PHYS 1370 **Introduction to Quantum Mechanics I** **Fall 2008**

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Homework for week of 10/27/2008

Deadline 11/03/2008

Problem 1: *(2 points)*
Find the momentum space wave function, $\Phi(p,t)$, for a particle in the ground state of the harmonic oscillator.

Problem 2: *(3 points)*
Show that

$$\langle x\rangle = \int \Phi^* \left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\right) \Phi \, dp.$$

Hint: Notice that $x\, e^{ipx/\hbar} = -i\hbar \frac{d}{dp}\, e^{ipx/\hbar}$.

Thus we find more generally

	position space	momentum space
position operator $\hat{x}$	x	$\frac{\hbar}{i}\frac{\partial}{\partial x}$
momentum operator $\hat{p}$	$i\hbar\frac{\partial}{\partial p}$	p

In principle you can do all calculations in momentum space just as well as in position space.

Problem 3: *(3 points)*

a) Show that the following commutator identity holds:

$$[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}.$$

b) Prove the following relation between the uncertainty in position ($\hat{x} = x$) and energy ($\hat{H} = \hat{p}^2/(2m) + V(x)$):

$$\sigma_x \sigma_H \geq \frac{\hbar}{2m}|\langle p\rangle|.$$

What does this relation tell you about stationary states?

Problem 4: *(4 points)*
Apply

$$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\left\langle [\hat{H},\hat{Q}]\right\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t}\right\rangle \tag{1}$$

to the special cases $\hat{Q}=1$, $\hat{Q}=\hat{H}$, $\hat{Q}=\hat{x}$, and $\hat{Q}=\hat{p}$.

Problem 5: Virial theorem. *(2 points)*
Use (1) to show that

$$\frac{d}{dt}\langle xp\rangle = 2\langle T\rangle - \left\langle x\frac{dV}{dx}\right\rangle.$$

where T is the kinetic energy, $H = T + V$. In a stationary state the left hand side is zero, so

$$2\langle T\rangle = \left\langle x\frac{dV}{dx}\right\rangle. \qquad \text{(Virial theorem)}$$

Use this to prove $\langle T\rangle = \langle V\rangle$ for stationary states of the harmonic oscillator.

Problem 6: *(3 points)*

a) For a function $f(x)$ that can be expanded in a Taylor series, show that

$$f(x+x_0) = e^{i\hat{p}x_0/\hbar}f(x)$$

(with x_0 constant). For this reason, $\hat{p}/\hbar$ is called the *generator of space translations.*
Note: The exponential of an operator is defined by the power series
$e^{\hat{Q}} = 1 + \hat{Q} + \frac{1}{2}\hat{Q}^2 + \frac{1}{3!}\hat{Q}^3 + \ldots$

b) If $\Psi(x,t)$ satisfies the (time-dependent) Schrödinger equation, show that

$$\Psi(x,t+t_0) = e^{-i\hat{H}t_0/\hbar}\Psi(x,t)$$

(with a constant t_0). $-\hat{H}/\hbar$ is called the *generator of time translations.*

c) Show that the expectation value of a dynamical variable $Q(x,p)$ at time $t+t_0$ can be written as

$$\langle Q\rangle_{t+t_0} = \left\langle \Psi(x,t)\left| e^{i\hat{H}t_0/\hbar}\hat{Q}(\hat{x},\hat{p})e^{-i\hat{H}t_0/\hbar}\right| \Psi(x,t)\right\rangle.$$