

Homework for week of 10/27/2008

Deadline 11/03/2008

Problem 1:

(2 points)

Find the momentum space wave function, $\Phi(p, t)$, for a particle in the ground state of the harmonic oscillator.

Problem 2:

(3 points)

Show that

$$\langle x \rangle = \int \Phi^* \left(-\frac{\hbar}{i} \frac{\partial}{\partial p} \right) \Phi dp.$$

Hint: Notice that $x e^{ipx/\hbar} = -i\hbar \frac{d}{dp} e^{ipx/\hbar}$.

Thus we find more generally

	position space	momentum space
position operator \hat{x}	x	$\frac{\hbar}{i} \frac{\partial}{\partial x}$
momentum operator \hat{p}	$i\hbar \frac{\partial}{\partial p}$	p

In principle you can do all calculations in momentum space just as well as in position space.

Problem 3:

(3 points)

a) Show that the following commutator identity holds:

$$[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}.$$

b) Prove the following relation between the uncertainty in position ($\hat{x} = x$) and energy ($\hat{H} = \hat{p}^2/(2m) + V(x)$):

$$\sigma_x \sigma_H \geq \frac{\hbar}{2m} |\langle p \rangle|.$$

What does this relation tell you about stationary states?

Problem 4:

(4 points)

Apply

$$\frac{d}{dt}\langle Q \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{Q}] \rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle \quad (1)$$

to the special cases $\hat{Q} = 1$, $\hat{Q} = \hat{H}$, $\hat{Q} = \hat{x}$, and $\hat{Q} = \hat{p}$.**Problem 5:** Virial theorem.

(2 points)

Use (1) to show that

$$\frac{d}{dt}\langle xp \rangle = 2\langle T \rangle - \left\langle x \frac{dV}{dx} \right\rangle.$$

where T is the kinetic energy, $H = T + V$. In a stationary state the left hand side is zero, so

$$2\langle T \rangle = \left\langle x \frac{dV}{dx} \right\rangle. \quad (\text{Virial theorem})$$

Use this to prove $\langle T \rangle = \langle V \rangle$ for stationary states of the harmonic oscillator.**Problem 6:**

(3 points)

- a) For a function
- $f(x)$
- that can be expanded in a Taylor series, show that

$$f(x + x_0) = e^{i\hat{p}x_0/\hbar} f(x)$$

(with x_0 constant). For this reason, \hat{p}/\hbar is called the *generator of space translations*.

Note: The exponential of an operator is defined by the power series

$$e^{\hat{Q}} = 1 + \hat{Q} + \frac{1}{2}\hat{Q}^2 + \frac{1}{3!}\hat{Q}^3 + \dots$$

- b) If
- $\Psi(x, t)$
- satisfies the (time-dependent) Schrödinger equation, show that

$$\Psi(x, t + t_0) = e^{-i\hat{H}t_0/\hbar} \Psi(x, t)$$

(with a constant t_0). $-\hat{H}/\hbar$ is called the *generator of time translations*.

- c) Show that the expectation value of a dynamical variable
- $Q(x, p)$
- at time
- $t + t_0$
- can be written as

$$\langle Q \rangle_{t+t_0} = \left\langle \Psi(x, t) \left| e^{i\hat{H}t_0/\hbar} \hat{Q}(\hat{x}, \hat{p}) e^{-i\hat{H}t_0/\hbar} \right| \Psi(x, t) \right\rangle.$$