| \ge \frac{|t|}{\pi}$

From Lipschitz regularity of f we thus find:

$|F_x(t)| \le C\pi$. In other words, $t \mapsto F_x(t)$ is continuous, bounded on $[-\pi,\pi]$, so it is integrable. By Riemann-Lebesgue, we deduce:

$$S_n[f](x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{F_x(t)}_{\text{integrable}}\sin((n+\tfrac{1}{2})t)\,dt \longrightarrow 0 \quad \text{as } n \to +\infty.$$