PHYS 1370 **Introduction to Quantum Mechanics I** **Fall 2008**

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Homework for week of 09/29/2008

Deadline 10/06/2008

Problem 1: *(4 points)*

a) Show that

$$\Psi(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + \frac{a^2}{2}(1+e^{-2i\omega t}) + \frac{i\hbar t}{m} - 2axe^{-i\omega t}\right)\right] \tag{1}$$

satisfies the time-dependent Schrödinger equation for the harmonic oscillator potential. Here a is a real constant with the dimension of length.

b) Eq. (1) can be rewritten as

$$\Psi(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega}{2\hbar}(x - x_c(t))^2 + \frac{i}{\hbar}\, x\, p_c(t) + i f(t)\right], \tag{2}$$

where $x_c(t) = a\cos(\omega t)$ and $p_c(t) = m\,\dot{x}_c(t)$ describe a solution for the *classical* harmonic oscillator, and $f(t)$ is a real function that only depends on t. Find $|\Psi(x,t)|^2$ and describe the motion of the wave packet.

c) Compute $\langle x\rangle$ and $\langle p\rangle$ (it is not necessary to perform any integral explicitly for this), and check that Ehrenfest's theorem

$$\frac{d\langle p\rangle}{dt} = -\left\langle \frac{\partial V}{\partial x}\right\rangle. \tag{3}$$

is satisfied.

Problem 2: The gaussian wave packet. *(6 points)*

A free particle has the initial wave function

$$\Psi(x,0) = Ae^{-ax^2}, \tag{4}$$

where A and a are constants, and $a > 0$.

a) Normalize $\Psi(x,0)$.

b) Find $\Psi(x,t)$.

Hint: Integrals of the form

$$\int_{-\infty}^{+\infty} e^{-(ax^2+bx)}\, dx$$

can be handled by "completing the square": let $y \equiv \sqrt{a}[x + b/(2a)]$, and note that $(ax^2 + bx) = y^2 - b^2/(4a)$.

In the end you should find

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-ax^2/(1+2i\hbar at/m)}}{\sqrt{1+2i\hbar at/m}}. \tag{5}$$

c) Find $|\Psi(x,t)|^2$. It is useful to introduce $\theta \equiv 1 + 2i\hbar at/m$ as an abbreviation. Sketch $|\Psi|^2$ as a function of x, at $t = 0$ and for some large t. Qualitatively, what happens to $|\Psi|^2$ as time goes on?

d) Calculate $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, $\langle p^2\rangle$, σ_x and σ_p. Does the uncertainty principle hold? At what time t does the system come closest to the uncertainty limit?

Problem 3: *(4 points)*

The delta function can be defined by

$$\int_{-\infty}^{+\infty} f(x)\delta(x)\, dx = f(0) \tag{6}$$

for every regular function $f(x)$.

a) Show that $\delta(cx) = \frac{1}{|c|}\delta(x)$ for any real constant c. [Check both $c > 0$ and $c < 0$.]

b) Let $\theta(x)$ be the step function,

$$\theta(x) \equiv \begin{cases} 1, & \text{for } x > 0, \\ 0, & \text{for } x \leq 0. \end{cases} \tag{7}$$

Show that $d\theta/dx = \delta(x)$.

c) Using Plancherel's theorem, show that the Fourier transform of $\delta(x)$ is given by

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikx}\, dk. \tag{8}$$

Note: technically, the delta function does not meet the requirements for Plancherel's theorem, and the integral on the right-hand side of eq. (8) seems ill-defined. This formula is still very useful, though, for many applications.